Symmetry-resolved entanglement and negativity in systems with U(1) symmetry

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Cortona Young 2021

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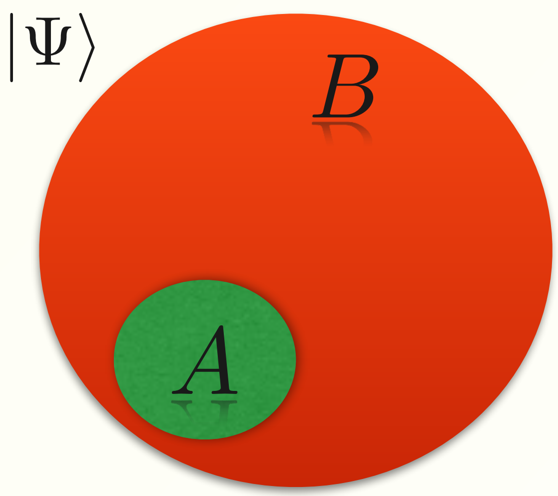
In collaboration with Pasquale Calabrese and Riccarda Bonsignori

Reduced Density Matrix (RDM) and useful definitions

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B, \qquad \rho = |\Psi\rangle\langle\Psi| \qquad \rho_A = \mathrm{Tr}_B \rho$$



A measure of the entanglement between A and B are the Rényi entropies

$$S_n \equiv \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n$$

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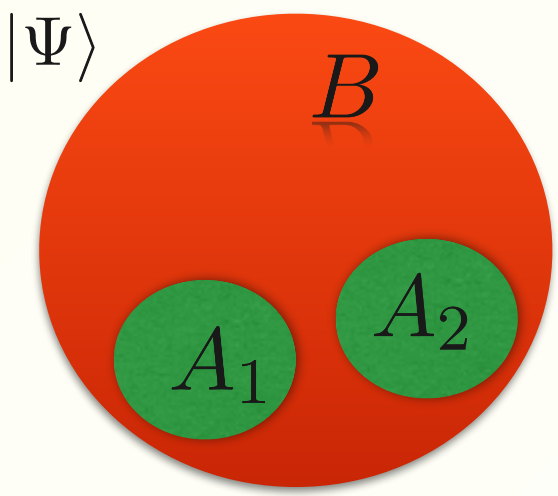

$S_{A_1 \cup A_2}$ gives the entanglement between A and B

A measure of the entanglement between A and B are the Rényi entropies

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What is the entanglement between the two non-complementary parts A_1 and A_2?

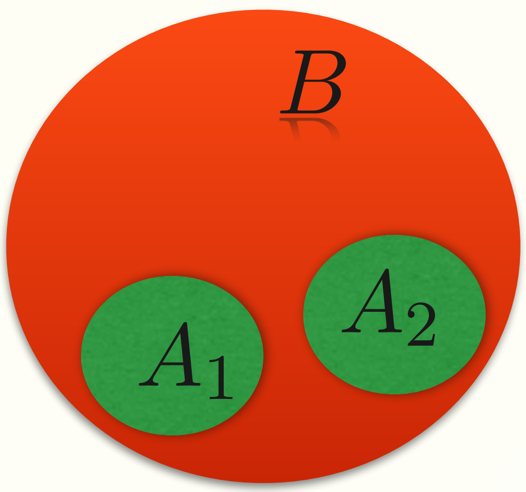
A computable measure of entanglement is the **negativity**

G. Vidal, R. F. Werner, Phys. Rev. A 65, 032314 (2002).

Fermionic partial transpose

occupation-number basis:

$$|\{n_j\}_{j\in A_1},\{n_j\}_{j\in A_2}\rangle = (f^\dagger_{m_1})^{n_{m_1}}\dots(f^\dagger_{m_{\ell_1}})^{n_{m_{\ell_1}}}(f^\dagger_{m'_1})^{n_{m'_1}}\dots(f^\dagger_{m'_{\ell_2}})^{n_{m'_{\ell_2}}}|0\rangle$$



fermionic partial transpose

$$(|\{n_j\}_{A_1},\{n_j\}_{A_2}\rangle\langle\{\bar{n}_j\}_{A_1},\{\bar{n}_j\}_{A_2}|)^{R_1} =$$
$$(-1)^{\phi(\{n_j\},\{\bar{n}_j\})}(|\{\bar{n}_j\}_{A_1},\{n_j\}_{A_2}\rangle\langle\{n_j\}_{A_1},\{\bar{n}_j\}_{A_2}|)$$

warning: $\rho_A^{R_1}$ not Hermitian →

focus on the composite object $\rho_A^{R_1}(\rho_A^{R_1})^\dagger$

$$\mathcal{N} \equiv \frac{\mathrm{Tr}|\rho_A^{R_1}|-1}{2} = \frac{\mathrm{Tr}\sqrt{\rho_A^{R_1}(\rho_A^{R_1})^\dagger}-1}{2}$$

H. Shapourian, K. Shiozaki, and S. Ryu Phys. Rev. B 95, 165101 (2017).

Two-sites examples: charge operator

$$|\Psi\rangle = \alpha\,|100\rangle + \beta\,|010\rangle + \gamma\,|001\rangle$$

a particle in one out of three boxes, $A = A_1 \cup A_2, B,$

$$\begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array}$$

$$\rho_A = \begin{pmatrix} |\gamma|^2 & 0 & 0 & 0 \\ 0 & |\beta|^2 & \alpha^*\beta & 0 \\ 0 & \beta^*\alpha & |\alpha|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

block-diagonal structure

$$\rho_A \cong \left(|\gamma|^2\right)_{\tilde{q}=0} \oplus \begin{pmatrix} |\beta|^2 & \alpha\beta^* \\ \beta\alpha^* & |\alpha|^2 \end{pmatrix}_{\tilde{q}=1} \oplus (0)_{\tilde{q}=2}$$

$$\tilde{Q} = Q_{A_1} \otimes \mathbb{1}_{A_2} + \mathbb{1}_{A_1} \otimes Q_{A_2}$$

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$$\rho_A(\tilde{q}) = \frac{\mathcal{P}_{\tilde{q}}\rho_A\mathcal{P}_{\tilde{q}}}{\mathrm{Tr}(\mathcal{P}_{\tilde{q}}\rho_A)}$$

symmetry resolved entanglement

$$S(\tilde{q}) = -\mathrm{Tr}(\rho_A(\tilde{q})\log\rho_A(\tilde{q}))$$

Two-sites examples: charge imbalance operator

$$|\Psi\rangle = \alpha\,|100\rangle + \beta\,|010\rangle + \gamma\,|001\rangle$$

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$$\rho_A^{R_1} = \begin{array}{c} \\ \\ \end{array} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array}$$

$$\rho_A^{R_1} = \begin{pmatrix} |\gamma|^2 & 0 & 0 & i\alpha\beta^* \\ 0 & |\beta|^2 & 0 & 0 \\ 0 & 0 & |\alpha|^2 & 0 \\ i\beta\alpha^* & 0 & 0 & 0 \end{pmatrix}$$

block-diagonal structure

$$\rho_A^{R_1} \cong \left(|\alpha|^2\right)_{q=-1} \oplus \begin{pmatrix} |\gamma|^2 & i\alpha\beta^* \\ i\beta\alpha^* & 0 \end{pmatrix}_{q=0} \oplus \left(|\beta|^2\right)_{q=1}$$

$$Q = Q_{A_2} - Q_{A_1}^{R_1}$$

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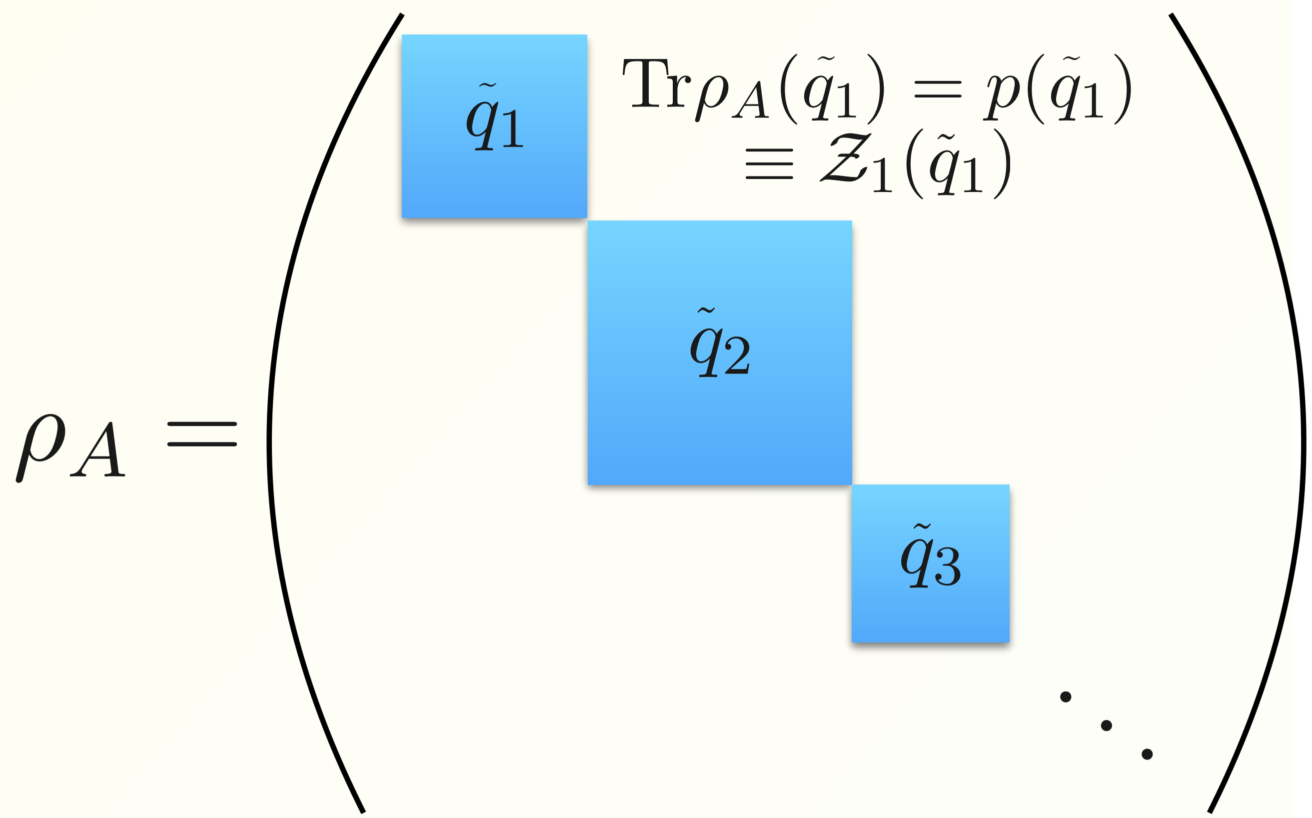
$$Q = Q_{A_2} - Q_{A_1}^{R_1}$$

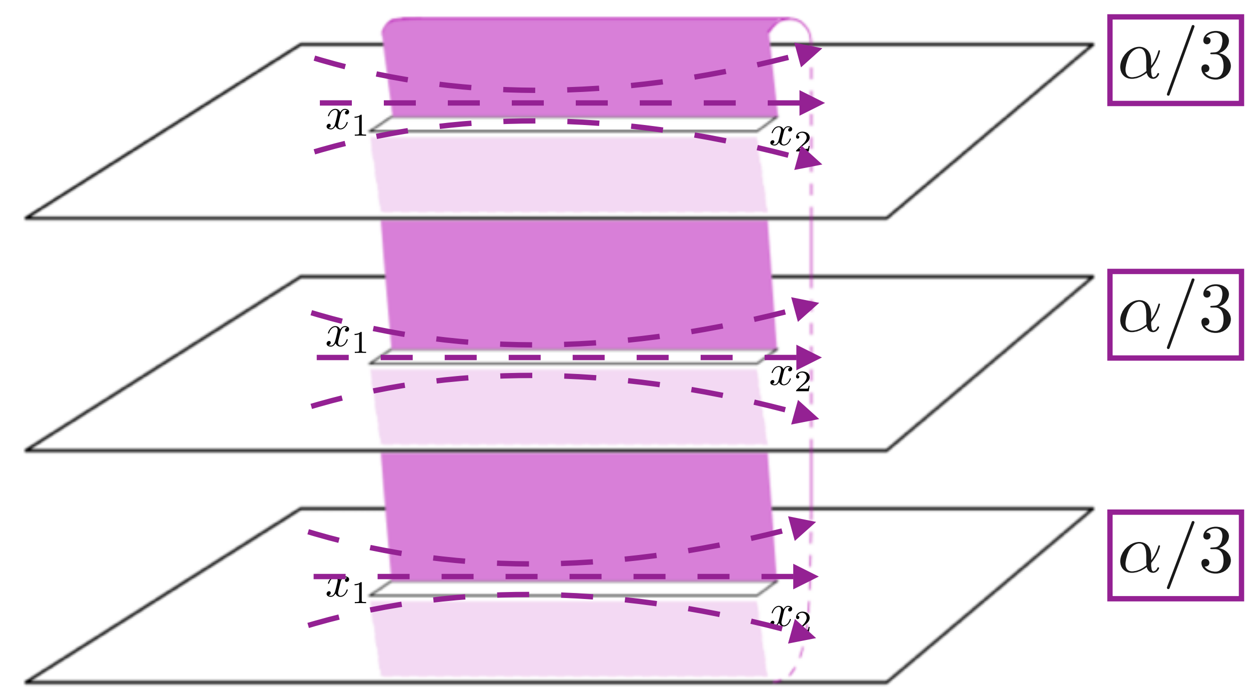
charge imbalance resolved negativity

$$\mathcal{N}(q) = \frac{\mathrm{Tr}|(\rho_A^{R_1}(q))| - 1}{2}, \qquad \rho_A^{R_1}(q) = \frac{\mathcal{P}_q \rho_A^{R_1} \mathcal{P}_q}{\mathrm{Tr}(\mathcal{P}_q \rho_A^{R_1})}$$

$$\mathcal{N} = \sum_q p(q)\mathcal{N}(q), \qquad p(q) = \mathrm{Tr}(\mathcal{P}_q \rho_A^{R_1})$$

Replica trick





q-moments of the RDM: $\quad \mathcal{Z}_n(\tilde{q}) \equiv \mathrm{Tr}(\mathcal{P}_{\tilde{q}}\rho_A^n) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\tilde{q}\alpha} Z_n(\alpha)$

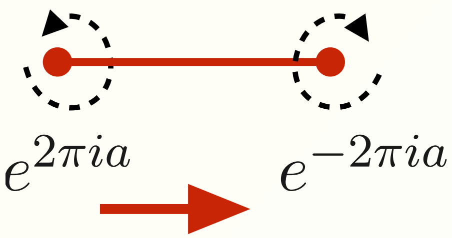
Symmetry Resolved Rényi entropies (SREE): $\quad S_n(\tilde{q}) \equiv \frac{1}{1-n} \log \mathrm{Tr}\rho_A^n(\tilde{q})$

Charged moments of the RDM: $\quad Z_n(\alpha) \equiv \mathrm{Tr}\rho_A^n e^{i\tilde{Q}_A\alpha}$

M. Goldstein and E. Sela, Phys. Rev. Lett. 120, 200602 (2018).

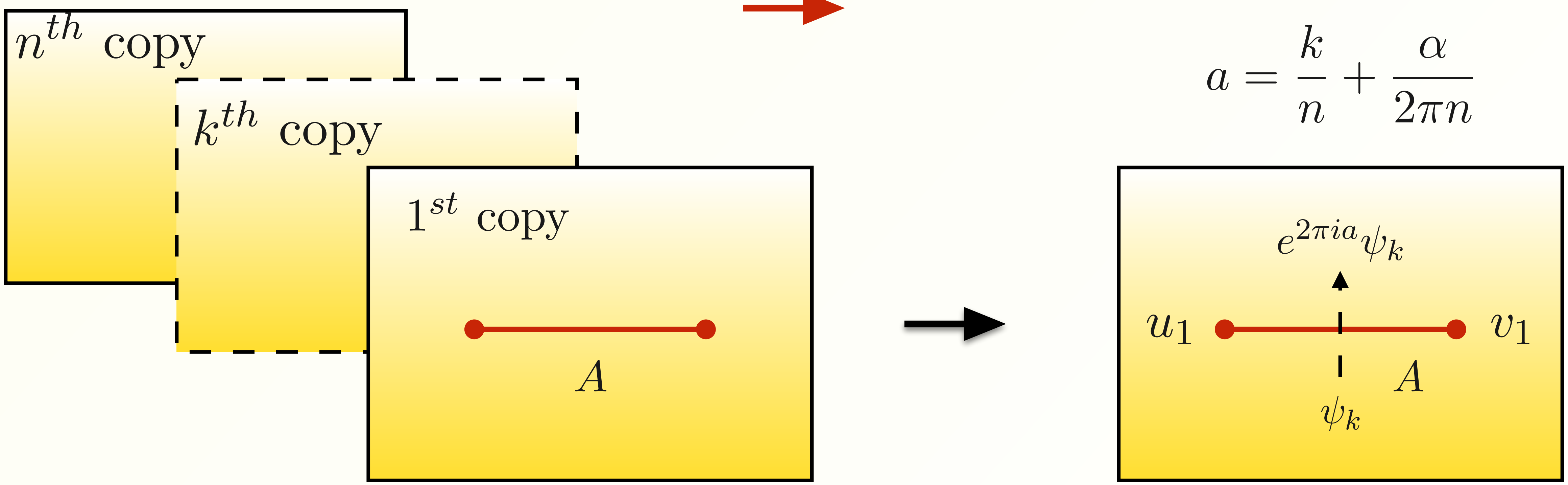
Diagonalisation in replica space: charged moments

$$\mathcal{L} = \bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi$$



$$\mathcal{L}_k = \bar{\tilde{\psi}}_k \gamma^{\mu}(\partial_{\mu} + i A_{\mu}^{k})\tilde{\psi}_k$$

$$a = \frac{k}{n} + \frac{\alpha}{2\pi n}$$



$$j_k^{\mu} \to \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_{\nu}\phi_k$$

$$\boxed{Z_{k,n}(\alpha) = \langle e^{i\int d^2x A_{\mu}^{k} j_k^{\mu}}\rangle}$$

bosonisation technique

$$V_a(x) = e^{-ia\phi_k(x)}$$

$$\boxed{Z_{k,n}(\alpha) = \langle V_{\frac{k}{n}+\frac{\alpha}{2\pi n}}(u_1)V_{-\frac{k}{n}-\frac{\alpha}{2\pi n}}(v_1)\rangle}$$

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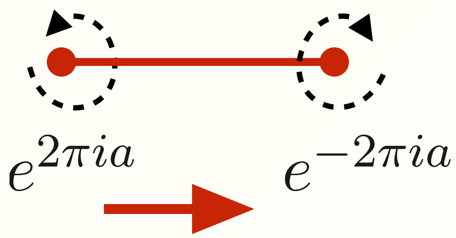

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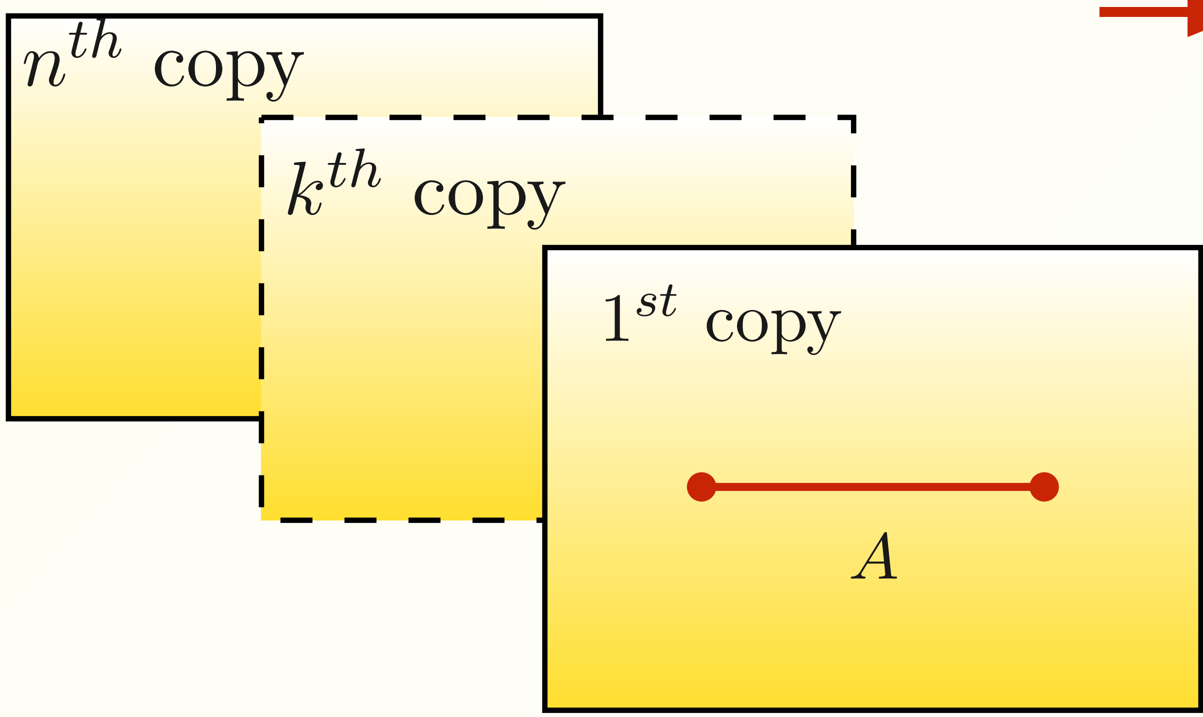

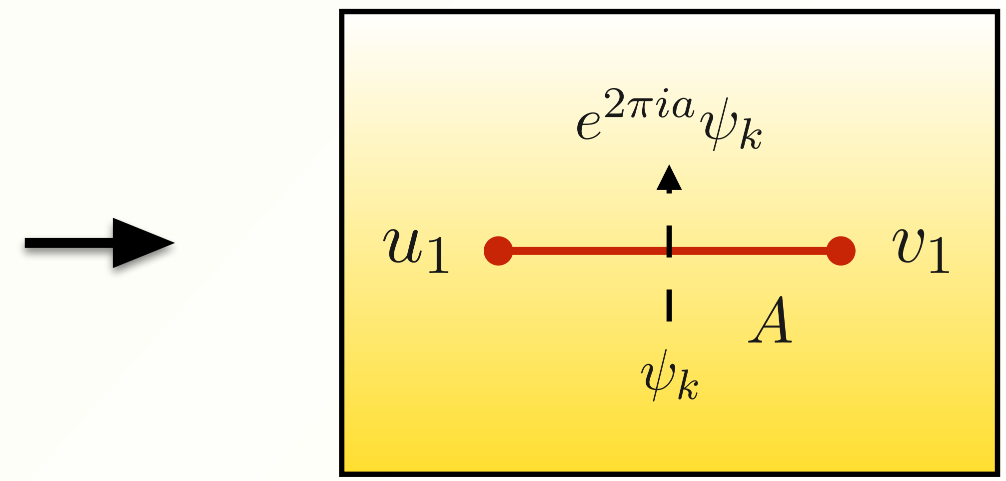

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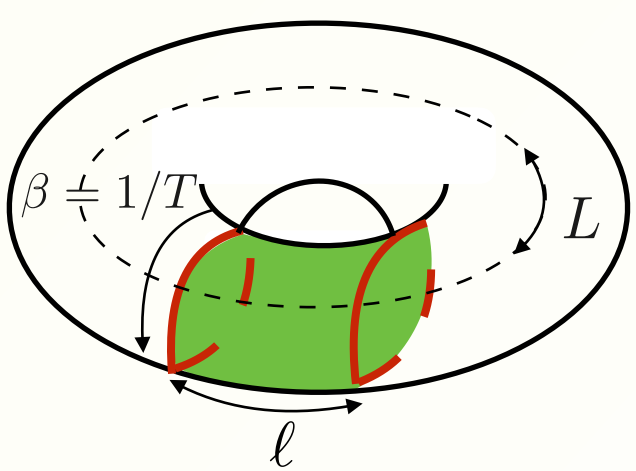
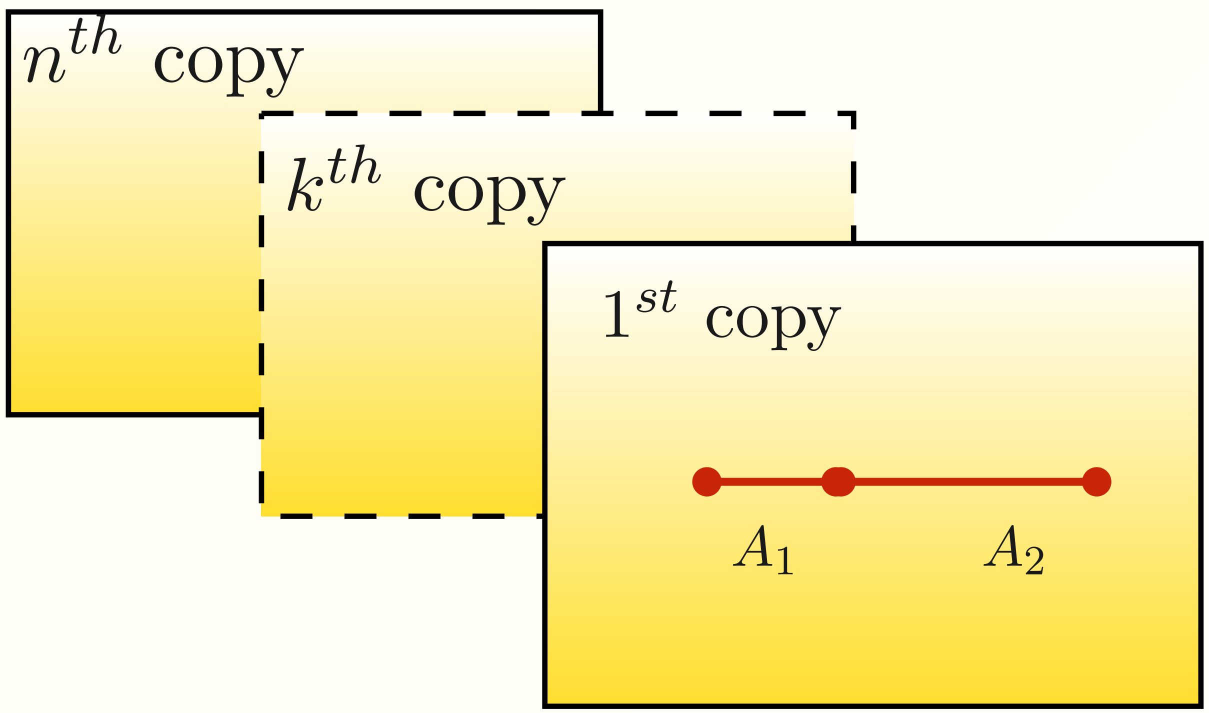


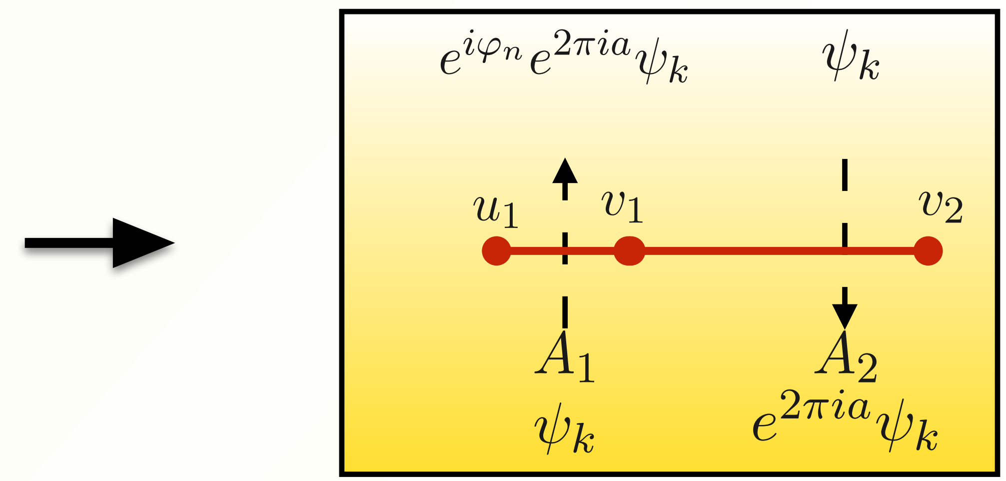
$$\boxed{Z^{(\nu)}_{k,n}(\alpha) = \langle V_{\frac{k}{n}+\frac{\alpha}{2\pi n}}(u_1) V_{-\frac{k}{n}-\frac{\alpha}{2\pi n}}(v_1)\rangle_{\beta}}$$

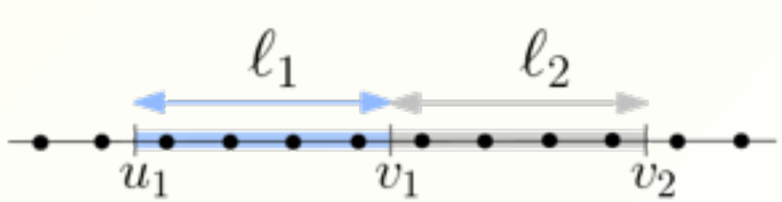
Diagonalisation in replica space: charged moments for negativity

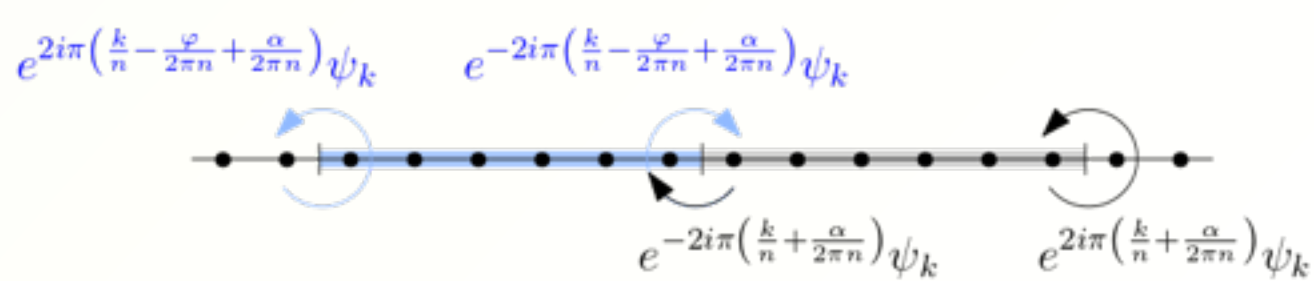
$$\mathcal{L}_k = \bar{\tilde{\psi}}_k \gamma^\mu (\partial_\mu + i A_\mu^k) \tilde{\psi}_k$$



$$a = \frac{k}{n} + \frac{\alpha}{2\pi n}$$







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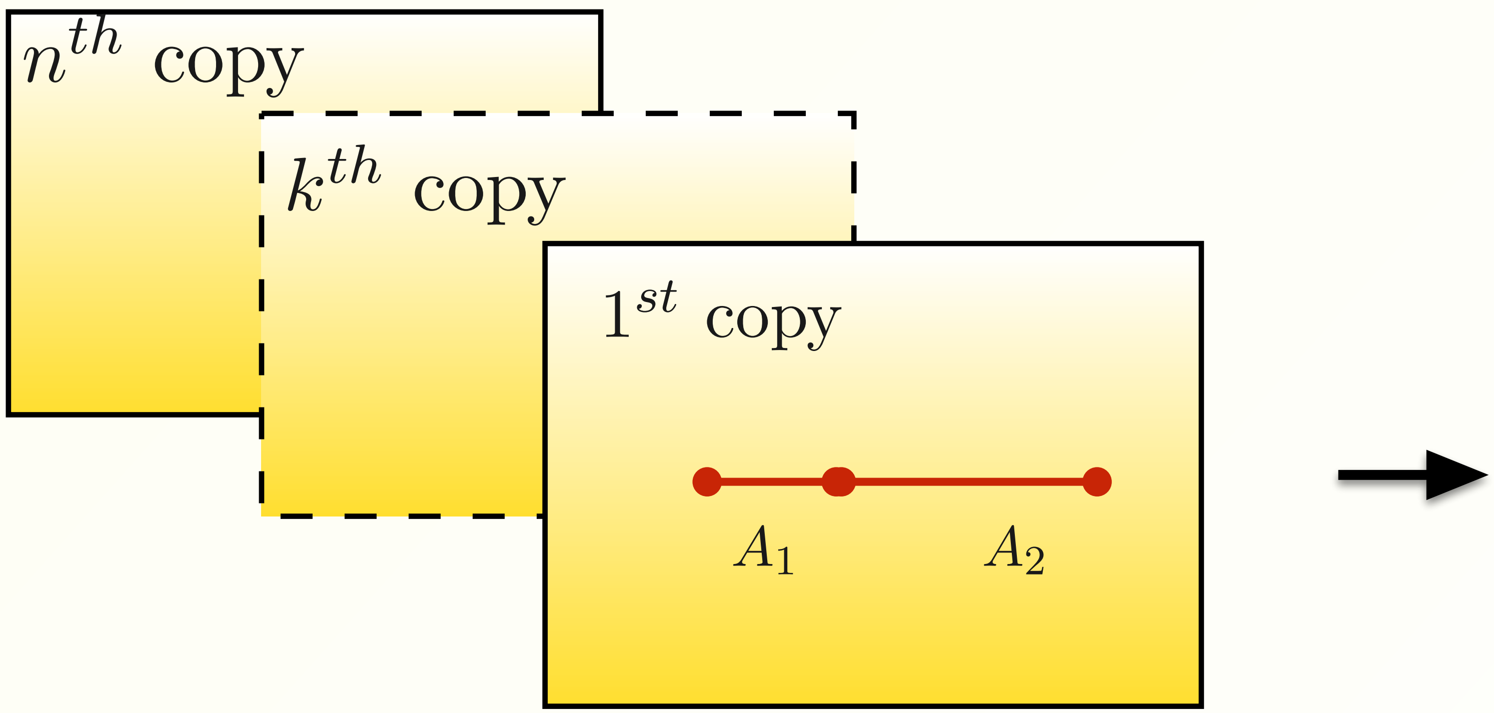

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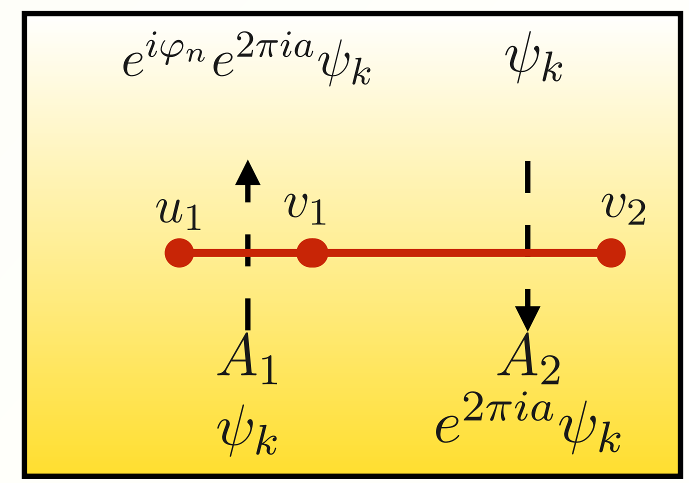
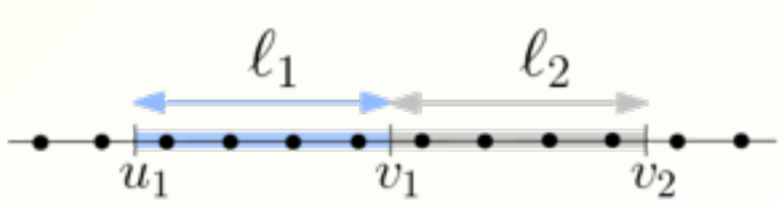
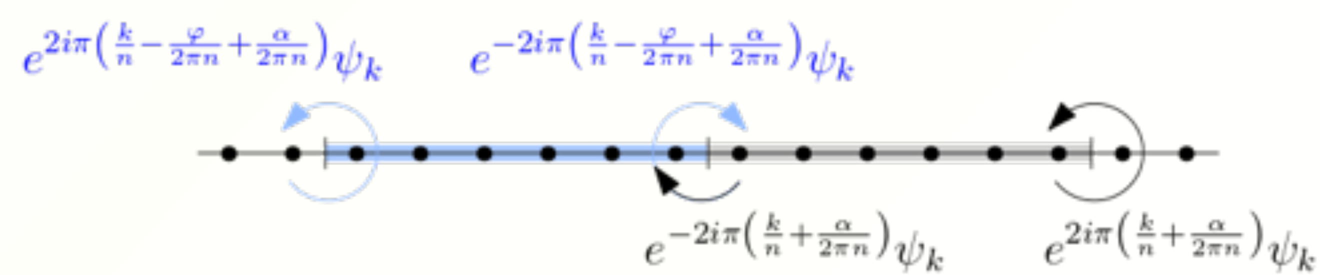


$$Z_{R_1,k}(\alpha) = \left\langle V_{\frac{k}{n} + \frac{\alpha}{2\pi n} - \frac{\varphi_n}{2\pi}}(u_1) V_{-\frac{2k}{n} - \frac{\alpha}{\pi n} + \frac{\varphi_n}{2\pi}}(v_1) V_{\frac{k}{n} + \frac{\alpha}{2\pi n}}(v_2) \right\rangle$$





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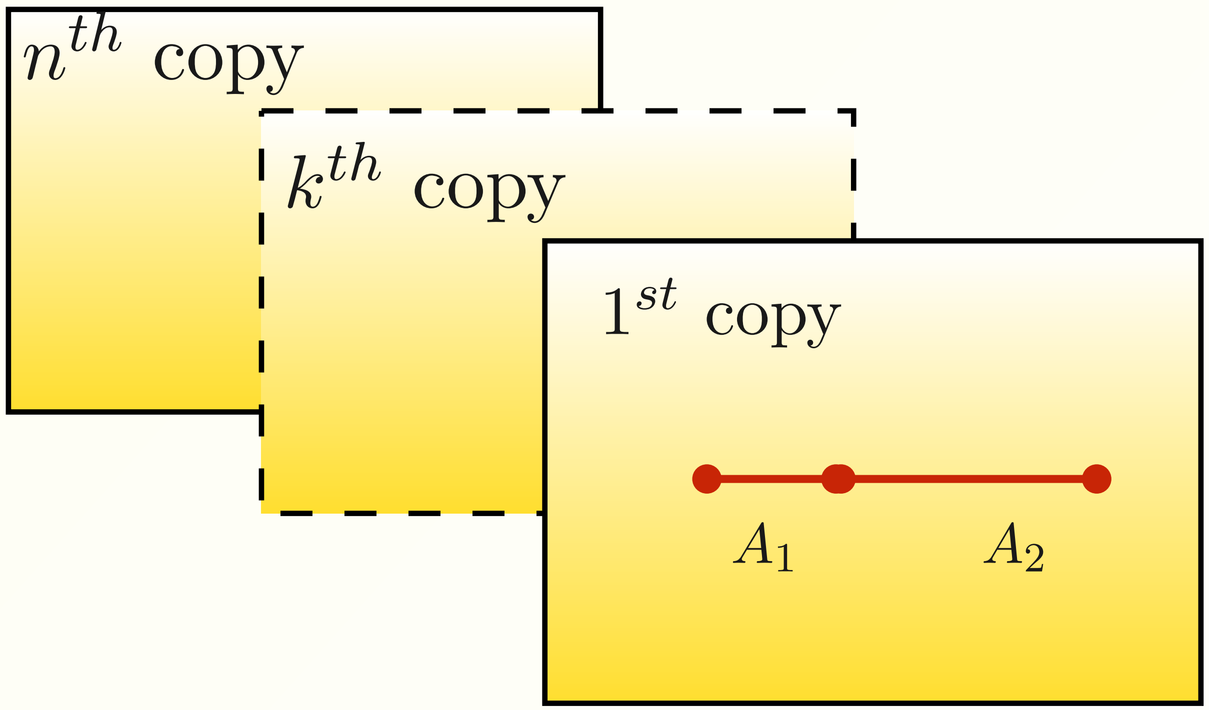

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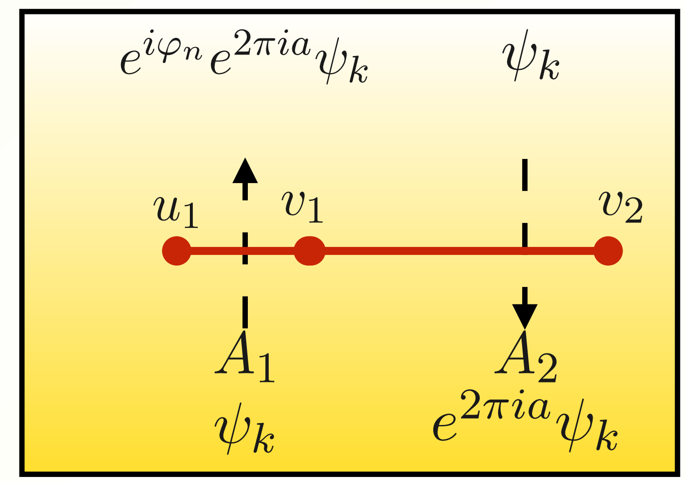
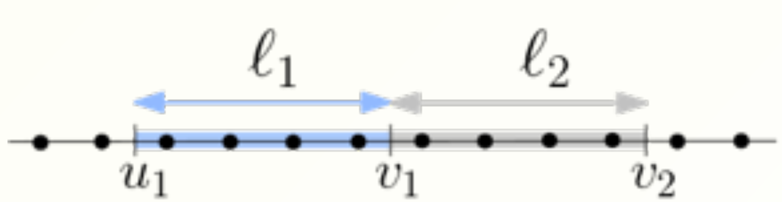
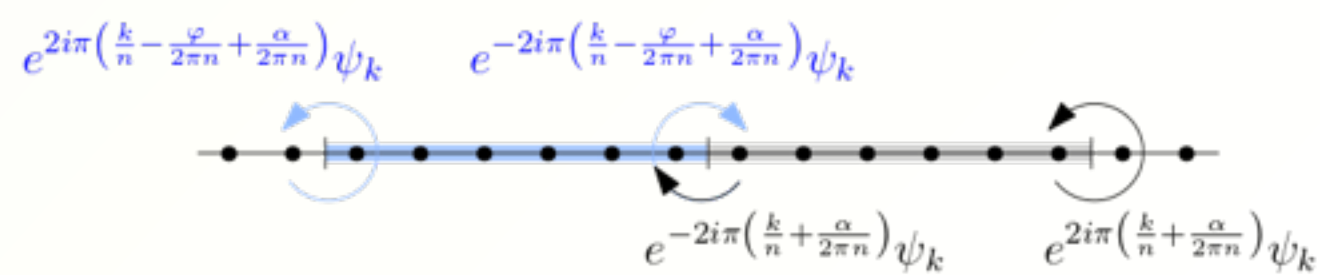


$$Z^{(\nu)}_{R_1,k}(\alpha) = \left\langle V_{\frac{k}{n}+\frac{\alpha}{2\pi n}-\frac{\varphi_n}{2\pi}}(u_1) V_{-\frac{2k}{n}-\frac{\alpha}{\pi n}+\frac{\varphi_n}{2\pi}}(v_1) V_{\frac{k}{n}+\frac{\alpha}{2\pi n}}(v_2) \right\rangle_\beta$$

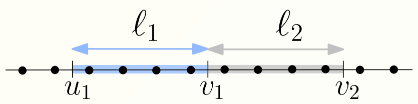


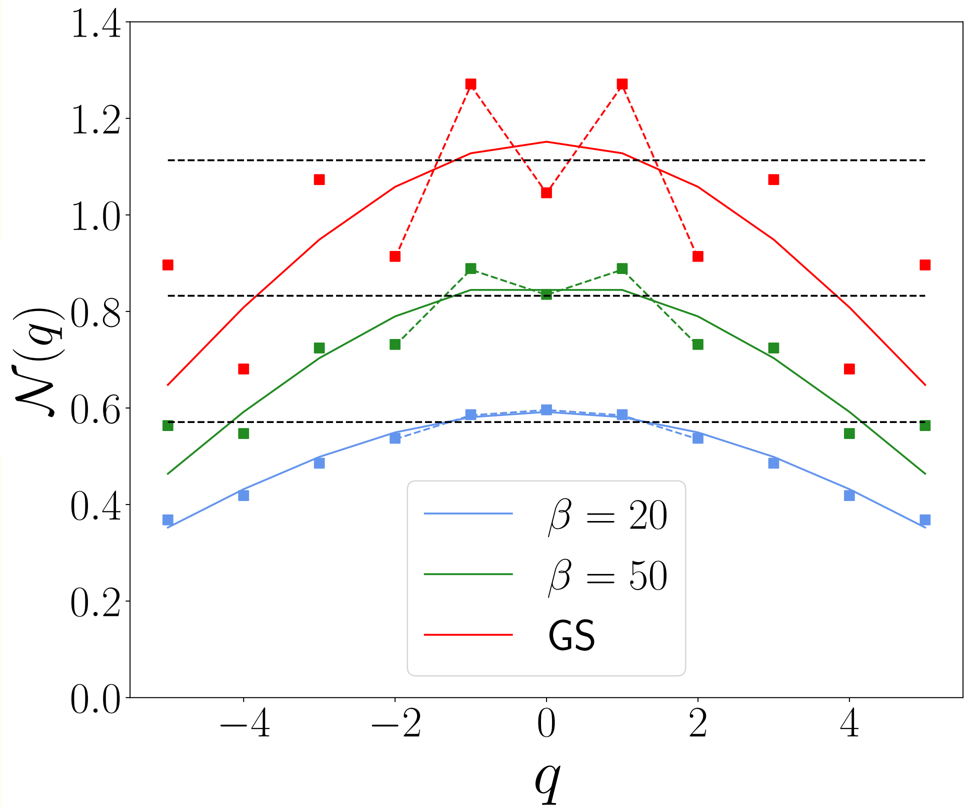


From charge moments to charge imbalance resolved negativity through Fourier transform

charged moments of $\rho_A^{R_1}$

① $N_n(\alpha) = \prod_{k=-(n-1)/2}^{(n-1)/2} Z_{R_1,k}(\alpha)$





Fourier transforms

② $$\mathcal{Z}_{R_1,n}(q) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} N_n(\alpha), \quad p(q) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} N_1(\alpha)$$

charge imbalance resolved negativity

③ $$R_n(q) = \frac{\mathcal{Z}_{R_1,n}(q)}{p^n(q)}, \qquad \mathcal{N}(q) = \frac{1}{2}\Big(\lim_{n_e \to 1} R_{n_e}(q) - 1\Big)$$

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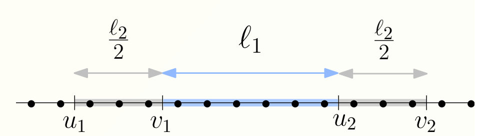

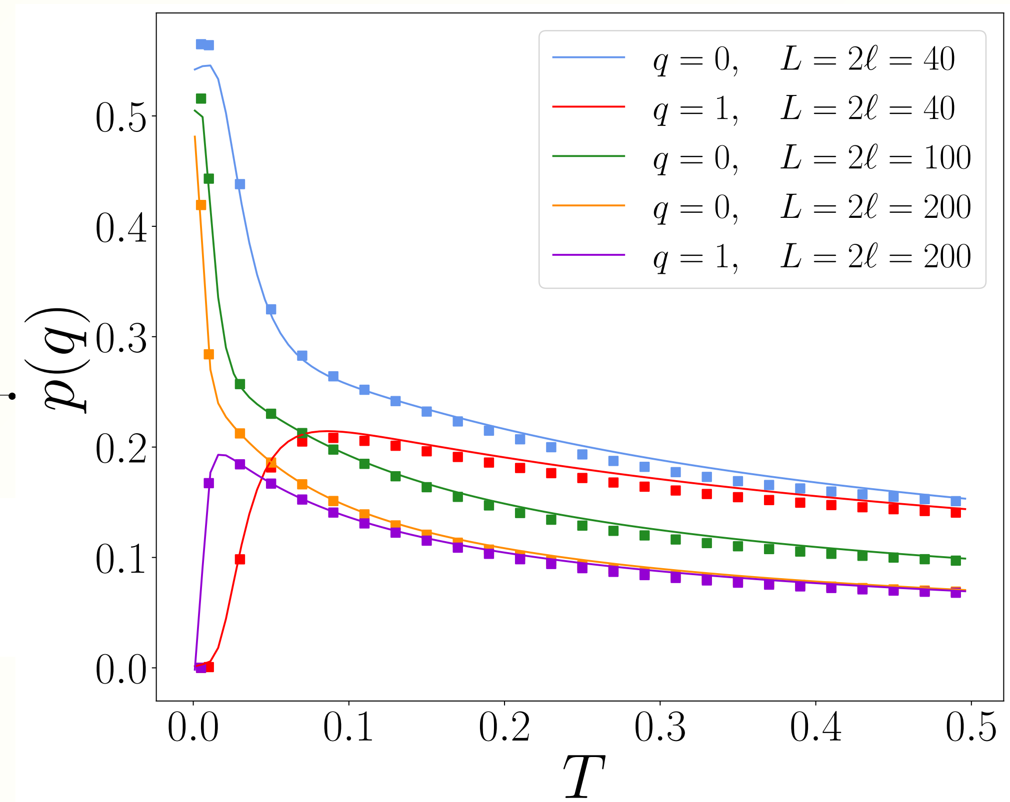

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Take-home messages...

- Entanglement negativity in systems with a conserved local charge is decomposable into symmetry sectors.
- Negativity equipartition.
- The normalized symmetry resolved negativity diverges (for some sectors) in the limit of pure states.

...and future perspectives

- $\mathbb{Z}_2$ symmetry of Majorana fermions at finite temperature through negativity.
- Symmetry resolved entanglement in non-abelian symmetries.

Thank you!