Complexity of mixed Gaussian states from Fisher information geometry

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Circuit Complexity

Reference state $|\psi_{\mathrm{R}}\rangle$ Target state $|\psi_{\mathrm{T}}\rangle$

Quantum circuit:

$$|\psi_{\mathrm{T}}\rangle = U\,|\psi_{\mathrm{R}}\rangle = Q_{i_D}\dots Q_{i_2}Q_{i_1}|\psi_{\mathrm{R}}\rangle$$

Set of elementary (unitary) gates: $\{Q_i\}_{i=1,\dots}$

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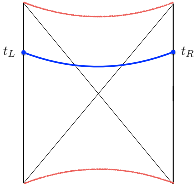
Complexity: minimum number of allowed gates that is needed to construct the target state starting from the assigned reference state

Holographic Complexity

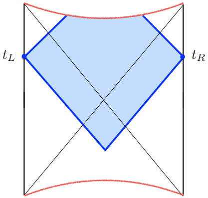
Holographic entanglement entropy is not able to capture the late-time dynamics behind the horizon of a black hole

[Stanford,Susskind,'14]



$$C_{\mathrm{V}} = \frac{\mathcal{V}}{\ell G_N^{(d+1)}}$$

[Brown, Roberts, Susskind, Swingle, Zhao,'16]



$$C_{\mathrm{A}} = \frac{I_{\mathrm{WDW}}}{\pi}$$

Complexity in quantum field theories still to be understood

Gaussian states in harmonic lattices

We compute the complexity for lattice models with a known continuum limit

$$\widehat{H} = \sum_{i=1}^{N} \left(\frac{1}{2m}\,\hat{p}_i^2 + \frac{m\omega^2}{2}\,\hat{q}_i^2 \right) + \sum_{\langle i,j \rangle} \frac{\kappa}{2} (\hat{q}_i - \hat{q}_j)^2 \qquad [\hat{q}_i, \hat{p}_j] = \mathrm{i}\delta_{ij}$$

- The continuum limit is a Klein-Gordon field theory
- In the continuum, when $\omega \to 0$, we have a conformal field theory

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We restrict our attention to the **Gaussian states** (with vanishing first moments)

$$\hat{\rho} \quad \longrightarrow \quad \gamma$$

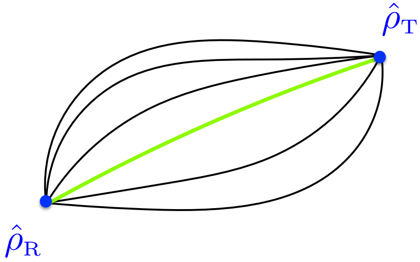
γ is the **covariance matrix**

The entries of γ are given by $\langle\{\hat{q}_i, \hat{q}_j\}\rangle$, $\langle\{\hat{p}_i, \hat{p}_j\}\rangle$ and $\langle\{\hat{q}_i, \hat{p}_j\}\rangle$ and it satisfies the uncertainty principle

$$\gamma + \mathrm{i}\frac{J}{2} \geqslant 0$$

Complexity as geodesic distance [Nielsen, '06]

The **quantum circuit** connecting two **mixed** states $\hat{\rho}_{\mathrm{R}}$ and $\hat{\rho}_{\mathrm{T}}$ can be seen as a **curve** connecting two points in a manifold of quantum mixed states



Optimal circuit (circuit with the minimum number of gates) $\longrightarrow$ Geodesics

Complexity $\longrightarrow$ Length of the geodesics

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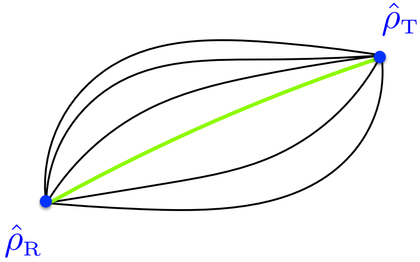

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Compute the complexity becomes a geometric problem with ambiguities due the choice of the **cost function**

Complexity of pure states: [Jefferson, Myers, '17; Hackl, Myers, '18; Chapman, Eisert, Hackl, Heller, Jefferson, Marrochio, Myers, '19]

Complexity from Fisher-Rao distance

Assumption: we restrict the space of all the states we can reach through a quantum circuit to the set of bosonic Gaussian **mixed** state

Reference state γ_{R}

Target state γ_{T}

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Reference state $\gamma_{\rm R}$ Target state $\gamma_{\rm T}$

The geodesics connecting $\gamma_{\rm R}$ and $\gamma_{\rm T}$ provides the optimal circuit: [Bhatia, Jain,'15]

$$G_s(\gamma_{\rm R}, \gamma_{\rm T}) \equiv \gamma_{\rm R}^{1/2} \left(\gamma_{\rm R}^{-1/2} \gamma_{\rm T} \gamma_{\rm R}^{-1/2} \right)^s,$$

$0 \leqslant s \leqslant 1$

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The complexity is given by the Fisher-Rao distance [Rao, '45; GDG, Tonni, '20]:

$$d(\gamma_{\mathrm{R}},\gamma_{\mathrm{T}}) \equiv \sqrt{\mathrm{Tr}\{[\log(\gamma_{\mathrm{T}}\gamma_{\mathrm{R}}^{-1})]^2\}} \qquad \mathcal{C} = \frac{1}{2\sqrt{2}}\,d(\gamma_{\mathrm{R}}\,,\gamma_{\mathrm{T}})$$

Explicit expressions for spectrum and basis complexity in [GDG, Tonni, '20]

Subsystem complexity at equilibrium [GDG, Tonni, '20]

$$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \quad \Rightarrow \quad \hat{\rho} = |\psi\rangle\langle\psi| \quad \Rightarrow \quad \hat{\rho}_A = \mathrm{Tr}_B \hat{\rho} \quad \longrightarrow \quad \gamma_A$$

$$\mathcal{C}_A = \frac{1}{2\sqrt{2}} \sqrt{\mathrm{Tr}\left\{ [\log(\gamma_{A,\mathrm{T}} \gamma_{A,\mathrm{R}}^{-1})]^2 \right\}}$$

Subsystem complexity in harmonic lattices [GDG, Tonni, '20]

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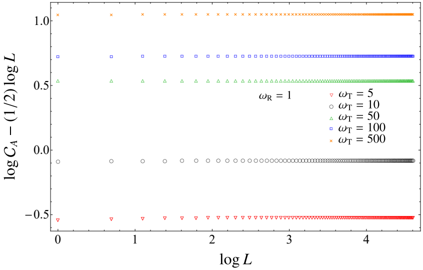
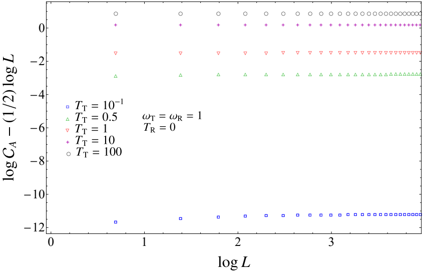
Subsystem complexity in harmonic lattices [GDG, Tonni, '20]

A is an interval in infinite line





Global quantum quench

$$\widehat{H} = \sum_{i=1}^{N} \left(\frac{1}{2m} \hat{p}_i^2 + \frac{m\omega^2}{2} \hat{q}_i^2 + \frac{\kappa}{2} (\hat{q}_i - \hat{q}_{i-1})^2 \right) = \widehat{H}(\omega, \kappa, m)$$

Global quench protocol [Calabrese, Cardy,'05]

$t = 0$

$|\Psi_0\rangle$ ground state of $\widehat{H}(\omega_0, \kappa, m)$

$[\widehat{H}(\omega_0, \kappa, m), \widehat{H}(\omega, \kappa, m)] \neq 0 \implies$

$t > 0$

$|\Psi(t)\rangle = e^{-i\widehat{H}(\omega,\kappa,m)t} |\Psi_0\rangle$

non-trivial time evolution
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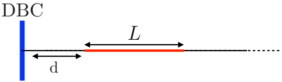
- When $\omega = 0 \quad \longrightarrow \quad$ CFT description in the continuum [Calabrese, Cardy,'05]
- Vaidya spacetimes are employed as the gravitational duals of global quantum quenches in the CFT on their boundary

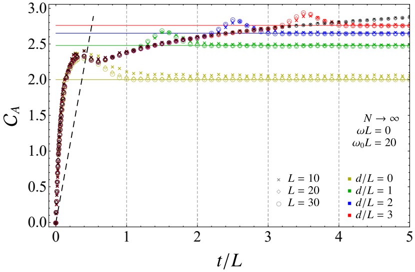
Temporal evolution of the subsystem complexity [GDG, Tonni, '21]

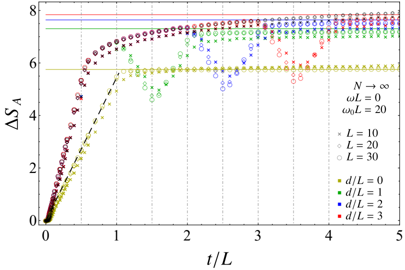
$\gamma_{A,\mathrm{R}}$ $\rightarrow$ initial state $(t = 0)$

$\gamma_{A,\mathrm{T}}$ $\rightarrow$ state at time t after the quench

$$\Delta S_A = S_A(t) - S_A(0)$$







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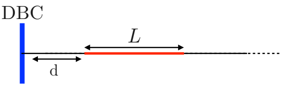
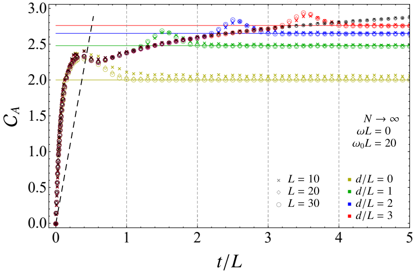
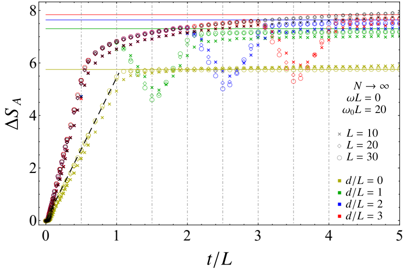


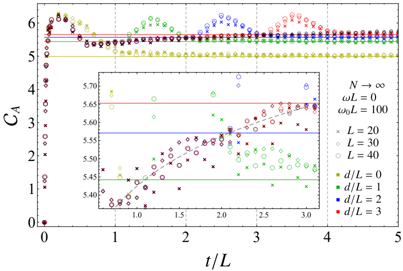
- Different initial growth
- Saturation when $t > L + d$: relaxation to the generalised Gibbs ensamble

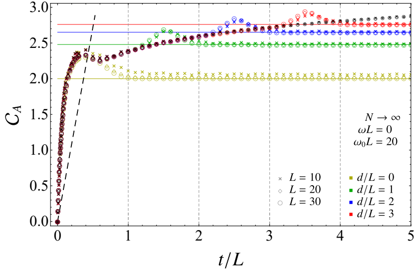




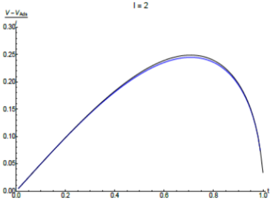
Qualitative comparison with holographic results

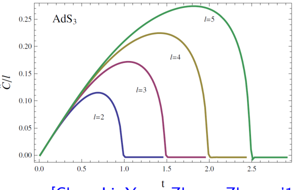
[GDG, Tonni, '21]





Holographic subsystem complexity: CV proposal





[Auzzi, Nardelli, Schaposnik Massolo, Tallarita, Zenoni '19]

[Chen, Li, Yang, Zhang, Zhang, '18]

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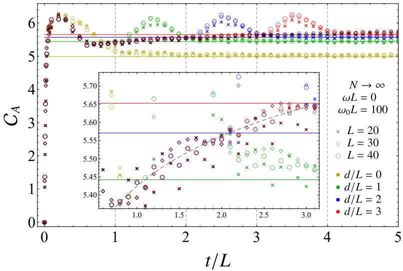
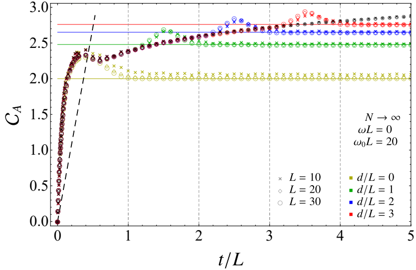
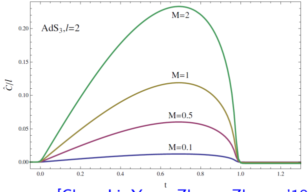




Holographic subsystem complexity: CV proposal

- Similar qualitative behaviours when $0 < t < L/2$: initial linear growth and local maximum
- Different behaviours when $t > L/2$



[Chen, Li, Yang, Zhang, Zhang, '18]

Conclusions

- Complexity for circuits made by **mixed states** exploiting Fisher-Rao distance
- Temporal evolution of subsystem complexity and entanglement entropy after a **global quench**
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Future perspectives

- Local quench
- Non-Gaussian states
- Complexity of fermionic mixed states

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Thank you!