Correlators from unitarity in 1D CFT

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based on ongoing work with L.Bianchi,G.Bliard, V.Forini and G.Peveri

Why 1D CFTs?

1D CFTs have many interesting realizations, such as:

- conformal line defects in higher-dimensional CFTs
- lines of fixed points in SYK models
- boundary correlators of QFTs in AdS_2

Why Unitarity?

- Unitarity often allows to compute amplitudes from their discontinuity (e.g. optical theorem).
- In perturbation theory, the discontinuity at a given order is often fixed in terms of lower order data.
- Unitarity methods have been crucial for an efficient evaluation of scattering amplitudes in 4D QFTs.
- Unitarity has been successfully applied to higher-dimensional CFTs in combination with crossing symmetry.
  [Aharony, Alday, Bissi, Caron-Huot, Meltzer, Perlmutter,...]

Correlators in 1D CFT

- Two and three-point functions are fixed by conformal symmetry

.

$$< O_\Delta(x_1) O_\Delta(x_2) >= \frac{C}{x_{12}^{2\Delta}}$$

$$< O_{\Delta_1}(x_1) O_{\Delta_2}(x_2) O_{\Delta_3}(x_3) >= \frac{f_{\Delta_1 \Delta_2 \Delta_3}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

- The 4-point function depends on $\mathcal{G}(z)$ where $z = \frac{x_{12}x_{34}}{x_{13}x_{24}}$

$$< O_\Delta(x_1) O_\Delta(x_2) O_\Delta(x_3) O_\Delta(x_4) >= \frac{C^2}{x_{12}^{2\Delta} x_{34}^{2\Delta}} \mathcal{G}(z)$$

4-point functions

- Using the OPE

$$O_{\Delta_1}(x_1)O_{\Delta_2}(x_2) = \sum_h f_{\Delta_1\Delta_2 h}\, x_{12}^{h-\Delta_1-\Delta_2}(O_h(x_2)+cx_{12}^2\partial^2 O_h+\ldots)$$

we find

$$\mathcal{G}(z) = \sum_h a_h G_h(z)$$

where $G_h = z^h {}_2F_1(h,h,2h,z)$ is the conformal block and $a_h = |f_{\Delta\Delta h}|^2$.

Crossing symmetry and Regge boundedness

- Expanding $\mathcal{G}(z)$ with the OPE in different channels

$$\mathcal{G}(z) = \frac{z^{2\Delta}}{(1-z)^{2\Delta}} \mathcal{G}(1-z)$$

- Unitarity and the OPE imply for $z \to \infty$ (Regge limit)

$$|z^{-2\Delta} \mathcal{G}(z)| < \infty$$

I_h and the OPE data

- $\mathcal{G}(z)$ can also be expanded in a complete set as

$$\begin{aligned}\mathcal{G}(z) &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{dh}{2\pi i} \frac{\Gamma(1-h)\Gamma^2(\frac{h}{2})}{\sqrt{\pi}\Gamma(h-\frac{1}{2})\Gamma^2(\frac{1-h}{2})} I_h G_h(z) + \sum_{m\in 2\mathrm{N}} \tilde{I}_m G_m(z) \\ &= \sum_{poles\ \ h^*} Res(\frac{\Gamma(1-h^*)\Gamma^2(\frac{h^*}{2})}{\sqrt{\pi}\Gamma(h^*-\frac{1}{2})\Gamma^2(\frac{1-h^*}{2})} I_{h^*}) G_{h^*}(z) \\ &= \sum_{h^*} a_{h^*} G_{h^*}(z)\end{aligned}$$

- I_h encodes the OPE data of the 4-point function!

$$I_h \to (h, a_h) \to \mathcal{G}(z)$$

The OPE inversion formula [Mazac '18]

For crossing symmetric and Regge-bounded $\mathcal{G}(z)$

$$I_h = 2\int_0^1 dz H_h^{\Delta}(z) dDisc(\mathcal{G}(z))$$

- $H_h^{\Delta}(z)$ is known explicitly for all integer Δ in the case of bosons and for half-integer Δ for fermions.
- In the bosonic case one may need to regularize $\mathcal{G}(z)$ and add two extra terms

$$\int_{C_-} dz H_h^{\Delta}(z)\mathcal{G}(z) + \int_{C_+} dz H_h^{\Delta}(z)\mathcal{G}(z)$$

$C_{\pm}$ are infinitesimal semicircular contours centered in $z = 1$

The double discontinuity

The crucial ingredient of the formula is

$$dDisc(\mathcal{G}(z)) = \mathcal{G}(z) - \frac{\mathcal{G}^{\circlearrowleft}(z) + \mathcal{G}^{\circlearrowright}(z)}{2}$$

where $\mathcal{G}^{\circlearrowleft}(z)$ and $\mathcal{G}^{\circlearrowright}(z)$ are the analytic continuations of $\mathcal{G}(z)$ around $z = 1$, from above and below.

- It represents the thermal expectation value $< [O(x_3); O(x_2)][O(x_1); O(x_4)] >$
- $\mathcal{G}^{\circlearrowleft}(z)$ and $\mathcal{G}^{\circlearrowright}(z)$ correspond to the out-of-time-order contributions in the double commutator.

Why is this formula useful?

$$I_h = 2\int_0^1 dz z^{-2} H_h^{\Delta}(z) dDisc(\mathcal{G}(z))$$

- In perturbation theory $dDisc(\mathcal{G}(z))$ can be computed at any order from lower order data!
- We can use unitarity to find the OPE data of $\mathcal{G}(z)$ without doing complicated diagrams

$$dDisc(\mathcal{G}(z)) \to I_h \to (h, a_h) \to \mathcal{G}(z)$$

Perturbation theory

Start from Generalized Free Field (GFF)

$$\mathcal{G}^{(0)}(z) = 1 + z^{2\Delta} + \frac{z^{2\Delta}}{(1-z)^{2\Delta}}$$

Its OPE data are

$$h^{(0)} = 2\Delta + 2n$$

$$a_h^{(0)} = \frac{2\Gamma^2(2\Delta+2n)\Gamma(4\Delta+2n-1)}{\Gamma^2(2\Delta)\Gamma(4\Delta+4n-1)\Gamma(2n+1)}$$

Perturbation theory

Now consider a perturbative expansion around GFF

$$\mathcal{G}(z) = \mathcal{G}^{(0)}(z) + \epsilon\mathcal{G}^{(1)}(z) + \epsilon^2\mathcal{G}^{(2)}(z) + ...$$

where

$$h = 2\Delta + 2n + \epsilon\gamma^{(1)} + \epsilon^2\gamma^{(2)} + ...$$
$$a_h = a_n^{(0)} + \epsilon a_n^{(1)} + \epsilon^2 a_n^{(2)} + ...$$

dDisc in perturbation theory

The first few terms read

$$\mathcal{G}^{(1)}(z) = \sum_n a_n^{(1)} G_{2\Delta+2n}(z) + a_n^{(0)}\gamma_n^{(1)} \partial G_{2\Delta+2n}(z)$$

$$\mathcal{G}^{(2)}(z) = \sum_n a_n^{(2)} G_{2\Delta+2n}(z) + (a_n^{(0)}\gamma^{(2)} + a_n^{(1)}\gamma_n^{(1)}) \partial G_{2\Delta+2n}(z) + \frac{1}{2} a_n^{(0)} (\gamma_n^{(1)})^2 \partial^2 G_{2\Delta+2n}(z)$$

At any order in perturbation theory it is fixed in term of lower order data!

$$dDisc(\mathcal{G}^{(1)}(z)) = 0$$

$$dDisc(\mathcal{G}^{(2)}(z)) = \epsilon^2 \pi^2 \sum_n a_n^{(0)} (\gamma_n^{(1)})^2 \frac{z^{2\Delta_\phi}}{(1-z)^{2\Delta_\phi}} G_{2\Delta_\phi+2n}(1-z)$$

Example: Φ^4 in AdS_2

Consider for example a theory of a scalar in AdS_2

$$S = \int dx^2 \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \Delta(\Delta-1)\phi^2(x) + \frac{g}{4!}\phi^4(x)$$

- The boundary 4-point functions define a 1D CFT.
- We can check some results against explicit computations or results from a functional approach.[Mazac,Paulos '18]

Tree level

For simplicity we set

$$\Delta = 1$$

At tree level the double discontinuity is zero, therefore

$$\begin{aligned} I_h^{(1)} &= \int_{C_-} dz H_h(z)\mathcal{G}^{(1)}(z) + \int_{C_+} dz H_h(z)\mathcal{G}^{(1)}(z) \\ &= \frac{8\pi^3}{\sin(\pi h)} \end{aligned}$$

And the 4-point function is

$$\mathcal{G}^{(1)}(z) = 2z^2 \left(\frac{log(1-z)}{z} + \frac{log(z)}{1-z} \right)$$

One loop

At one loop the double discontinuity is given by

$$dDisc(\mathcal{G}^{(2)}(z)) = \pi^2 \frac{z^2}{(1-z)^2} \log^2 z$$

and the contour integrals are given by

$$\begin{aligned} &\int_{C_-} dz H_h(z) \mathcal{G}^{(2)}(z) + \int_{C_+} dz H_h(z) \mathcal{G}^{(2)}(z) = \\ &= a_0^{(1)} \gamma_0^{(1)} \frac{8\pi^3}{\sin(\pi h)} \\ &= -2 I_h^{(1)} \end{aligned}$$

One loop coefficient

$$
\begin{aligned}
I_h^{(2)} = 2 \Bigg(\pi^3 \csc(\pi h) \Bigg(& -\frac{\Gamma(h)^2\, {}_3\tilde{F}_2(h-1, h, h; 2h, h+1; 1)}{\Gamma(-h)} \\
& + \frac{\sin(\pi h)\Gamma(2-h)\Gamma(1-h)^2\, {}_3\tilde{F}_2(1-h, 1-h, -h; 2-2h, 2-h; 1)}{\pi} \\
& - (h-1)h \left({}_3\tilde{F}_2{}^{(\{0,0,1\},\{0,0\},0)}(\{2-h, h+1, 0\}, \{2,2\}, 1) + \pi \cot(\pi h) \right) \\
& + \frac{\sin(\pi h)}{\pi h^2 - \pi h} - 2(h-1)h H_{h-2} + 2(h-2)h - \Gamma(2-h)\Gamma(h+1) + \frac{2\pi^2}{3} \\
& - 16 + 4\psi^{(2)}(2) + 4\pi^3 \csc(\pi h) \Big(\gamma\, {}_4\tilde{F}_3(1,1,1,2; 3, 2-h, h+1; 1) \\
& - {}_5\tilde{F}_4{}^{(\{0,0,0,0,0\},\{0,0,0,1\},0)}(\{1,1,1,2,2\}, \{3, 2-h, h+1, 2\}, 1) \\
& + \frac{2\pi \csc(\pi h) \left(2\pi^2 \left(-\psi^{(0)}\left(\frac{2-h}{2}\right) - \psi^{(0)}\left(\frac{h+1}{2}\right) - \gamma + \psi^{(0)}\left(\frac{1}{2}\right) \right) \right)}{(h-1)h} \\
& - 2 I_h^{(1)}
\end{aligned}
$$

One loop 4-point function

- In this case the coefficient is very complicated and we cannot resum the OPE expansion, we can only find the OPE data.
- We can find the 4-point function imposing an Ansatz using functions up to trascendentality four. [Ferrero et al., '19]
- We can impose crossing,unitarity and fix the remaining unknowns from OPE data extracted from $I_h^{(2)}$.

$$
\begin{aligned}
\mathcal{G}^{(2)}(z) = \frac{1}{(1-z)^2} \Big[& 4(z-2)z^3 \mathrm{Li}_4(1-z) + 4\left(z^2-1\right)(1-z)^2 \mathrm{Li}_4(z) \\
& - 2(1-z)^2 \mathrm{Li}_3(1-z)\left(\left(z^2-1\right)\log(1-z) + \left(z^2+2\right)\log(z)\right) \\
& - 2z^2 \mathrm{Li}_3(z)\left(\left(z^2-2z+3\right)\log(1-z) + (z-2)z\log(z)\right) \\
& + \left(3.29(2z-1) - (z-1)^2\left(z^2+1\right)\log^2(z)\right)\log^2(1-z) \\
& + 4(2z-1)\mathrm{Li}_4\left(\frac{z}{z-1}\right) - 1.08z^2\left(z^2-2z-6\right) - (z-1)z^2\log(z) \\
& + (1-z)^2\left(3.29\left(z^2+2\right)\log(z) + z\right)\log(1-z) \\
& + 1.20\left(2\log(z) - 2\left(2z^3-3z^2+4z-1\right)\log\left(\frac{z}{1-z}\right)\right) \\
& + \frac{1}{6}(2z-1)\log^4(1-z) - \frac{1}{3}(4z-2)\log(z)\log^3(1-z) \\
& - \frac{1}{2}\left(z^2\log^2\left(\frac{z}{1-z}\right) + \frac{z^2\log^2(z)}{(1-z)^2} + \log^2(1-z)\right)
\end{aligned}
$$

Conclusions

- We have sketched how to compute correlators in 1D CFTs using unitarity techniques.
- The OPE data is encoded in a coefficient I_h, which can be determined from the double discontinuity of the 4-point function

$$dDisc(\mathcal{G}(z)) \to I_h \to (h, a_h) \to \mathcal{G}(z)$$

- $dDisc(\mathcal{G}(z))$ is fixed at any order in perturbation theory in terms of lower order data.
- We have seen an explicit application in a perturbative expansion of a scalar theory in AdS_2.

Outlook

- We would like to find a dispersion formula to find $\mathcal{G}(z)$ directly from the double discontinuity.
- It would be interesting to extend this approach to the case of non identical operators and in the presence of a global symmetry.
- By doing that, one could apply this unitarity method to interesting 1D CFTs such as line defects in higher dimensional theories.

Thank you for your attention!