Open string methods and Gopakumar-Vafa invariants

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based on *A. Collinucci, AS, R. Valandro [21]*
A. Collinucci, M. De Marco, AS, R. Valandro [21]

Context

Punchline:

Get insight and concretely handle computations of physically relevant mathematical objects using string theory techniques

Outline and motivation

- What are the Gopakumar-Vafa topological invariants (a.k.a **GV invariants**)?
- Why is it interesting for physicists to count them?
- Idea: find an easy and physics-based way of **computing (genus zero) GV invariants for a class of non-toric CY**

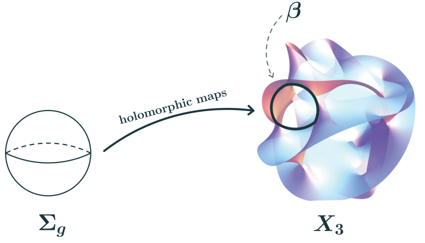
What are GV invariants?

GV invariants are topological invariants of **Calabi-Yau spaces**

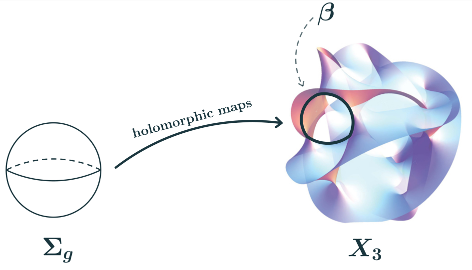
To "define" them, consider:

- a Riemann surface Σ_g of genus g
- a Calabi-Yau threefold X_3
- a 2-cycle β inside the CY

How many **holomorphic** maps from Σ_g to $\beta \in X_3$?



GV invariants



- How many **holomorphic** maps from Σ_g to $\beta \in X_3$? $\quad \boldsymbol{n_\beta^g}$

$\boldsymbol{n_\beta^g}$ are the **GV invariants**

We are interested in genus 0 GV invariants $\boldsymbol{n_\beta^0}$ (i.e. the Riemann surface is just a sphere)

GV invariants

- Why do physicists care about them?

$\boldsymbol{n^g_\beta}$ gives the number of BPS states of M2-branes wrapping the curve β

- $\boldsymbol{n^0_\beta}$ gives instantonic corrections to the Kähler potential in type II theories:

$$K = \underbrace{P(moduli)}_{perturbative} + \underbrace{\sum_\beta P'(n^0_\beta, moduli)}_{instanton\ correction}$$

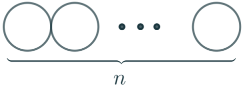
How to compute the GV invariants?

We are interested in singular non-toric CY threefolds that arise as one-parameter families of deformations of ADE singularities

- ADE surfaces are singular spaces classified in terms of the exceptional divisors pattern in their resolution

Example:

$$A_n : x^2 + y^2 = z^{n+1}$$



We consider singular threefolds where only 1 $\mathbb{P}^1$ can be blown up (simple flops)

ADE singularities

- Deforming ADE singularities with terms depending on a single parameter w gives a (possibly singular) threefold.

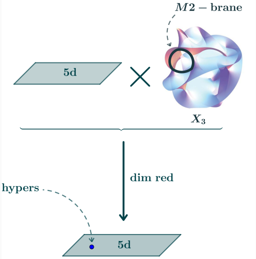
Example: **conifold**

$$\underbrace{x^2 + y^2 = z^2}_{A_1 \text{singularity}} - \underbrace{w^2}_{\text{def}} \qquad \text{SINGULAR}$$

Consider **M-theory** on a singular threefold arising from a one-parameter deformed ADE singularity

The EFT in 5d contains **hypermultiplets** descending from M2-branes wrapped on holomorphic curves (*Witten* [96])

These are the states that we want to count



M-theory/IIA duality

- Write the threefold in the form of a $\mathbb{C}^*$-fibration:

$$uv = \text{det}\, T(z, w)$$

- Example (**conifold**): $x^2 + y^2 = z^2 - w^2 \quad \Rightarrow \quad uv = z^2 - w^2$

M-theory		**type IIA**
$\updownarrow$	$\Longleftrightarrow$	$\updownarrow$
$uv = \text{det}\, T(z, w)$		D6 branes on $\text{det}\, T(z, w) = 0$

- What is $T(z, w)$?

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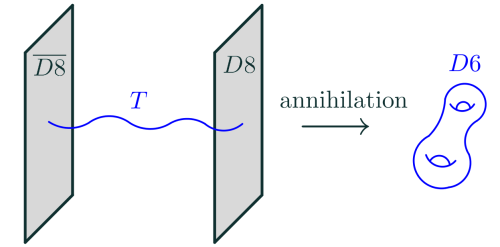
Tachyon condensation

Tachyon condensation formalism

(*Sen* [99], *Collinucci, Savelli* [16])



$$0 \longrightarrow \tilde{E} \xrightarrow{\quad\quad T \quad\quad} E \longrightarrow \text{coker}(T) \to 0$$

- $\tilde{E}$ and E are vector bundles on $\overline{D8}$ and $D8$ respectively

Objective: compute **open string states** in the D6 brane system left over after $D8 - \overline{D8}$ partial annihilation

Half-way recap

Strategy to compute the GV invariants:

- Consider CY threefolds defined as **one-parameter families of ADE deformations**
- Use **M-theory/IIA duality** to give a description of the corresponding D6 brane system
- Build the **tachyon** that describes the D6 brane system
- Compute the **fluctuations of the tachyon** that correspond to the **open string modes**
- Identify the open string modes with the **GV invariants**

Let's see this in a concrete example!

Reid's pagodas

Reid's pagodas are a natural generalization of the conifold:

- They are defined by:

$$\underbrace{x^2 + y^2 = z^{2k}}_{A_{2k-1}} - \underbrace{w^2}_{\text{def}} \qquad \text{with } k \in \mathbb{Z}$$

  admitting a single $\mathbb{P}^1$ in the resolution

- Changing variables they can be immediately rearranged as a C^*-fibration:

$$uv = (z^k + w)(z^k - w)$$

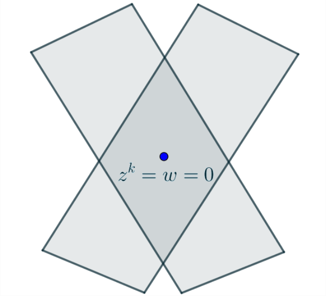
- We must find the tachyon, i.e. a matrix T such that:

$$\det T = (z^k + w)(z^k - w)$$

Reid's pagodas

- The minimal tachyon that does the work is:

$$T = \begin{pmatrix} z^k + w & 0 \\ 0 & z^k - w \end{pmatrix} \qquad \text{det } T = (z^k + w)(z^k - w)$$



Reid's pagodas

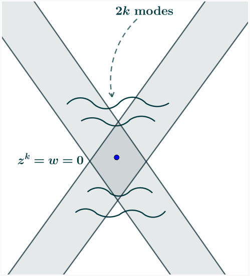
- To find the open string modes we must consider fluctuations δT of the tachyon and mod them by gauge equivalences:

$$\delta T \sim \delta T + g_{D8} \cdot T + T \cdot g_{\overline{D8}}$$

where $(g_{D8}, g_{\overline{D8}}) \in \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$

- We find a total of $2k$ modes, corresponding to $\boldsymbol{k}$ **hypermultiplets in 5d**
- On the other hand the GV invariants are: $\boldsymbol{n_{\beta}^{g=0} = n^{0}_{[\mathbb{P}^1]} = k}$ and all the others vanish

AGREEMENT!



Conclusions

The **tachyon** is a multipurpose tool, that allows us to:

- find a physics-based way to compute genus 0 GV invariants for a wide class of non-toric CY (A_n and D_n series with simple flops or non-resolvable singularities)
- Investigate the structure of the **Higgs branch** of the 5d theory allowing a check with existing results (*Closset, Schafer-Nameki, Wang* [20])

Outlook

- multiple flops singularities
- switching on T-brane entries
- apply to other more involved single-flop and non-resolvable singularities (connecting with existing literature)

Thank you!