Integrability and cycles of deformed $\mathcal{N} = 2$ gauge theory

Based on arXiv:1908.08030 *with Davide Fioravanti*

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Outline

Introduction to deformed Seiberg-Witten theory

- The partition function for 4D $\mathcal{N} = 2$ SYM theories has been obtained through equivariant localisation techniques, deforming spacetime through two super-gravity parameters, ϵ_1 and ϵ_2 (the *Omega background*, needed for computing instanton contributions). [Nekrasov:2004, Nekrasov-Okounkov:2006, Nekrasov:2009]
- When both $\epsilon_1\,, \epsilon_2 \to 0$, the logarithm of the partition function reproduces the Seiberg-Witten prepotential $\mathcal{F}_{\mathrm{SW}}$. [Seiberg-Witten:1994]
- An intermediate limit which we will study is the Nekrasov-Shatashvili (NS): $\epsilon_1 = \hbar$, $\epsilon_2 \to 0$ [Nekrasov-Shatashvili:2009]. More specifically, having in mind the AGT corresponding Liouville field theory (and precisely its level 2 degenerate field equation), we may think of it as a quantisation/deformation of the quadratic SW differential for pure $(N_f = 0)$ $SU(2)$ SYM which takes up the form of the Mathieu equation

$$-\frac{\hbar^2}{2}\frac{d^2}{dz^2}\psi(z) + [\Lambda^2 \cos z - u]\psi(z) = 0\,. \tag{1}$$

where u parametrizes the moduli space of vacua and Λ is a scaling parameter. [Alday-Gaiotto-Tachikawa:2010; Gaiotto:2013; Awata-Yamada:2010]

- The deformed prepotential $\mathcal{F}_{\mathrm{NS}}$ (logarithm of the partition function) may be derived by eliminating u between the two deformed cycles (periods)

$$a(\hbar,u,\Lambda)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{P}(z;\hbar,u,\Lambda)\,dz\,,\quad a_D(\hbar,u,\Lambda)=\frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0}^{\arccos(u/\Lambda^2)-i0}\mathcal{P}(z;\hbar,u,\Lambda)\,dz \tag{2}$$

of the quantum SW differential $\mathcal{P}(z)=-i\frac{d}{dz}\ln\psi(z)$. (In gauge theory also $a=2\langle\tilde{\Phi}\rangle$, where $\tilde{\Phi}$ is the scalar field).

- In particular, we may expand asymptotically, around $\hbar=0$, $\mathcal{P}(z)\doteq\sum_{n=-1}^{\infty}\hbar^n\mathcal{P}_n(z)$, and then the NS-deformed periods (modes) are [Mironov-Morozov:2010]

$$a^{(n)}(u,\Lambda)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{P}_{2n-1}(z;u,\Lambda)\,dz\,,\quad a_D^{(n)}(u,\Lambda)=\frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0}^{\arccos(u/\Lambda^2)-i0}\mathcal{P}_{2n-1}(z;u,\Lambda)\,dz\,. \tag{3}$$

- Alternatively, we can use Matone's formula connecting $\mathcal{F}_{\mathrm{NS}}$, a, and u (still valid upon deformation). [Matone:1995; Flume-Fucito-Morales-Poghossian:2004]

Introduction to ODE/IM with 2 irregular singularities

- Since early '80s , 2D Liouville field theory has been recognised as the effective theory of 2D quantum gravity. [Polyakov:1981] It is also an integrable model.
- In the approach of ODE/IM correspondence [Dorey-Tateo:1999; Lukyanov-Bazhanov-Zamolodchikov:1999; Gaiotto-Moore-Neitzke:2010], it was discovered by Alexei Zamolodchikov [Zamolodchikov:2012] that the solution $\psi(y)$ of the following ODE (Generalized Mathieu Equation)

$$\left\{-\frac{d^2}{dy^2} + e^{2\theta}(e^{y/b} + e^{-yb}) + P^2\right\}\psi(y) = 0\,, \tag{4}$$

  can be used to construct Q, Y, and T functions and functional relations of the Lioville integrable model at L. coupling b and L. momentum P [Zamolodchikov:2006].
- w.r.t. "usual" ODE/IM, with polynomial potential, this equation has 2 irregular singularities ($y \to \pm\infty$). Thus its study is interesting also for ODE/IM itself.

Liouville ODE/IM details

- In ODE/IM, one defines the Q function

$$Q(\theta, P^2) = W[U_0, V_0] = -i \lim_{y \to +\infty} \frac{V_0(y;\theta)}{U_1(y;\theta)} \,. \tag{5}$$

where U_0 and V_0 are the solution of the GME (4) with b.c.

$$U_0(y) \simeq \frac{1}{\sqrt{2}} \exp\Big\{-\theta/2 - y/4b\Big\} \exp\Big\{-2be^{\theta + y/2b}\Big\} \qquad \mathrm{Re}\, y \to +\infty \,; \tag{6}$$

$$V_0(y) \simeq \frac{1}{\sqrt{2}} \exp\Big\{-\theta/2 + yb/4\Big\} \exp\Big\{-\frac{2}{b} e^{\theta - yb/2}\Big\} \qquad \mathrm{Re}\, y \to -\infty \,. \tag{7}$$

- All Baxter's functions and functional relations of ODE/IM can be derived by considerations on linear relations among the solutions generated by the following discrete symmetries of the GME (4) (where $q = b + 1/b$)

$$\Lambda_b : \theta \to \theta + i\pi b/q \quad y \to y + 2\pi i/q \,,\ \Omega_b : \theta \to \theta + i\pi/(bq) \quad y \to y - 2\pi i/q \,, \tag{8}$$

as $U_k = \Lambda_b^k U_0$ and $V_k = \Omega_b^k V_0$, with U_k invariant under Ω_b and V_k under Λ_b.

- Def. $Y(\theta) = Q(\theta + i\pi a/2) Q(\theta - i\pi a/2)$, $(a = 1 - 2\frac{b}{q})$, Y-system

$$Y(\theta + i\pi/2) Y(\theta - i\pi/2) = \Big(1 + Y(\theta + ia\pi/2)\Big)\Big(1 + Y(\theta - ia\pi/2)\Big). \quad (9)$$

This functional equation can be inverted into the TBA for $\varepsilon(\theta) = -\ln Y(\theta)$

$$\varepsilon(\theta) = \frac{8\sqrt{\pi^3}\, q}{\Gamma(\frac{b}{2q})\Gamma(\frac{1}{2bq})} e^{\theta} - \quad (10)$$

$$- \int_{-\infty}^{\infty} \left[\frac{1}{\cosh(\theta - \theta' + ia\pi/2)} + \frac{1}{\cosh(\theta - \theta' - ia\pi/2)} \right] \ln\left[1 + \exp\{-\varepsilon(\theta')\}\right] \frac{d\theta'}{2\pi}, \quad (11)$$

with boundary condition $\varepsilon(\theta, P^2) \simeq +4qP\theta$, $P > 0$, at $\theta \to -\infty$.

- Def. two (dual under $b \to 1/b$) T functions through TQ-relations $(p = \frac{b}{q})$

$$T(\theta)Q(\theta) = Q(\theta + i\pi p) + Q(\theta - i\pi p) \qquad \tilde{T}(\theta)Q(\theta) = Q(\theta + i\pi(1-p)) + Q(\theta - i\pi(1-p)), \quad (12)$$

- It can be derived in ODE/IM also the periodicity of T

$$T(\theta + i\pi(1-p)) = T(\theta) \qquad \tilde{T}(\theta + i\pi p) = \tilde{T}(\theta) . \quad (13)$$

Integrability-Gauge fundamental correspondence

- The self-dual ($b = 1$) GME is known in literature as modified Mathieu equation:

$$\left\{-\frac{d^2}{dy^2} + 2e^{2\theta}\cosh y + P^2\right\}\psi(y) = 0\ , \tag{14}$$

- This equation can be related to the Mathieu equation which quantises the SW differential in the NS limit (1) by the independent variable change $z = -iy - \pi$ and the parameters correspondence

$$\frac{\hbar}{\Lambda} = \frac{\epsilon_1}{\Lambda} = e^{-\theta} \qquad \frac{u}{\Lambda^2} = \frac{1}{2}\frac{P^2}{e^{2\theta}}\ . \tag{15}$$

- We (D. Fioravanti and D. Gregori - arXiv:1908.08030) have found that the two deformed Seiberg-Witten cycle periods for pure ($N_f = 0$) SU(2) $\mathcal{N} = 2$ supersymmetric gauge theory are connected to the Baxter's Q and T functions of the Liouville integrable model at the self dual point by the very simple relations:

$$Q(\theta, P^2) = \exp 2\pi i a_D(\hbar, u)\,, \tag{16}$$

$$T(\theta, P^2) = 2\cos 2\pi a(\hbar, u)\,. \tag{17}$$

Proof of the fundamental identification $Q = \exp 2\pi i a_D$

We have proven relation (16) analytically by studying the properties of the solution $\mathcal{P}(y) = -i\frac{d}{dy}\ln\psi(y)$ of the Riccati equation

$$\mathcal{P}^2(y,\hbar,u) - i\frac{d\mathcal{P}(y,\hbar,u)}{dy} = -(\frac{2u}{\hbar^2} + \frac{2\Lambda^2}{\hbar^2}\cosh y), \quad (18)$$

with boundary condition given by the (double) limit $y \to +\infty$ of the Seiberg-Witten leading $\hbar \to 0$ order: $\mathcal{P}_{-1}(y) = -i\sqrt{\frac{2u}{\hbar^2} + \frac{2\Lambda^2}{\hbar^2}\cosh y} \sim -i\frac{\Lambda}{\hbar}e^y$, as $y \to +\infty$.

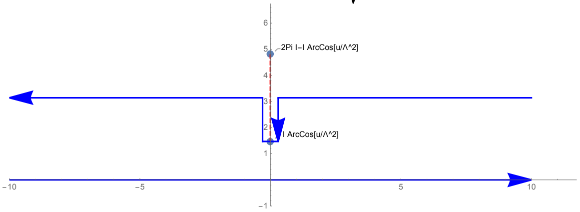


Figure: Integration contour in the y complex plane for the proof of relation (16).

$$\underbrace{\int_{-\arccos(u/\Lambda^2)-i0}^{+\arccos(u/\Lambda^2)-i0} \mathcal{P}(z)\,dz}_{2\pi i a_D(\hbar,u)} = \underbrace{\int_{-\infty}^{+\infty} \mathcal{P}_{reg}(y)\,dy}_{\ln Q(\theta,P^2)}, \quad (19)$$

$$\mathcal{P}_{reg}(y) = \mathcal{P}(y) + 2ie^{\theta}\cosh\frac{y}{2} - \frac{i}{4}\tanh y\,. \quad (20)$$

Proof of the fundamental identification $T = 2\cos 2\pi a$

- Alexei Zamolodchikov proved a relation between the $b = 1$ T and the Floquet index ν of the Mathieu equation, such that $\psi(z + 2\pi) = e^{2\pi\nu}\psi(z)$. [Zamolodchikov:2012]

$$T(\theta, P^2) = 2\cosh 2\pi\nu = 2\left[1 - 2\Delta(0)\sin^2 \pi P\right] . \qquad (21)$$

- ν was computed from the Hill determinant $\Delta(0)$ and T was computed through the analytic continuation of the TBA (11) for $\ln Q = -\frac{1}{2}\varepsilon$, by the TQ relation (12).

$$\Delta(0) = \lim_{n\to\infty} \det \mathcal{A}_n(0) \qquad \xi_n = \frac{e^{2\theta}}{P^2 - n^2} \qquad \mathcal{A}_n(\nu) = \begin{pmatrix} 1 & \xi_{-n} & 0 & & & & & \\ \xi_{-n+1} & 1 & \xi_{-n+1} & & & & & \\ & & \vdots & \ddots & & & & \\ & & & 0 & \xi_0 & 1 & \xi_0 & \\ & & & & & \ddots & \vdots & \\ & & & & & & \xi_n & 1 \end{pmatrix} . \qquad (22)$$

- He did not recognise that $\nu = ia$ (NS limit of $\mathcal{N} = 2$ SYM not yet existing), but we did it, thus establishing another connection to gauge theory.

Gauge TBA

- The self-dual Liouville QQ relation for $Q(\hbar, u) \equiv Q(\theta, u) \equiv Q(\theta, P^2)$ translates in the gauge variables as

$$1 + Q^2(\theta, u) = Q(\theta - i\pi/2, -u)Q(\theta + i\pi/2, -u) \qquad (23)$$

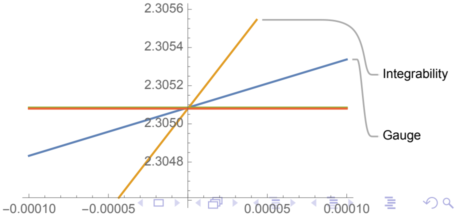
and can be inverted to obtain a TBA for the dual period $(e^{-\varepsilon(\theta,u)} = Q^2(\theta, u))$

$$\varepsilon(\theta, u) = -4\pi i a_D^{(0)}(u)\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\{-\varepsilon(\theta', -u)\}\right]}{\cosh(\theta - \theta')}\frac{d\theta'}{2\pi} \qquad (24)$$

- The TBA enjoys the $\mathbb{Z}_2$ R-symmetry of the moduli space $u \leftrightarrow -u$, which is nothing but the discrete symmetry Λ_1 or Ω_1 used for the ODE/IM construction.
- The solution $\varepsilon(\theta, u)$ of the gauge TBA (24) and that $\varepsilon(\theta, P^2)$ of the integrability TBA (11) match

$$\varepsilon(\theta_0, u) = \varepsilon(\theta_0, P^2) \text{ when } P^2 = 2ue^{2\theta_0}/\Lambda^2 , \qquad (25)$$

that is, we verified numerically the relation (16).



Non perturbative $\mathbb{Z}_2$ symmetry relations

- The self-dual Liouville *TQ* relation and *T* periodicity relation read in gauge variables

$$T(\theta, u) = \frac{Q(\theta - i\pi/2, -u) + Q(\theta + i\pi/2, -u)}{Q(\theta, u)}, \tag{26}$$

$$T(\theta, u) = T(\theta - i\pi/2, -u). \tag{27}$$

- These relations appear to be a non-perturbative exact generalizations of the perturbative $\mathbb{Z}_2$ *R*-symmetry relations for the periods [Bilal-Ferrari:1996, Basar-Dunne:2015]

$$a_D^{(n)}(-u) = i(-1)^n \left[-\mathrm{sgn}\,(\mathrm{Im}\, u)\, a_D^{(n)}(u) + a^{(n)}(u)\right], \tag{28}$$

$$a^{(n)}(u) = \mathrm{sgn}\,(\mathrm{Im}\, u)\, a_D^{(n)}(u) - i(-1)^n a_D^{(n)}(-u). \tag{29}$$

Perturbative periods and local integrals of motion

- The $\theta \to +\infty$ asymptotic expansions of Q in finite gauge u and integrability P variables are

$$Q(\theta, u) \doteq \exp\left\{2\pi i \sum_{n=0}^{\infty} e^{\theta(1-2n)} \Lambda^{2n-1} a_D^{(n)}(u, \Lambda)\right\} \quad , \tag{30}$$

$$Q(\theta, P^2) \doteq \exp\left\{-e^{\theta} \frac{8\sqrt{\pi^3}}{\Gamma^2(\frac{1}{4})} - \sum_{n=1}^{\infty} e^{\theta(1-2n)} C_n I_{2n-1}(P^2)\right\} \quad . \tag{31}$$

where $I_{2n-1}(b=1, P^2) = \sum_{k=0}^{n} \Upsilon_{n,k} P^{2k}$

- Since in Seiberg Witten theory u is finite as $\theta \to +\infty$, it is necessary that also $P^2(\theta) = 2\frac{u}{\Lambda^2} e^{2\theta} \to +\infty$. In this double limit, the LIMs resum to the perturbative periods.

$$2\pi i a_D^{(n)}(u, \Lambda) = -\Lambda^{1-2n} \sum_{k=0}^{\infty} 2^k C_{n+k} \Upsilon_{n+k,k} \left(\frac{u}{\Lambda^2}\right)^k \quad . \tag{32}$$

Quantum Picard-Fuchs equations

- We have derived from the series (32) the quantum Picard-Fuchs equations at all perturbative orders (through an algorithm) for $a^{(n)}$ and $a_D^{(n)}$. For instance

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2}+4u\frac{\partial}{\partial u}+\frac{5}{4}\right\}a_D^{(1)}(u,\Lambda)=0\,, \tag{33}$$

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2}+6u\frac{\frac{u^2}{\Lambda^4}+\frac{111}{8}}{\frac{u^2}{\Lambda^4}+\frac{325}{32}}\frac{\partial}{\partial u}+\frac{21}{4}\frac{\frac{u^2}{\Lambda^4}+\frac{689}{32}}{\frac{u^2}{\Lambda^4}+\frac{325}{32}}\right\}a_D^{(2)}(u,\Lambda)=0\,, \tag{34}$$

$$\cdots \tag{35}$$

(the same equation holding for $a^{(n)}$).

- Since the analytic series (32) are essentially the P^2 coefficients of the LIMs, we can interpret in integrability the quantum Picard-Fuchs equations as fixing the LIMs for $b=1$.

Generalisations

- We have described our gauge-integrability correspondence just for the simplest $SU(2)$ with $N_f = 0$ case, but we have evidence that it is of much more general validity.
  - For the case of $SU(3)$ with $N_f = 0$, D.Fioravanti, R.Poghossian and H.Poghosyan have found a relation between the 3 periods and the T function of the A_2 Toda Integrable model, which is a generalisation of (17). [Fioravanti-Poghossian-Poghosyan:2020]
  - For the case of $SU(2)$ with $N_f = 1$, D. Fioravanti, D.Gregori and H. Shu have found a relation between the 2 periods and the Y of the Integrable Perturbed Hairpin model. [Fioravanti-Gregori-Shu:to appear]
- A different kind of connection between Q and Y functions and gauge periods has been found in [Grassi-Gu-Marino:2020, Grassi-Hao-Neitzke:2021]. However, it turns out that they are different, since they involve different periods, generated by the instanton prepotential rather than by the cycle integrals.

Conclusions and perspectives

- In conclusion, the powerful ODE/IM correspondence has been revealing a very suggestive connexion between the quantum integrable models and ϵ_1-deformed Seiberg-Witten $\mathcal{N} = 2$ supersymmetric gauge theories.
- The ODE/IM correspondence yields a natural quantisation scheme for general SW theory: (the suitable power of) the SW differential becomes quantised as differential operator or oper whose cycles (periods) or monodromies are encoded into the connexion coefficients (for instance, of the ODE/IM).
- It would be also interesting to explore:
  - the implications for the cycles and periods as described in [Bourgine-Fioravanti:2018A,B];
  - the correspondence with higher $SU(2)$ flavours $N_f = 2, 3, 4$, which is related also to the computation of quasi-normal-modes frequencies of black holes merging. [Aminov-Grassi-Hatsuda:2020, Bianchi-Consoli-Grillo-Morales:2021]

THANK YOU!