Lectures Florence

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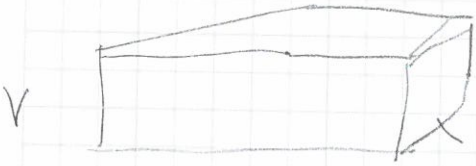
Meson propagator in nuclear matter

1. Strong-interaction matter

The structure of 'ordinary matter' is governed by the electromagnetic interaction (mean-interparticle spacing ~ nanometers)

~~it comes in various states → picture~~

(a) basic thermodynamics



$N \sim 10^{23}$ particles (Avogadro's number)

exchange energy and particles with a reservoir
in equilibrium with the reservoir

1. law of thermodynamics (grand-canonical ensemble)

$$E = TS - pV + \sum_i \mu_i N_i$$

or $$\varepsilon = Ts - p + \sum_i \mu_i n_i$$

infinite # of degrees of freedom
→ true singularities)

or grand potential/Volume (thermodynamic limit $V \to \infty$, $N \to \infty$, N/V fixed)

$$\Omega(T, \mu_i) = \varepsilon - Ts - \sum_i \mu_i n_i \quad , \quad p = -\Omega$$

$$s = -\frac{\partial \Omega}{\partial T} \quad , \quad n_i = -\frac{\partial \Omega}{\partial \mu_i}$$

(static) susceptibilities $$\chi_{\mu_i \mu_j} = -\frac{\partial^2 \Omega}{\partial \mu_i \partial \mu_j} \qquad \chi_{TT} = -\frac{\partial \Omega^2}{\partial T^2}$$

(proportional to the specific heat)

$$c_V = -T \frac{\partial^2 \Omega}{\partial T^2}$$

in stat physics Ω is related to the grand - canonical partition function

$$Z(V,T,\mu) = \mathrm{Tr}\, e^{-(H-\mu N)/T} = \sum_n \langle n| e^{-(H-\mu N)/T} |n\rangle$$

$$Z = e^{-\Omega V/T} \qquad \downarrow = \mathrm{Tr}\, e^{-1/T \int_V d^3x\,(H-\mu\psi^{+}\psi)}$$

$$\Rightarrow \quad \underline{\Omega(T,\mu) = -\lim_{V\to\infty} \frac{T}{V} \ln Z(V,T,\mu)}$$

for relativistic systems:

action $\quad S = \int dt \int d^3x\, \mathcal{L}(x) \leftarrow$ Lorentz invariant Lagrange density

$\leftarrow \mathcal{L}(\phi(x), \partial_\mu \phi(x))$

in a QFT

$$\lim_{\Delta t\to\infty} \langle \phi_1| e^{-iH\Delta t} |\phi_2\rangle = \int \mathcal{D}[\psi,\bar\psi,\phi]\, e^{iS}$$

$\Delta t = t_2 - t_1$

use formal similarity $e^{-iH\Delta t}$ and $e^{-H/T}$ by $i\Delta t \to 1/T$

$-i\Delta t \to \tau$

$$\Rightarrow \quad Z(V,T,\mu) = \int \mathcal{D}[\bar\psi,\psi,\phi..]\, e^{-\int_0^{1/T} d\tau \int_V d^3x\,(\mathcal{L}^E - i\mu\psi^{+}\psi)}$$

boundary conditions: bosons $\quad \phi(\tau + 1/T, \vec{x}) = \phi(\tau, \vec{x})$

fermions $\quad \psi(\tau + 1/T, \vec{x}) = -\psi(\tau, \vec{x})$

(b) phase transitions (picture → states of matter)

- in general $\Omega(K)$ continuous and convex function of a set of parameters $K = \{k_i\}$, but not necessarily analytic

- phase boundaries define a point, a line, surface etc on which $\Omega(K)$ non-analytic in any one of the k_i variables

- $\frac{\partial \Omega}{\partial k_i}$ discontineous   1. order

  $\frac{\partial \Omega}{\partial k_i}$ contineous   2. order $\frac{\partial^2 \Omega}{\partial k_i \partial k_j}$ discontineous

- order parameter field $\sigma(x)$, (magnetization, density, ...)

$$Z = \int \mathcal{D}\sigma \, e^{-\Gamma_{eff}(K;\sigma(x))}$$

Landau functional

dominated by the minimum of Γ_{eff}

$$\Gamma_{eff}^{min} = S_{eff} = \mathcal{L}_{eff} \cdot V$$

for homogeneous systems

$$\mathcal{L}_{eff}(K;\sigma) = \sum_n a_n(K)\sigma^n$$

Landau function (Landau 'free energy')

1. order $\mathcal{L}_{eff} = \frac{a}{2}\sigma^2 - \frac{c}{3}\sigma^3 + \frac{b}{4}\sigma^4 - h\sigma$

2. order $\mathcal{L}_{eff} = \frac{a}{2}\sigma^2 + \frac{b}{4}\sigma^4 - h\sigma$

- 2nd-order transition $\mathcal{L}_{eff}(T;\sigma)$

$a = a_t \frac{T - T_c}{T_c}$, $a_t > 0$, $b > 0$, $h \geq 0$

stationarity condition: $\frac{\partial \mathcal{L}_{eff}}{\partial \sigma} = 0 \rightarrow a\sigma + b\sigma^3 = h$

for $h = 0$

$$\bar{\sigma}|_{h=0} = \begin{cases} 0 & (T \geq T_c) \\ \pm\left(-\frac{a}{b}\right)^{1/2} & (T \leq T_c) \end{cases}$$

for finite h smooth cross-over $\rightarrow \bar{\sigma}(T = T_c) = (h/b)^{-1/3}$

(T_c where $a = 0$)

Specific heat

$$c_V(T, h=0) = -T \frac{\partial^2 \mathcal{L}_{eff}(\bar{\sigma}, T)}{\partial T^2}\Big|_{h=0} = \begin{cases} 0 & (T \geq T_c) \\ \frac{a_t^2}{2b} \frac{T}{T_c^2} & (T < T_c) \end{cases}$$

'magnetic susceptibility'

$$\chi_M(T, h)\Big|_{h=0} = \frac{\partial \bar{\sigma}}{\partial h}\Big|_{h=0} = \begin{cases} \frac{1}{a} \sim |T - T_c|^{-1} & (T \geq T_c) \\ \frac{1}{-2a} \sim |T - T_c|^{-1} & (T < T_c) \end{cases}$$

Critical exponents :

$$\bar{\sigma}(T \to T_c^-, h=0) \sim |T - T_c|^{\beta} \checkmark$$

$$c_V(T \to T_c^{\pm}, h=0) \sim |T - T_c|^{-\alpha_\pm} \checkmark$$

$$\bar{\sigma}(T = T_c^-, h \to 0) \sim h^{1/\delta} \checkmark$$

$$\chi_M(T \to T_c^{\pm}, h=0) \sim |T - T_c|^{-\gamma_\pm} \checkmark$$

in MFA : $\alpha_\pm = 0$, $\beta = 1/2$, $\gamma_\pm = 1$, $\delta = 3$

exact results depend on 'universility class' (only det. by symmetries of the system)

for O(4)

$\alpha_\pm - 0.244$, $\beta = 0.384$, $\gamma = 1,477$, $\delta = 4.851$

will later present a theory that goes beyond Landau theory!
to include quantum- and thermal fluctuation (FRG)

QCD - matter

What happens is you altimately compress and heat matter such that $10^{-15}m$
the mean-interparticle spacing is of the order of fm rather than nm? 10^{-9} m, $T \sim 10^{9}-10^{12}$

→ early universe & neutron stars

~~particles b~~

- interactions governed by strong interaction (and gravity).
- particles become relativistic → quantum field theory

QCD: Yang-Mills part

$$\mathcal{L}_{QCD} = \underbrace{-\frac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu}_{a}}_{} + \sum_{f} \bar{q}_{i,f} (i\gamma^{\mu} D^{ij}_{\mu} - m_f \delta^{ij}) q_{j,f}$$

$$D^{ij}_{\mu} = \partial_{\mu} \delta^{ij} - i g_s \lambda^{ij}_{a} A^{a}_{\mu}$$ - adjoint representation of $SU(3)_c$

$$G^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g f^{abc} A^{b}_{\mu} A^{c}_{\nu} \qquad \alpha_s = \frac{g_s^2}{4\pi} = \frac{12\pi}{(33-2N_f)\ln Q^2/\Lambda^2}$$

$$m_f = \left(\begin{array}{ccc|ccc} m_u & & & & & \\ & m_d & & & 0 & \\ & & m_s & & & \\ \hline & & & m_c & & \\ & 0 & & & m_b & \\ & & & & & m_t \end{array}\right)$$

light sector

$m_u, m_d, m_s = 3, 7, 120$ MeV
$m_c, m_b, m_t = 1.4, 4.5, 175$ GeV

$\mathcal{L}_{QCD} = \mathcal{L}_{light} + \mathcal{L}_{heavy}$

⊛ Chiral symmetry: light quarks $m_u, m_d, m_s = 0$

$$q_{L,R} = \frac{1}{2}(1 \mp \gamma_5) q$$

$\mathcal{L}^{0}_{light}$ invariant under global $U(3)_L \times U(3)_R$ trafos in flavor space

$$q_{L,R} \to e^{-i\theta^{a}_{L,R} \lambda_a/2} q_{L,R}$$ in flavor space!

Conserved Noether currents: $J^{\mu}_{L,R;a} = \bar{q}_{L,R} \gamma^{\mu} \lambda_a/2\, q_{L,R} \qquad \partial_{\mu} J^{\mu}_{L,R;a} = 0$

alternatively: $J^{\mu}_{V;a} = J^{\mu}_{L,a} + J^{\mu}_{R;a} = \bar{q} \gamma^{\mu} \lambda_a/2\, q$

$$J^{\mu}_{A;a} = J^{\mu}_{L,a} - J^{\mu}_{R,a} = \bar{q} \gamma^{\mu} \gamma_5 \lambda_a/2\, q$$

Charges: $Q_{V;a} = \int d^3x\, q^{+}(x) \lambda_a/2\, q(x), \quad Q_{A,a} = \int d^3x\, q^{+}(x) \lambda_a/2\, \gamma_5\, q(x)$

Commute with H^{0}_{QCD} $\quad [Q_{V,A;a}, H^{0}_{QCD}] = 0$

⊛ When dealing with phasetransitions, symmetries and their breaking pattern are important (e.g. crystallization, spontaneous magnetization, etc)

spontaneous breaking

'Wigner - Weyl' realization

$$Q_V^a|0\rangle = 0 = Q_A^a|0\rangle$$

→ parity doublets in hadron spectrum

Nambu - Goldstone realization

$$U(3)_L \times U(3)_R \sim SU(3)_V \times SU(3)_A \times \underset{\uparrow\ \text{strictly conserved}}{U(1)_B} \times \overset{\text{broken by quantum anomaly}\ \downarrow}{U(1)_A}$$

$$Q_V^a|0\rangle = 0 \qquad Q_A^a|0\rangle \neq 0$$

$$SU(3)_V \times SU(3)_A \Rightarrow SU(3)_V$$

→ massless 'Goldstone' bosons

$$|\pi_a\rangle = Q_A^a|0\rangle \,, \qquad H^0_{QCD}|\pi_a\rangle = Q_A^a H^0_{QCD}|0\rangle = 0$$

physical interpretation : scalar quark-antiquark pairs condense

$$\langle 0|\bar q q|0\rangle \equiv \langle \bar q q\rangle \neq 0$$

formal connection :

$$P_a(x) = \bar q(x)\gamma^5 \lambda_a q(x)$$

$$[Q_A^a, P^b] = -\delta^{ab}\bar q q$$

$$\Rightarrow Q_A^a|0\rangle \neq 0 \rightarrow \langle \bar q q\rangle \neq 0$$

weak pion decay

$$\langle 0|J^\mu_{A,a}|\pi_b(p)\rangle = -\delta_{ab} F_\pi p^\mu e^{-ipx}$$

$$F_\pi = 92.4 \pm 0.3 \text{ MeV}$$

Center symmetry Z(3) is the center of the gauge group SU(3)

$$A_\mu(\tau + 1/T, \vec x) = A_\mu(\tau, \vec x)\,, \quad q(\tau + 1/T, \vec x) = -q(\tau/\vec x) \qquad A_\mu = (\lambda_a/2)A_\mu^a$$

QCD Lagrangian invariant under local gauge transformations $g \in SU(3)_c$

$$g(x) = e^{i g_s \theta_c^a(x)\lambda_a/2}$$

$$A_\mu(x) \xrightarrow{g} A_\mu^g = g(x)\left(A_\mu(x) + \frac{i}{g_s}\partial_\mu\right)g^+(x)$$

$$q(x) \xrightarrow{g} {}^g q(x) = g(x) q(x)$$

periodicty puts constraint on allowed Gauge trafos:

$$g(\tau+1/T,\vec{x}) = h\, g(\tau,\vec{x}) \qquad h \in SU(3) \text{ constant matrix}$$

('twist' matrix)

then for gluons

$$^{g}A_\mu(\tau+1/T,\vec{x}) = h\, {}^{g}A_\mu(\tau,\vec{x})\, h^{+g}$$

Since $^{g}A_\mu$ is periodic $\quad h = z\, 1 \qquad z = e^{2\pi i n/3}$, $n = 1,2,3$

these are the center elements of SU(3)

for quarks

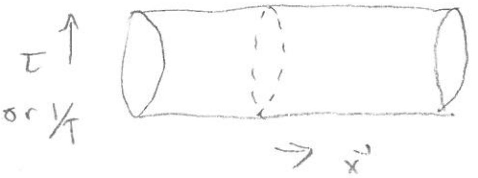
$$^{g}q(\tau+1/T,\vec{x}) = g(\tau+1/T)\, q(\tau+1/T,\vec{x}) = -z\, g(\tau,\vec{x})\, q(\tau,\vec{x}) = -z\, {}^{g}q(\tau,\vec{x})$$

$\Rightarrow z=1$ from antisymmetry

→ Center symmetry disappears

Static quarks:

the propagation of infinitly heavy quarks ~~are~~ is described by a 'Polyakov loop'



$$L(\vec{x}) = \mathrm{Tr}_c\left[\mathcal{P}\, e^{i\int_0^{1/T} d\tau\, A_4(\tau,\vec{x})}\right]$$

complex scalar field

transforms non-trivially under the gauge group

$$^{g}L(\vec{x}) = z\, L(\vec{x})$$

expectation value (YM only)

$$\langle L(\vec{x})\rangle = \frac{1}{Z_{YM}}\int \mathcal{D}[A_\mu^a]\, L(\vec{x})\, e^{-S^E_{YM}} = e^{-\beta F_q(\vec{x})}$$

$\beta = 1/T$

$$\left(\langle O\rangle = \frac{1}{Z}\int \mathcal{D}[\cdot\cdot]\, O\, e^{-S^E}\right)$$

$F_q(\vec{x})$: free energy of a static test quark at position $\vec{x}$

- small T : color confined → $F_q = \infty$ $\langle L\rangle = 0$
- high T : ~~quarks~~ color deconfined F_q finite $\langle L\rangle \neq 0$

hence $\langle L\rangle$ measure of confinement, order parameter

@ high T Z(3) symmetry spontaneously broken

FRG

method to deal with phase transition beyond the MFA (Landau theory)
includes quantum fluctuations and is widely used in stat. Physics
here thermal field theory

scalar field: $\varphi(x)$ $\qquad \mathcal{L} = \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)$

$$S = \int d^4x\, \mathcal{L}(\varphi, \partial_\mu\varphi) \qquad \left(V(\varphi) = \frac{g}{4!}\varphi^4\right)$$

generating functional

$$Z[J] = \int \mathcal{D}[\varphi]\, e^{i\int d^4x \left[\overset{\frac{1}{2}(\partial_0\varphi)^2 - \frac{1}{2}(\nabla\phi)^2}{\frac{1}{2}\partial^\mu\varphi\partial_\mu} - \frac{m^2}{2}\varphi^2 - \frac{g}{4!}\varphi^4 + \overset{\text{classical source}}{J\varphi}\right]}$$

$$= Z[0]\sum_{n=0}^{\infty}\frac{i^n}{n!}J(x_1)\cdots J(x_n)\langle 0|T\{\varphi(x_1)\cdots\varphi(x_n)\}|0\rangle$$

partition function
$-it \to t$ Wick rotation (stat mechanics problem)

$$Z'[J] = \int\mathcal{D}[\varphi]\, e^{-\int d^4x \underbrace{\left[\frac{1}{2}(\partial_{i=1,4}\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{g}{4!}\varphi^4 + \overset{\text{classical source}}{J\varphi}\right]}_{H = T + V \text{ in 4D Euclidean space}}}$$

$$= \int\mathcal{D}[\varphi]\, e^{-S^E[\varphi] + \int d^4x\, J(x)\varphi(x)}$$

coarse-graining (Wilson) $\qquad \varphi(x) = \int\frac{d^4q}{(2\pi)^4}\tilde{\varphi}(q)e^{iq\cdot x} \quad \varphi_{q<k}(x) = \int_0^k\frac{d^4q}{(2\pi)^4}\tilde{\varphi}(q)$

$$\varphi(x) = \varphi_{q\le k}(x) + \varphi_{q>k}(x)$$

only include fluctuations with $q > k$

$$Z[J] = \int\mathcal{D}[\varphi_{q\le k}]\underbrace{\int\mathcal{D}[\varphi_{q>k}]\, e^{-S[\varphi] + \int d^4x\, J\varphi}}_{Z_k[J] \,:\, \text{scale-dependent partition function}}$$

obviously: $\lim\limits_{k\to 0} Z_k[J] = Z[J]$

to suppress infrared mode introduce regulator term

$$\Delta S_k[\varphi] = \frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\varphi(-q)R_k(q)\varphi(q)$$

$\lim\limits_{k\to 0} R_k(q) = 0$

$\lim\limits_{k\to\Lambda} R_k(q) = \infty$

such that $\qquad$ (acts as mass term)

$$Z_k[J] = \int\mathcal{D}\varphi\, e^{-S[\varphi] - \Delta S_k[\varphi] + \int d^4x\, J(x)\varphi(x)}$$

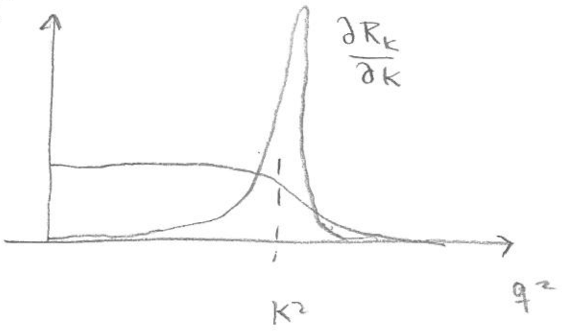
scale-dependent effective action: $\qquad \phi(x) \equiv \langle\varphi(x)\rangle \qquad W[J] = \log Z[J]$

Legendre-transform:

$$\Gamma_k[\phi] \equiv \sup_J\left(\int d^4x\, J(x)\phi(x) - \log Z_k[J]\right) - \Delta S_k[\phi]$$

regulator function

$R_k(q)$



Γ_K interpolates between $K = \Lambda$ (no fluctuations) and $K = 0$ (full quantum action)

$$\lim_{K \to \Lambda} \Gamma_K[\phi] = S_c[\phi]$$

$$\lim_{K \to 0} \Gamma_K[\phi] = \Gamma[\phi] \qquad \Gamma[\phi] = \frac{V}{T} \Omega(T, \mu)$$

'flow equation' (without derivation)

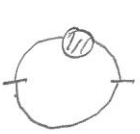
Wetterich eq $$\partial_K \Gamma_K[\phi] = \frac{1}{2} \mathrm{Tr} \left\{ \partial_K R_K \left(\Gamma_K^{(2)}[\phi] + R_K \right)^{-1} \right\}$$

With

$$\Gamma_K^{(2)}(q) \frac{\delta \Gamma_K[\phi]}{\delta \phi[-q] \delta \phi(q)}$$

$\partial_K \Gamma_K = \frac{1}{2}$

$\Gamma_k^{(2)}$



these equations are exact, but functional equations!

Idea is to turn them into ordinary non-linear PDE

several posibilies – here 'derivative expansion'

$$\Gamma_K[\phi] = \int d^4x \left\{ U_K(\phi) + \frac{1}{2} Z_K(\phi) (\partial_\mu \phi)^2 + \cdots \right\}$$

LPA : Keep only the lowest order U_K (i.e. $Z_K(\phi) = 1$)
reproduces critical exponents well in many cases.

The O(4) model (same universality class as 2-flavor QCD) ⊛ (6a)

$$\phi = (\phi_1 \cdots \phi_4) = (\vec{\pi}, \sigma)$$

in LPA

$$\Gamma_K[\phi] = \int d^4x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + U_K(\phi) \right\} \qquad \phi^2 = \sigma^2 + \vec{\pi}^2$$

$$\Gamma_{K;ij}^{(2)}(q) = \Gamma_{K,\pi}^{(2)}(q) \left\{ \delta_{ij} - \frac{\phi_i \phi_j}{\phi 2} \right\} + \Gamma_{K;\sigma}^{(2)}(q) \frac{\phi_i \phi_j}{\phi^2}$$

$$\partial_K U_K = I_\sigma^k + 3 I_\pi^k \qquad I_i^k = \frac{1}{2} \mathrm{Tr}_q \left(\partial_K R_K(q) \left[\Gamma_{k,i}^{(2)}(q) + R_K(q) \right]^{-1} \right.$$

with $$R_k(q) = (k^2 - q^2)\, \theta(k^2 - q^2)$$

$$\Rightarrow I_i^K = \frac{k^4}{3\pi^2} \frac{1}{2E_i^K} \; ; \quad E_\pi^K = \sqrt{k^2 + 2U_k'} ; \quad E_\sigma^K = \sqrt{k^2 + 2U_k' + 4U_k'' \phi^2}$$

QCD and chiral symmetry

Construct an effective Lagrangian that respects chiral symm

$$\sigma(x) = \bar q(x)\, q(x) \qquad \text{Lorentz scalar}$$
$$\vec\pi(x) = i\bar q(x)\,\vec\tau\gamma_5\, q(x) \qquad \text{" pseudoscalar}$$

recall $$U_L(2)\times U_R(2) = U_V(1)\otimes \underbrace{SU_V(2)\otimes SU_A(2)}_{\text{isomorphic to } O(4)} \otimes U_A(1)$$

$q = \begin{pmatrix} u \\ d \end{pmatrix}$ iso doublet

$$\Lambda_V:\quad q \to e^{-i\vec\tau\cdot\vec\theta_V/2}\, q$$
$$\bar q \to e^{+i\vec\tau\cdot\vec\theta_V/2}\, \bar q$$

$$\Lambda_A:\quad q \to e^{-i\gamma_5\vec\tau\cdot\vec\theta_A/2}\, q$$
$$q \to e^{+i\gamma_5\vec\tau\cdot\vec\theta_A/2}\, \bar q$$

Λ_V almost exact, while Λ_A broken by quark mass

$$\Lambda_A:\ m\bar q q \to m\bar q q - i m\vec\theta_A\, \bar q\vec\tau\gamma_5 q$$

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now

$$\Lambda_V:\ \sigma \to \sigma \qquad\qquad (\sigma = \bar q q,\ \vec\pi = i\bar q\vec\tau\gamma_5 q)$$
$$\Lambda_V:\ \vec\pi \to \vec\pi + \vec\theta_V \times \vec\pi$$

$$\Lambda_A:\ \sigma \to \sigma - \vec\theta_A\cdot\vec\pi$$
$$\Lambda_A:\ \vec\pi \to \vec\pi + \vec\theta_A\,\sigma$$

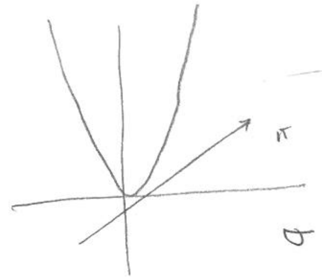
then

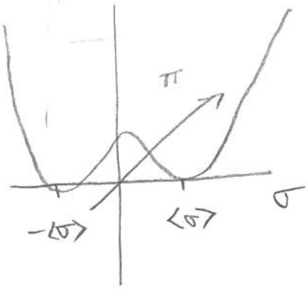
$$\Lambda_{V,A}:\ \sigma^2 + \vec\pi^2 \to \sigma^2 + \vec\pi^2$$
$$\Lambda_{V,A}:\ \partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec\pi\,\partial^\mu\vec\pi \to \partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec\pi\cdot\partial^\mu\vec\pi$$

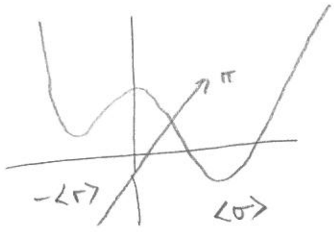
$$\Rightarrow\ \mathcal{L}(\sigma,\vec\pi) = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec\pi\cdot\partial^\mu\vec\pi - \overbrace{\sigma^2 + \vec\pi^2}^{U(\phi)}$$

where $U_k' = \frac{\partial U_k}{\partial \phi}\Big|_{\phi=\phi_0}$ etc $\qquad U_\Lambda(\phi) = \frac{g}{4}(\sigma^2+\pi^2)^2 - c\sigma$

$\langle\sigma\rangle \equiv \langle\bar{q}q\rangle \neq 0$ spontaneous symm. breaking







The QM-model

to describe QCD matter at finite μ fermions have to be included

chirally invariant interaction term $\qquad \mathcal{L}_{int} = -G(\bar{q}q\sigma + i\bar{q}\gamma^5\vec{\tau}q\vec{\pi})$

$$\Rightarrow \mathcal{L}_{QM} = \bar{q}(i\not{\partial} - G(\sigma + i\gamma^5\vec{\tau}\vec{\pi}))q + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + U(\sigma,\vec{\pi})$$

or in Euclidean space-time $(\gamma_j^M = i\gamma_j^E$, γ_0 and γ_5 unchanged$)$
$(t \to -it)$

$$\mathcal{L}_{QM}^E = \bar{q}(\not{\partial} + G(\sigma + i\gamma^5\vec{\tau}.\vec{\pi}))q + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + U(\vec{\sigma},\vec{\pi})$$

in LPA;

$$\Rightarrow \Gamma_k[\bar{q},q,\phi] = \int_0^{1/T} d\tau \int d^3x \left[\bar{q}(\not{\partial} + G(\sigma + i\gamma^5\vec{\tau}\vec{\pi}) - \mu\gamma_0)q + \frac{1}{2}(\partial_\mu\phi)^2 + U_k(\phi)\right]$$

with appropriate boundary conditions for bosons and fermions

flow equations

$$\partial_k U_k = \frac{k^4}{12\pi^2}\left[\frac{1}{E_{k;\sigma}}\coth\left(\frac{E_{k,\sigma}}{2T}\right) + \frac{3}{E_{k,\vec{\pi}}}\coth\left(\frac{E_{k,\pi}}{2T}\right)\right.$$
$$\left. - \frac{2N_cN_f}{E_{q,k}}\left(\tanh\left(\frac{E_{q,k}-\mu}{2T}\right) + \tanh\left(\frac{E_{q,k}+\mu}{2T}\right)\right)\right]$$

$E_{k,\pi}^2 = k^2 + 2U_k'$ → curvature mass

$E_{k,\sigma}^2 = k^2 + 2U_k' + 4U_k''\sigma^2$

$E_{k,q}^2 = k^2 + G^2\sigma^2$

$$U_{k\to 0}(T,\mu) = \Omega(T,\mu)\big|_{\langle\sigma\rangle}$$

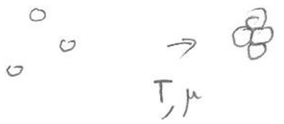
$$\partial_k U_k = \frac{1}{2} I_{k,\sigma} + \frac{3}{2} I_{k,\pi} - N_c N_f I_{k;q}$$

$$I_{k;q} = \frac{k^4}{3\pi^2}\,\frac{1 - n_F(E_{k,q}-\mu) - n_F(E_{k,q}+\mu)}{E_{k,q}}$$

Hadrons in the QCD medium

As ~~ma~~ ~~to~~ matter is heated and or/compressed the vacuum probarties of QCD change.

- confinement - deconfinement transition
- chiral symmetry restoration



concentrate on the latter

constituent mass

- QM-model predicts $M_q = m_q + G\langle\sigma\rangle = m_q + G\langle\bar{q}q\rangle_0$

Confirmed by QCD calculations

as $\langle\bar{q}q\rangle$ dimished, do all hadrons become light?

- $\langle\bar{q}q\rangle_0 \neq 0$ implies splitting of parity partner

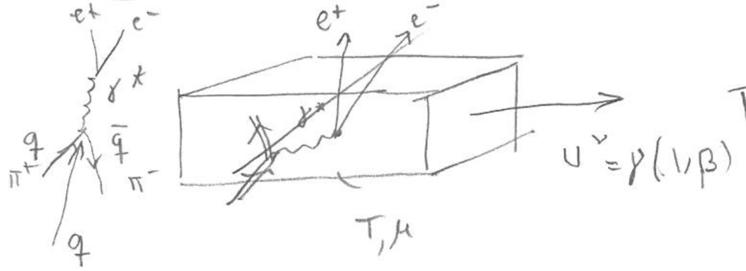
$\langle\bar{q}q\rangle \xrightarrow[T,\mu]{} 0$ degeneracy restored (π, σ) (ρ, a_1) etc

Are such effects observable?

QCD matter created in relativistic HIC $(E_{coll} \gg m_p)$

- real and virtual photons ideal probe for all stages of the matter evolution since $\lambda_\gamma \gg R_{\text{fire ball}}$.

Electromagnetic correlation function



$$\Pi^{\mu\nu}_{em}(q_0,\vec{q}) = -i\int d^4x\,\theta(x^0)\langle[J^\mu(x), J^\nu(0)]\rangle_{T,\mu}$$

Bose distribution in restframe

real photons: $$q_0\frac{dN_\gamma}{d^4x\,d^3q} = -\frac{\alpha_{em}}{\pi} f^B(q\cdot U, T)\,\text{Im}\,\Pi_{em}(q_0 = q; \mu, T)$$

dileptons: $$\frac{dN_{ee}}{d^4x\,d^4q} = -\frac{\alpha^2_{em}}{M^2\pi^3} L(M^2) f^B(q\cdot U; T)\,\text{Im}\,\Pi_{em}(M, q; \mu T)$$

$M^2 = q_0^2 - \vec{q}^{\,2}$, $L(M^2)$ lepton phase-space factor

$\Pi_{em} = \frac{1}{3}\Pi^{\mu\nu}_{em} g_{\mu\nu}$, $\Pi_{em} = \frac{1}{3}(\Pi^L_{em} + 2\Pi^T_{em})$

for real photons only transverse part

Vacuum

$\mathrm{Im}\,\Pi_{em}$ measurable through $e^+ + e^- \rightarrow$ hadrons , $s \equiv M^2$ because of Lorentz invariance

$$R(s) - \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = -\frac{12\pi}{s}\,\mathrm{Im}\,\Pi^V_{em}(s)$$

$$\frac{4\pi\,\alpha_{em}}{3s}$$

$$\Pi^{em}_L = \Pi^{em}_T$$

$q\bar{q}$ annihilation:

$$R(s) = N_c \sum_q e_q^2 \left(1 + \frac{\alpha_{em}}{\pi} + 1.411\left(\frac{\alpha_{em}}{\pi}\right)^2 - 12.8\left(\frac{\alpha_{em}}{\pi}\right)^3 + \cdots\right)$$

altogether

$$\mathrm{Im}\,\Pi^{vac}_{em}(M) = \begin{cases} \displaystyle\sum_{V=\rho,\omega,\varphi}\left(\frac{m_V^2}{g_V}\right)^2 \mathrm{Im}\,D^{vac}_V(M) & M < 1.5\ \mathrm{GeV} \\ \displaystyle -\frac{M^2}{12\pi}\left(1 + \frac{\alpha_s(M)}{\pi} + \cdots\right) N_c \sum_{q=u,d,s}(e_q)^2 & M \geqslant 1.5\ \mathrm{GeV} \end{cases}$$

electromagnetic current: $u, c, t\ ;\ \frac{2}{3}e$

$d, s, b\ :\ -\frac{1}{3}e$

quark content

$\rho^\pm$ $u\bar{d}$

$\rho^0 = (u\bar{u} - d\bar{d})/\sqrt{2}$

$\omega = (u\bar{u} + d\bar{d})/\sqrt{2}$

quark-basis: $j^\mu_{em} = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s$

hadron basis $j^\mu_\rho = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)\ ;\ j^\mu_\omega = \frac{1}{6}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d)\ ;\ j^\mu_\phi = -\frac{1}{3}(\bar{s}\gamma^\mu s)$

$$j^\mu_{em} = \frac{m_\rho^2}{g_\rho}\rho^\mu + \frac{m_\omega^2}{g_\omega}\omega^\mu + \frac{m_\phi^2}{g_\omega}\phi^\mu$$

$R \sim \left[\mathrm{Im}\,D_\rho + \frac{1}{9}\mathrm{Im}\,D_\omega + \frac{2}{9}\mathrm{Im}\,D_\phi\right] \rightarrow$ ρ-meson dominates!

ρ-meson in hadronic medium

$$D^{\mu\nu}_V = \left(M^2 - m_V^2 - \Sigma^L_V(q_0, q)\right)^{-1} P^{\mu\nu}_L + \left(M^2 - m_V^2 - \Sigma^T_V(q_0, q)\right)^{-1} P^{\mu\nu}_T$$

$$P^{\mu\nu}_L = \frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} - P^{\mu\nu}_T \quad , \quad P^{\mu\nu}_T = \begin{cases} 0 & \mu = 0 \text{ or } \nu = 0 \\ \delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} & ,\mu,\nu = 1,2,3 \end{cases}$$

In-medium spectral functions from FRG

how are the medium modifications of hadrons related to the change of vacuum structure of QCD? (deconfinement and chiral symmetry restoration)

- need to calculate equilibrium properties and spectral functions on the <u>same</u> footing! → FRG

- equilibrium FRG formulated and solve in Euclidean space-time
- spectral functions are real-time quantities in Minkowski space-time
- analytic continuation procedure needed!

- <u>some definitions</u> (scalar field $\phi(x)$)
and relations

retarded propagator: $D^R(\omega,\vec{q}) = -i\int d^4x\, e^{iqx}\,\theta(x_0)\langle[\phi(x),\phi(0)]\rangle$

advance propagator: $D^A(\omega,\vec{q}) = i\int d^4x\, e^{iqx}\,\theta(-x_0)\langle[\phi(x),\phi(0)]\rangle$

then spectral function: $\rho(\omega,\vec{q}) = \frac{i}{2\pi}\left(D^R(\omega,\vec{q}) - D^A(\omega,\vec{q})\right)$

Källén - Lehmann representation

$$D^R(\omega,\vec{q}) = -\int_{-\infty}^{\infty} d\omega'\,\frac{\rho(\omega',\vec{q})}{\omega'-\omega-i\varepsilon}$$

$$D^A(\omega,\vec{q}) = -\int_{-\infty}^{\infty} d\omega'\,\frac{\rho(\omega',\vec{q})}{\omega'-\omega+i\varepsilon}$$

complex values of $\omega \equiv z$

$$D(z,\vec{q}) = -\int_{-\infty}^{\infty} d\omega'\,\frac{\rho(\omega',\vec{q})}{\omega'-z}$$

$z = -iq_0$ ⇒ Euclidean propagator

$$D^E(q_0,\vec{q}) = -\int d\omega'\,\frac{\rho(\omega',\vec{q})}{\omega'+iq_0}$$

given $\mathcal{D}^E$, $\mathcal{D}^R$ and $\mathcal{D}^A$ can be obtained by analytic continuation

$$\mathcal{D}^R(\omega,\vec{q}) = -\mathcal{D}^E(q^0 \to i\omega - \varepsilon, \vec{q})$$
$$\mathcal{D}^A(\omega,\vec{q}) = -\mathcal{D}^E(q^0 \to i\omega + \varepsilon, \vec{q})$$

or

$$\mathcal{D}^R(\omega,\vec{q}) = \mathcal{D}(z \to \omega + i\varepsilon, \vec{q})$$
$$\mathcal{D}^A(\omega,\vec{q}) = \mathcal{D}(z \to \omega - i\varepsilon, \vec{q})$$

$$\Rightarrow \quad \rho(\omega,\vec{q}) = \frac{i}{2\pi}\left[\mathcal{D}(z\to\omega+i\varepsilon,\vec{q}) - \mathcal{D}(z\to\omega-i\varepsilon,\vec{q}\right]$$

also

$$\mathrm{Im}\,\mathcal{D}^R(\omega,\vec{q}) = -\mathrm{Im}\,\mathcal{D}^A(\omega,\vec{q})$$
$$\mathrm{Re}\,\mathcal{D}^R(\omega,\vec{q}) = \mathrm{Re}\,\mathcal{D}^A(\omega,\vec{q})$$

$$\Rightarrow \quad \rho(\omega,\vec{q}) = -\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{D}^R(\omega,\vec{q}) \qquad \circledast$$

Flow equations for two-point functions

recall scalar case

$$\partial_k\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k\left(\Gamma^{(2)}_{k(q)} + R_k\right)^{-1}\right\} = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k \mathcal{D}_k\right\}$$

$$\Gamma^{(2)}_{k(q)} = \frac{\delta\Gamma_k}{\delta\phi(-q)\,\delta\phi(q)}$$

Flow eq. for two-point function (Euclidean)

$$\partial_k\Gamma^{(2)}_k = \frac{1}{2}\frac{\delta^2}{\delta\phi^2}\mathrm{Tr}\left\{\partial_k R_k \mathcal{D}_k\right\}$$
$$= \mathrm{Tr}\left\{\partial_k R_k \left(\mathcal{D}_k\Gamma^{(3)}_k\mathcal{D}_k\Gamma^{(3)}_k\mathcal{D}_k\right\} - \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k\left(\mathcal{D}_k\Gamma^{(4)}_k\mathcal{D}_k\right)\right\}$$

3-point function $\quad \Gamma^{(3)}_k = \frac{\delta^3\Gamma_k}{\delta\phi^3}$ $\qquad$ 4-point-function $\quad \Gamma^{(4)}_k = \frac{\delta^4\Gamma_k}{\delta\phi^4}$

$\Rightarrow$ hierachy of flow equations

Approximation:

extract $\Gamma^{(3)}_k$ and $\Gamma^{(4)}_k$ from local potential U_k only

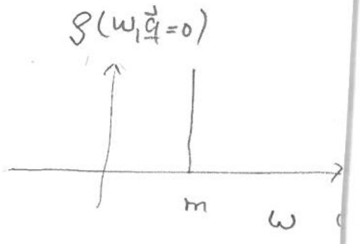
(no momentum dependence)

⊛ physical interpretation of the spectral function

free particle :

$$D^R(\omega,\vec{q}) = \frac{1}{(\omega+i\epsilon)^2 - q^2 - m^2}$$

$$\Rightarrow \rho(\omega,\vec{q}) = \mathrm{sgn}(\omega)\,\delta(\omega^2 - q^2 - m^2)$$



Thermodynamically consistent

$$\Gamma^{(2)}_{k,\pi}(q=0) = 2U_k'$$

$$\Gamma^{(2)}_{k,\sigma}(q=0) = 2U_k' + 4U_k''\sigma^2$$

also symmetry preserving ; $\Gamma^{(2)}_{k,\pi}(q=0) = 0$! Goldstone mode persists

Analytic continuation

to begin with flow eqns for 2-point functions in Euclidean space-time.
To obtain spectral functions need to analytically continue.

basics:

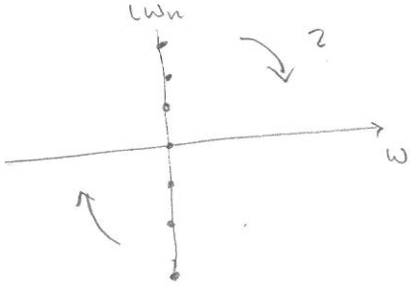
extend domain of a function $f(i\omega_n)$ into the entire complex plane

$$f(i\omega_n) \to F(z) \quad \text{with} \quad F(z)\big|_{z=i\omega_n} = f(i\omega_n)$$

(not unique)

Baym-Mermin boundary cond.:
(physically motivated)

- $F(z)$ bounded for $|z| \to \infty$
- $F(z)$ analytic outside the real axis



result:

$$\partial_k \Gamma_k^{(2),R}(\omega,\vec{q}) = -\lim_{\varepsilon \to 0} \partial_k \Gamma_k^{(2),E}(q_0 = -i(\omega + i\varepsilon), \vec{q})$$

$$\Rightarrow \rho(\omega,\vec{q}) = -\frac{1}{\pi}\,\mathrm{Im}\, D^R(\omega,\vec{q}) \qquad D^R(\omega,\vec{q}) = \Gamma^{(2),R}(\omega,\vec{q})^{-1}$$

$$\Rightarrow \rho(\omega,\vec{q}) = \frac{1}{\pi}\,\frac{\mathrm{Im}\,\Gamma^{(2),R}(\omega,\vec{q})}{\left(\mathrm{Re}\,\Gamma^{(2),R}(\omega,\vec{q})\right)^2 + \left(\mathrm{Im}\,\Gamma^{(2),R}(\omega,\vec{q})\right)^2}$$

Since $$D^R(\omega,\vec{q}) = \left((\omega+i\Gamma)^2 - q^2 - m^2 - \Pi^R\right)^{-1}$$

Γ contains physical decay channels

$$\Rightarrow \rho(\omega,\vec{q}) = \frac{1}{\pi}\,\frac{2\omega\Gamma}{(\omega^2-\Gamma^2-q^2-m^2-\Pi^R)^2+4\omega^2\Gamma^2}$$

→ results :
- flow of ρ_π^k and ρ_σ^k
- ρ_π and ρ_σ as a function T

Including vector - and axialvector mesons

gauged linear σ-model B.W. Lee and H.T. Nieh, Phys. Rev. 166, 1507 (1968)

$U(\phi^2)$ O(4) symmetric $\phi = (\sigma,\vec{\pi})$

- Vector mesons in the O(4) adjoint representation

$$V^\mu = \vec{\rho}^{\,\mu}\vec{T} + \vec{a}_1^{\,\mu}\vec{T}^5$$

$$(T_i)_{jk} = \begin{pmatrix} -\varepsilon_{ijk} & \vec{0} \\ \vec{0} & 0 \end{pmatrix}, \qquad \vec{T}^5 = \begin{pmatrix} 0_{3\times3} & -i\vec{e}_i \\ i\vec{e}_i^{\,T} & 0 \end{pmatrix}$$

$$T_i^L = \tfrac{1}{2}(T_i - T_i^5), \qquad T_i^R = \tfrac{1}{2}(T_i + T_i^5)$$

form representations of $SU(2)_L$ and $SU(2)_R$

- coupling of vector mesons to σ and $\vec{\pi}$ via minimal coupling $D_\mu = \partial_\mu - igV_\mu$

⇒ effective action

$$\Gamma_K = \int d^4x \Big[\bar{q}\big(\partial\!\!\!/ - \mu_0\gamma_0 + G_s(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi}) + iG_v(\gamma_\mu\vec{\tau}\cdot\vec{\rho}^{\,\mu} + \gamma_5\gamma_5\vec{\tau}\cdot\vec{a}_1^{\,\mu})\big)$$
$$+ U_K(\phi^2) - c\sigma + \tfrac{1}{2}(\partial_\mu\phi)^2$$
$$+ \tfrac{1}{8}\mathrm{Tr}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - igV_\mu\phi\partial_\mu\phi - \tfrac{1}{2}g^2(V_\mu\phi)^2 + \tfrac{1}{2}m_{K,V}^2\,\mathrm{Tr}(V_\mu V_\mu)\Big]$$
$$+ \Delta\Gamma_{\pi a_1}$$

↑ $\pi - a_1$ mixing

Seminar questions

1. work out Landau theory for 2nd-order phase transitions

2. work out $[Q_A^a, p^b] = -\delta^{ab}\,\bar{q}q$

3. Transformation properties of the O(4) model

vector $\sigma \to \sigma$

$\vec{\pi} \to \vec{\pi} + \vec{\theta}_V \times \vec{\pi}$

axialvector: $\vec{\sigma} \to \sigma - \vec{\theta}_A \cdot \vec{\pi}$

$\vec{\pi} \to \vec{\pi} + \vec{\theta}_A \sigma$

4. rewrite j^μ_{em} in the hadronic basis