# Invariant polynomials and integrable systems on compact Hermitian symmetric spaces

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# About geometric quantization

**Classical...**

Thanks to symplectic geometry, it is possible to enlarge the classical mechanics to phase spaces where canonical conjugate coordinates are just locally defined.

**...vs. quantum**

From a quantum point of view, one can extend the quantization to classical theory on these phase spaces recurring to the geometric quantization

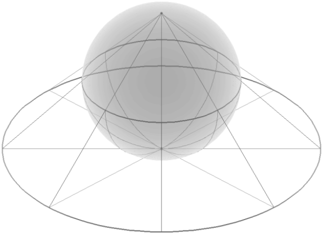
Starting from a classical theory on such a kind of symplectic manifold, one obtains a semi-classical interpretation for quantum objects.

# Example: $S^2$

On the $S^2$ sphere, we can define a Poisson bracket

$$\{\zeta, \bar{\zeta}\} = 1,$$

where $\zeta$ is the complex coordinate in the stereographic projection.



**Semiclassical interpretation of the spin**

The geometrical quantization of a sphere of area $2n\pi\hbar$, $n \in \mathbb{Z}$, reproduces the quantum spin representation $j = \frac{n}{2}$.

Furthermore, on this sphere we can define another Poisson algebra as

$$\{\zeta, \bar{\zeta}\} = 1 + |\zeta|^2,$$

which is degenerate in the singular point of projection. In this case a quantization will define different quantum objects.

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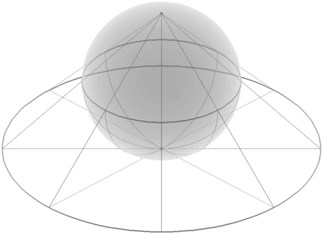

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# Plan of the talk

Our attention is focused on

Compact Hermitian Symmetric Spaces

a class of coadjoint orbits of Lie groups admitting an additional Poisson structure.

$\Downarrow$

The existence of an integrable models on HSS, i.e. a coordinate system such that

$$\frac{d}{dt}\phi_i = I_i, \quad \frac{d}{dt}I_i = 0,$$

$i = 1...(1/2\dim M)$

$\Rightarrow$

it is useful for geometric quantization of the *second* Poisson tensor [Bonechi Tarlini et al 2014].

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# Compact hermitian symmetric spaces

Compact hermitian symmetric spaces $M_\phi = G/H_\phi$ are a class of co-adjoint orbit of compact Lie groups. They are classified as

$$SU(n)/S(U(n-k)\times U(k)) \qquad SO(n+2)/(SO(n)\times SO(2))$$

$$SO(2n)/U(n) \qquad Sp(2n)/U(n)$$

$$E_6/(SO(10)\times SO(2)) \qquad E_7/E_6.$$

- $\mathfrak{h}_\phi$ has a one dimension center, generated by $\rho_\phi \in \mathfrak{t}$.
- $M_\phi$ are $G$-hamiltonian spaces with momentum map

$$\begin{aligned} \mu : G/H_\phi &\longrightarrow \mathfrak{g} \\ [g] &\longmapsto \mu(g) = g\rho_\phi g^{-1}, \end{aligned}$$

# KKS and Bruhat-Poisson structures

On each compact HSS $M_\phi$, one can define two compatible Poisson structures

$M_\phi$'s are co-adjoint orbits

$\Downarrow$

Kirillov-Kostant-Souriau symplectic form $\omega$ (as in $S^2$ example).

The standard Poisson structure $\Pi_G$ on $G$ induces

$\Downarrow$

$(G/H_\phi, \Pi)$

Bruhat-Poisson structure on $M_\phi$ (as in $S^2$ example).

### Poisson brackets

On $M_\phi$ we have two compatible Poisson brackets

$$\{f, g\}_\omega\,, \qquad \{f, g\}_\Pi\,.$$

$f, g \in \mathcal{C}^\infty(M_\phi)$.

# KKS and Bruhat-Poisson structures

Compatibility means that we can introduce a Nijenhuis tensor

$$N = \Pi \circ \omega,$$

such that

$$T(N)(v_1, v_2) = [Nv_1, Nv_2] - N\left([Nv_1, v_2] + [v_1, Nv_2] - N[v_1, v_2]\right) = 0,$$

for each vector fields $v_1, v_2$. A manifold endowed with this kind of structure is said a Poisson-Nijenhuis manifold.

**Lemma**

*On $M_\phi$ we can define a Poisson-Nijenhuis structure such that*

$$N^* d\mu = -[Jd\mu, \mu] + d\mu,$$

*where $J : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ defines the almost complex structure on $M_\phi$.*

# Thimm method and spectral problem

Now, we introduce two methods that we are going to combine in searching integrable models on these manifolds.
The first one [Thimm 1981, Guillemin Sternberg 1983] is the

### Thimm method

On a $G$-hamiltonian space, if (and only if) we select a chain of subalgebras

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \dots \supset \mathfrak{g}_{k+1} = 0,$$

satisfying some technical conditions, then the set of $G_i$-invariant functions

$$F\left(\mathfrak{g}_1, \dots, \mathfrak{g}_k\right) \subset \mathcal{C}^{\infty}(M_\phi)$$

defines a

complete integrable model

# Thimm method and spectral problem

The second one involves the $PN$-structure [Magri Morosi 1984].

### Spectral problem

On a $2n$-dimensional $PN$-manifold,

$N$ has at most $n$ different eigenvalues.

Moreover, in a neighborhood of a regular point a collection of functions $\{\lambda_i\}_{i=1\ldots n}$ that give eigenvalues of $N$ satisfies

$$N^* d\lambda_i = \lambda_i \, d\lambda_i,$$

$$\{\lambda_i, \lambda_j\}_\omega = \{\lambda_i, \lambda_j\}_\Pi = 0,$$

defining a

bihamiltonian integrable model

# Thimm method and spectral problem

The Thimm method gives a necessary and sufficient condition for the existence of an integrable model on a compact HSS, i.e. $\exists \, \{I_i\}_{i=1\ldots\frac{1}{2}\mathrm{dim}M_\phi}$ set of independent functions such those

$$\{I_i, I_j\}_\omega = 0.$$

Moreover, if the set of invariant functions $\{I_i\}$ satisfy

$$N^* dI_i = \sum_{jk} I_j \, dI_k,$$

then

$$\{I_i, I_j\}_\Pi = 0,$$

i.e. the set $\{I_i\}$ defines a

bihamiltonian integrable model.

# Classical Lie groups cases

It was showed in [Bonechi Qiu Tarlini 2018] that,

for Classical Lie groups, the HSS admit bihamiltonian integrable models.

Nevertheless, the proof used the eigenvalues of the moment map as involutive and independent functions, which are just locally smooth. This means that them hamiltonian fields are not globally defined and this fact is problematic in the quantization of the Bruhat-Poisson structure.

## Invariant polynomials

We are going to show that, on these manifolds, one can choose a subset of invariant polynomials with respect to the subalgebras of the Thimm chain as a set of involutive and independent functions of the bihamiltonian integrable model and these are globally smooth funtions.

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# Decomposable representation

A way to discuss the action of the Nijenhuis tensor on invariant polynomials is recurring to the existence of a so-called decomposable representation.

Let $R_V$ be a decomposable representation of $\mathfrak{g}$ with respect to $h_\phi$, i.e. $V = V_+ \oplus V_-$ such that $V_\pm$ are representations of $h_\phi$ with

$$\rho_\phi\big|_{V_\pm} = ia_\pm.$$

Moreover let us assume that $V$ is decomposable with respect to a Thimm chain

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_i \supset \ldots \supset \mathfrak{g}_{k+1} = 0,$$

i.e. for each step

$$V = W_+^i \oplus W_-^i, \qquad W_\pm^i \text{ representations of } \mathfrak{g}_i$$

such that $W_+^i$ is decomposable with respect to $\mathfrak{g}_{i+1}$.

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# Nijenhuis action on invariant polynomials

## Proposition

*If we select as invariant polynomials on compact HSS*

$$I_r^{(k)} = \frac{i^r}{r!}\mathrm{Tr}_{W_+^k}\left(\mu_{\mathfrak{g}_k}\right)^r,$$

*for each* $\mathfrak{g}_k$ *belonging to a decomposable Thimm chain, then*

$$N^* dI_r^{(k)} = 2a_+ dI_r^{(k)} + 2dI_{r+1}^{(k)}.$$

## Nota bene

When a decomposable representation exists, we can select a base of invariant polynomials on which we are able to compute the action of the Nijenhuis tensor.

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## $SU(n)/S(U(n-k)\times U(k))$

- Thimm chain :
  $\mathfrak{su}(n) \supset \mathfrak{su}(n-1)\oplus\mathfrak{u}(1) \supset \mathfrak{su}(n-2)\oplus\mathfrak{u}(1)^2 \supset \ldots \supset \mathfrak{u}(1)^n$
- decomposable representation: fundamental

## $SO(n)/SO(n-2)\times SO(2)$

- Thimm chain : $\mathfrak{so}(n) \supset \mathfrak{so}(n-2)\times\mathfrak{so}(2) \supset \mathfrak{so}(n-4)\times\mathfrak{so}(2)^2 \supset \ldots$
- decomposable representation: spin

## $SO(2n)/U(n)$

- Thimm chain : $\mathfrak{so}(2n) \supset \mathfrak{u}(n) \supset \mathfrak{u}(n-1)\oplus\mathfrak{u}(1) \supset \ldots \supset \mathfrak{u}(1)^n$
- decomposable representation: fundamental

## $Sp(2n)/U(n)$

- Thimm chain : $\mathfrak{sp}(2n) \supset \mathfrak{u}(n) \supset \mathfrak{u}(n-1)\oplus\mathfrak{u}(1) \supset \ldots \supset \mathfrak{u}(1)^n$
- decomposable representation: fundamental

# Conclusions

### An open problem

In the exceptional cases

$$E_6/(SO(10)\times SO(2)), \qquad E_7/E_6.$$

a decomposable representation does not exist.

Finally, we can conclude that

- the existence of a decomposable representation allows us to select a base of invariant polynomials on which we are able to compute the action of the Nijenhuis tensor.
- a decomposable representation exists only in classical cases.
- exceptional cases are open problems and we are actively working on them. So far, we have some positive partial results.

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# Thank you for your attention!