SUSY gauge theories, cluster integrable systems & q-Painleve equations

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Supersymmetric Quantum Field Theories in the Non-perturbative Regime

Galileo Institute, Florence, May, 2018

based on:

Cluster integrable systems and q-Painlevé equations,
JHEP 02 (2018) 077, arXiv:1711.02063

Cluster Toda chains and Nekrasov functions,
to appear in L.D.Faddeev volume of Theor. & Math. Phys.,
arXiv:1804.10145

with Misha Bershtein & Pasha Gavrylenko

OLD STORY (mid 90-s):

- Exact non-perturbative Seiberg-Witten solution of $\mathcal{N} = 2$ SUSY 4d gauge theories;

- Formulated (GKMMM95) in terms of *integrable systems* (algebraic, ...), pure SUSY gauge theories $\equiv$ affine or periodic Toda chains;

- 5d (Nekrasov96,...) generalization $\equiv$ "relativization" of an integrable system (compact 5-th dim's $R \equiv \frac{1}{c}$);

- Relativistic Toda chains on Poisson-Lie groups (Fock & AM 95-97) $\Rightarrow$ towards *cluster* integrable systems (5d $\equiv$ cluster).

NEW MILLENNIUM (2000 +):

- Seiberg-Witten prepotential as a limit of Nekrasov *instanton partition functions*;

- Nekrasov functions as conformal blocks (2d CFT) and partition functions of topological strings;

- 5d generalization "more effective", quantum mechanics on instanton moduli spaces, topological vertices etc;

- Relativistic Toda chains as cluster integrable systems: pure combinatorial approach (GK,. . .).

PRESENT DAYS (2012 +):

- Conformal blocks and isomonodromic deformation tau-functions (Painlevé equations etc): the “Kiev formulas” (GIL-PG-MB & АЩ);

- 5d SUSY gauge theories and q-deformed conformal algebras;

- Discrete integrable systems & q-difference equations: from cluster mutations.

Main motivation:

- *Non-perturbative* SUSY gauge theories: equations for partition functions (integrable systems etc);

- *Define* the partition functions as solutions – the way to take into account non-perturbative effects;

- Testing at the level of the instanton expansions.

Amazingly:

- 4d $\to$ 4+1=5d (with infinitely many KK modes) – more simple, 5-th dimension as “time” for the quantum mechanics on instanton moduli spaces, Nekrasov functions are better defined;

- q-deformation from dual 2d-point of view (q-Virasoro etc), and q-deformation of the Painlevé systems: more simple *discrete* dynamics, than continuous.

Lack of “physical intuition” is compensated.

Our (Bershtein-Gavrylenko-AM)

MAIN CONJECTURE (2017):

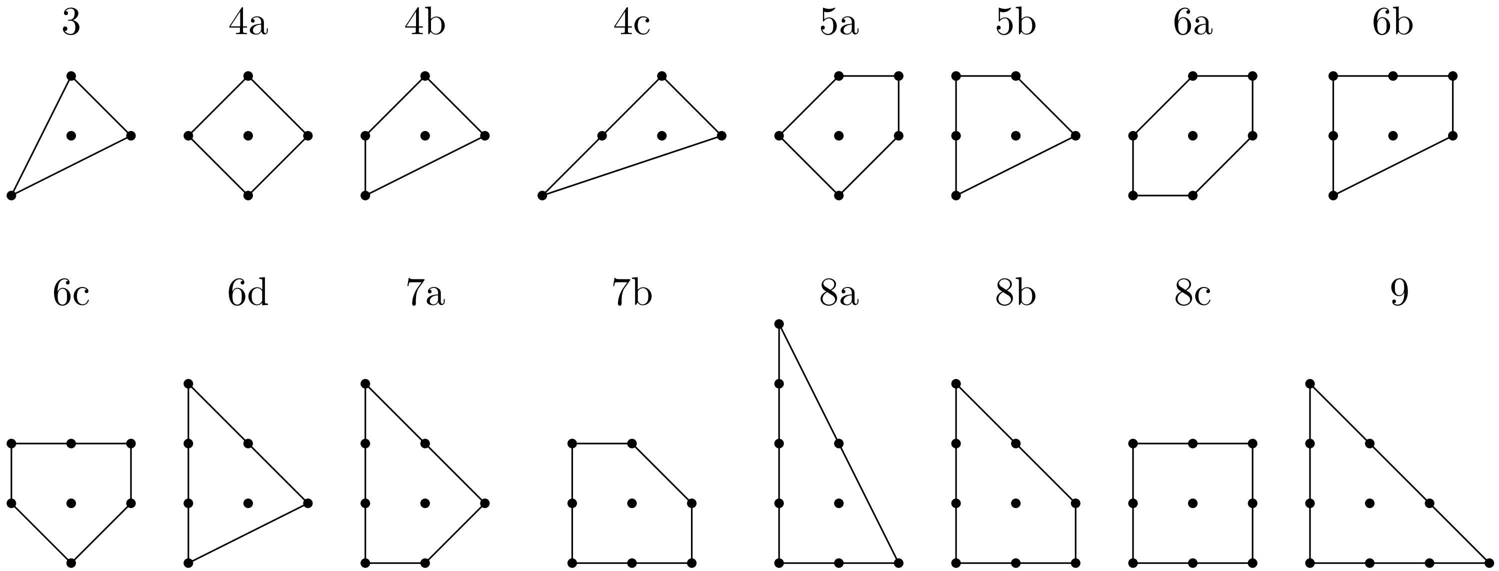
- Deautonomization of a *cluster integrable system* (defined by a Newton polygon Δ), leads to q-difference equations of the Painlevé type, generated by discrete flows, treated as sequences of quiver mutations;

- In tau-variables they can be written as a system of Hirota bilinear difference equations;

- The tau-functions are given by (Fourier-)dual 5d Nekrasov partition functions or partition functions of the topological string on 3d Calabi-Yau (also determined by the same polygon Δ as the SW curve).

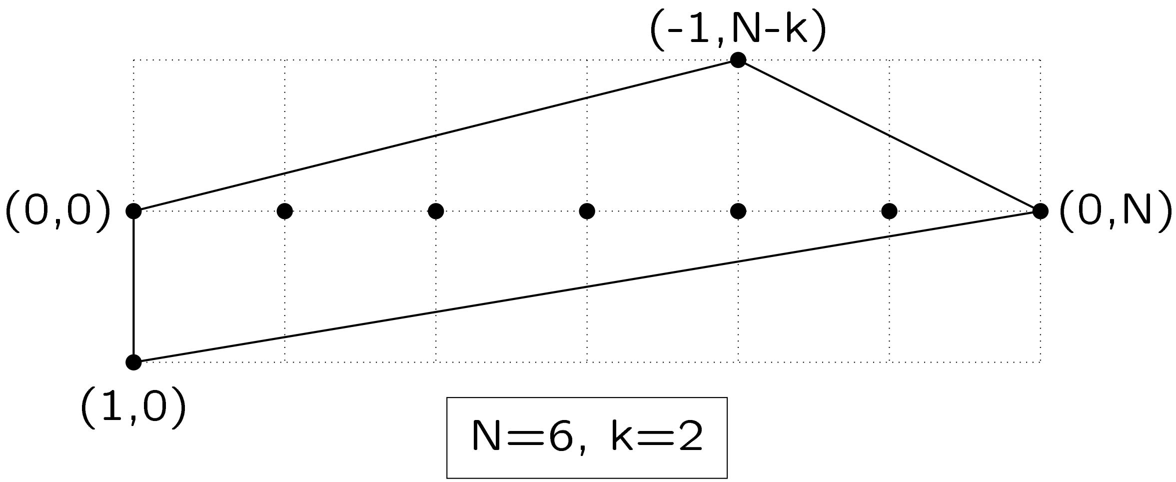
This Conjecture has been tested:

the *Painlevé case*: list of Newton polygons Δ with a single internal point and $3 \leq B \leq 9$ boundary points.



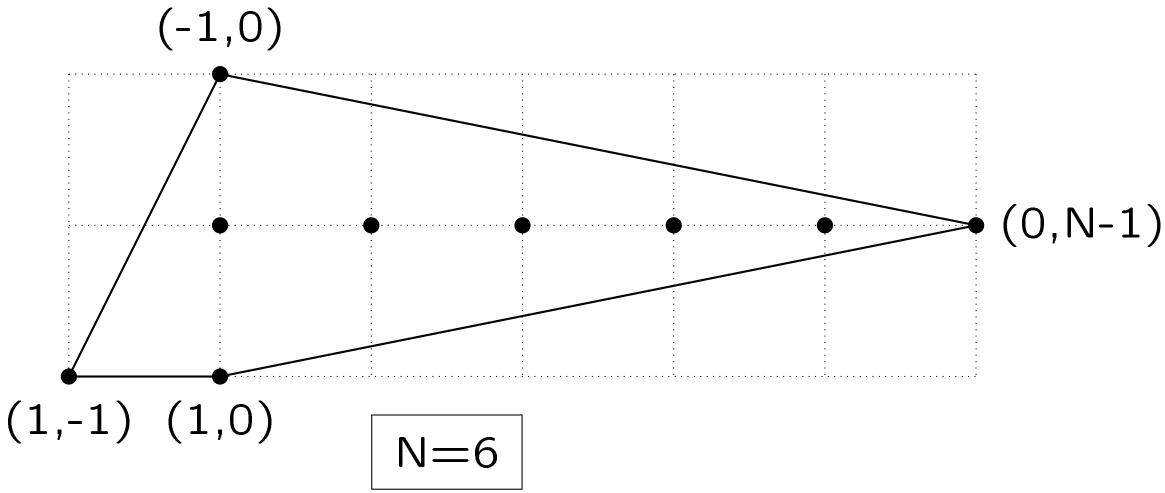
Here the SW curve $f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0$ is always a torus.

the *Toda case*: Newton polygons with $N-1$ internal points and $B=4$ boundary points.



$Y^{N,k}$-geometry, N-particle relativistic Toda chain (“true” for $k=0$) or 5d SUSY $SU(N)$ pure gauge theory with CS-term at level k.

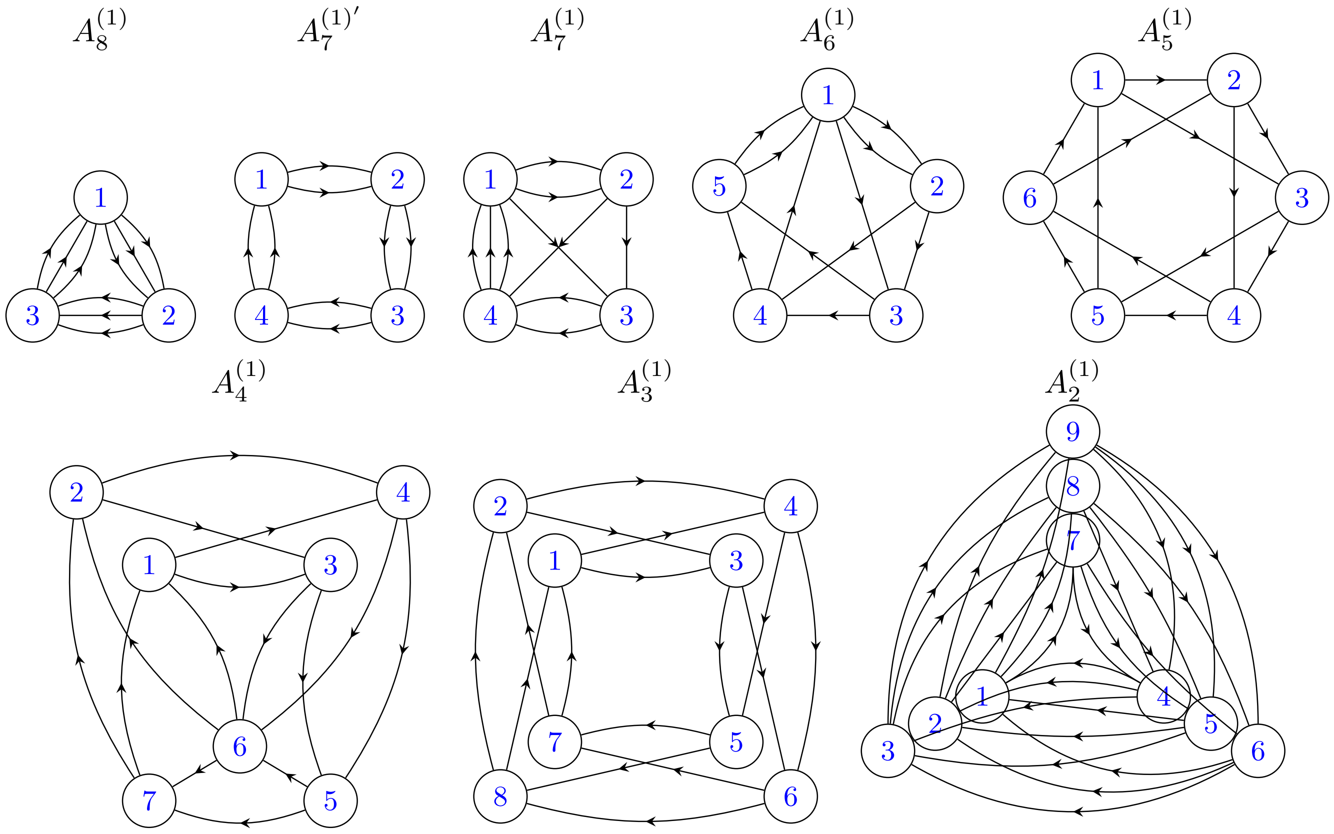
Exceptional case of $L^{1,2N-1,2}$-geometry.



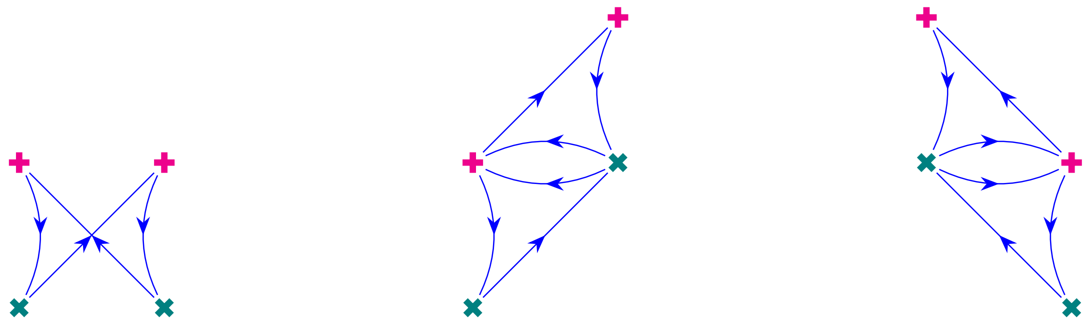
q-difference equations can be constructed in the same way.

The SW curve $f_{\Delta}(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0$ in Toda cases is hyperelliptic, the Krichever data $\frac{d\lambda}{\lambda} \wedge \frac{d\mu}{\mu}$.

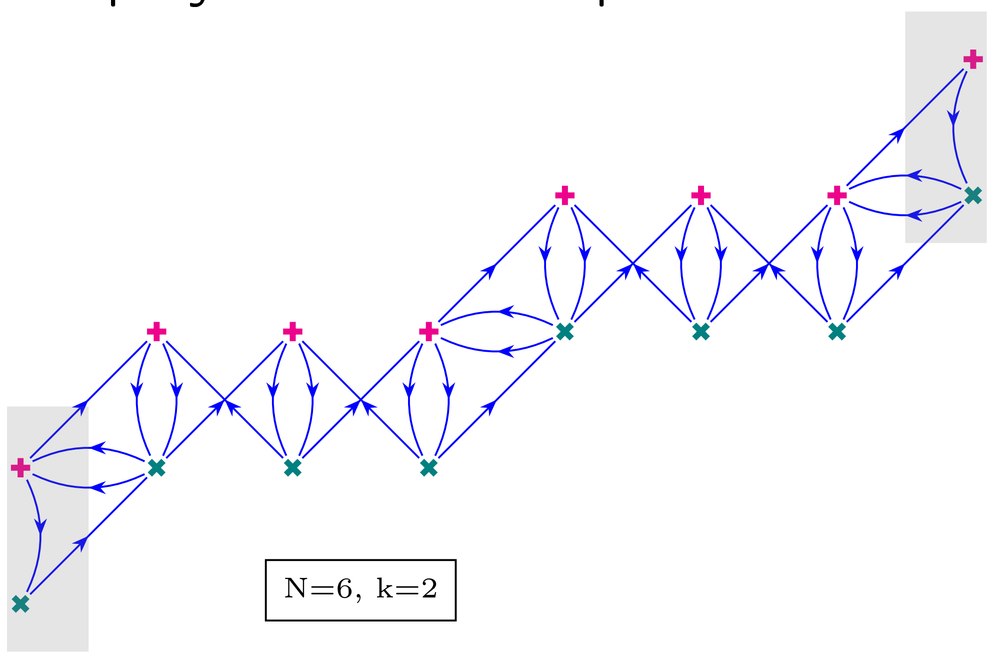
Quivers $\mathcal{Q}$ of the "Painlevé" cluster varieties (labeled their q-Painlevé names).



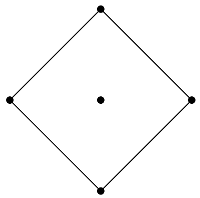
Building blocks for Toda quivers:

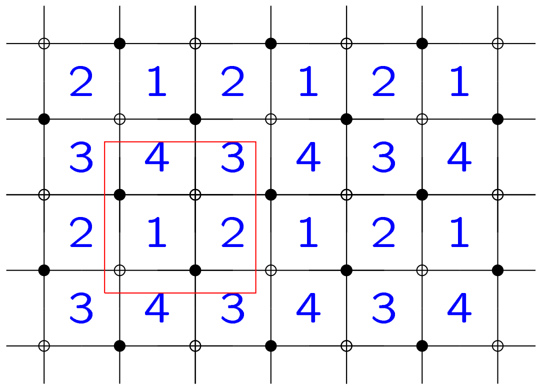


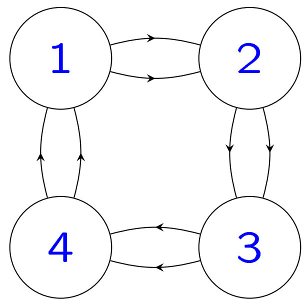
glued along the polyline "Motzkin paths"



$\bigcap$ better, than $\bigcup$: relativistic Toda (2-particle)





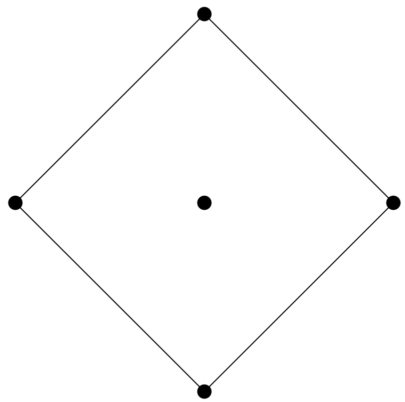


Here $q = x_1 x_2 x_3 x_4 (= 1)$ and $z = x_1 x_3$ are Casimir functions, if $y = x_1$, $x = x_2$, then $\{y, x\} = 2yx$. The Hamiltonian

$$H = \sqrt{yx} + \sqrt{\frac{y}{x}} + \frac{1}{\sqrt{yx}} + z\sqrt{\frac{x}{y}}$$

generates discrete (algebraic) flow: $(y, x) \mapsto (x\frac{(y+z)^2}{(y+1)^2}, y^{-1})$.

In detail (up to $SA(2,\mathbb{Z})$-tranform):

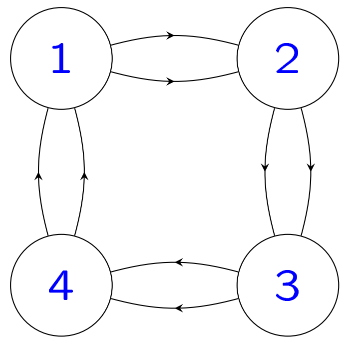


Newton polygon for the SW curve of 5d pure $SU(2)$ gauge theory:

$$f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = \lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} + u = 0 \qquad (1)$$

spectral curve for relativistic affine 2-particle Toda at $H = u$.

Realized on a cluster Poisson variety with the quiver:



just means that Poisson bracket is logarithmically constant

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad i, j = 1, \ldots, |\mathcal{Q}| \tag{2}$$

with the skew-symmetric matrix

$$\epsilon_{ij} = \#\text{arrows } (i \to j) = -\epsilon_{ji} \tag{3}$$

Obviously $q = x_1 x_2 x_3 x_4$ and $z = x_1 x_3$ are in the center of Poisson algebra.

Poisson maps include *mutations* of the graph:

$$\mu_k : \quad x_k \to \frac{1}{x_k}, \qquad x_i \to x_i \left(1 + x_k^{\mathrm{sgn}(\epsilon_{ik})}\right)^{\epsilon_{ik}}, i \neq k \tag{4}$$

Direct *quantization* of the cluster variety:

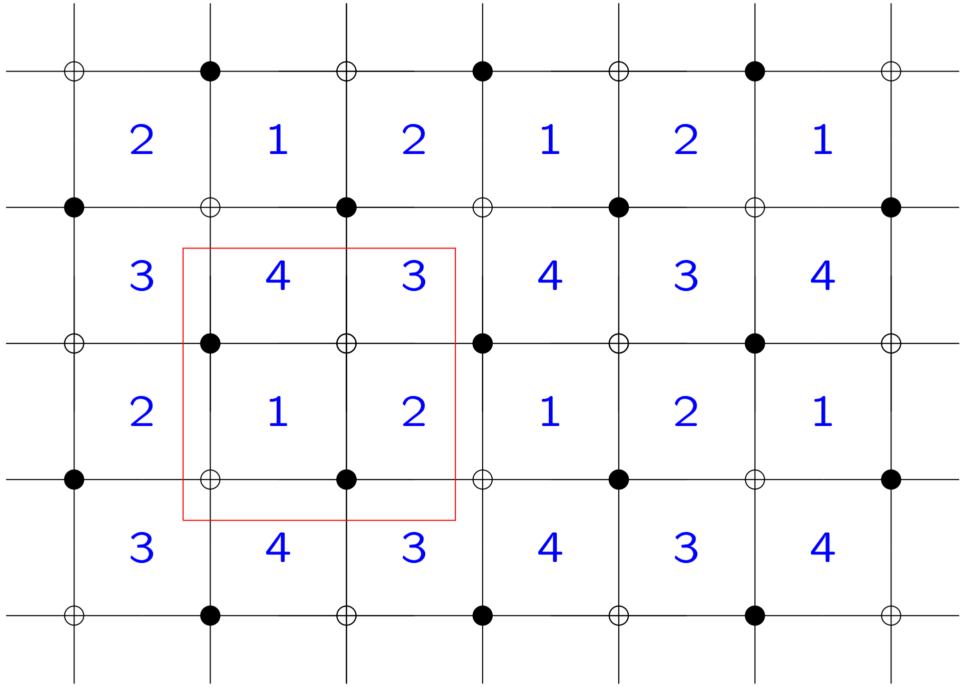
$$X_i X_j = p^{-2\epsilon_{ij}} X_j X_i, \quad i, j = 1, \ldots, |\mathcal{Q}| \tag{5}$$

with quantum mutations

$$\begin{aligned} X'_k &= X_k^{-1} \\ X_i'^{1/|\epsilon_{ik}|} &= X_i^{1/|\epsilon_{ik}|} \left(1 + p X_k^{\mathrm{sgn}\,\epsilon_{ik}}\right)^{\mathrm{sgn}\,\epsilon_{ik}} \end{aligned} \tag{6}$$

where $p = \exp(-i\hbar/2)$ is multiplicative quantum parameter (do not *mix* with q).

Finally, the dimer partition function on a bipartite graph



gives rise ... for $q = 1$ ... to an integrable system with a 5d SW spectral curve $Z_{\text{dimer}} \sim f_{\Delta} = \lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} + H(\vec{x})$.

Deautonomization $q \neq 1$:

discrete flow $T = (1,2)(3,4) \circ \mu_1 \circ \mu_3$ – a sequence of mutations in the opposite vertices of the quiver

$$(x_1, x_2, x_3, x_4) \mapsto \left(x_2 \frac{(x_3+1)^2}{(x_1^{-1}+1)^2}, x_1^{-1}, x_4 \frac{(x_1+1)^2}{(x_3^{-1}+1)^2}, x_3^{-1} \right) \tag{7}$$

or, for $q = x_1 x_2 x_3 x_4$, $z = x_2^{-1} x_4^{-1}$ and $F = x_1$, $G = x_2^{-1}$

$$T\colon (z, q, F, G) \mapsto \left(qz, q, \frac{(F+qz)^2}{(F+1)^2 G}, F \right). \tag{8}$$

Consider G, F as a functions of z such that $T : G \mapsto G(qz) = F(z)$, then

$$G(qz)G(q^{-1}z) = \frac{(G(z)+z)^2}{(G(z)+1)^2} \tag{9}$$

the second order q-difference equation (q-Painlevé equation of the type $A_7^{(1)'}$).

For tau-functions $G(z) = z^{1/2}\frac{\tau_3(z)^2}{\tau_1(z)^2}$: bilinear (non-autonomous Hirota) equations

$$\begin{aligned} \tau_1(qz)\tau_1(q^{-1}z) &= \tau_1(z)^2 + z^{1/2}\tau_3(z)^2 \\ \tau_3(qz)\tau_3(q^{-1}z) &= \tau_3(z)^2 + z^{1/2}\tau_1(z)^2 \end{aligned} \tag{10}$$

Generic equations for the (N,k)-theory

$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right)$$
$$j \in \mathbb{Z}/N\mathbb{Z} \tag{11}$$

are solved $\tau_j(z) = \tau_j^{N,k}(\vec{u},\vec{s};q|z)$ by the "Kiev-formula"

$$\tau_j^{N,k}(\vec{u},\vec{s};q|z) = \sum_{\vec{\Lambda}\in Q_{N-1}+\omega_j} s^{\Lambda} Z_{N,k}(\vec{u}q^{\vec{\Lambda}};q^{-1},q|z) \tag{12}$$

where the sum is over the A_{N-1} root lattice, $\{\omega_j\}$ are the fundamental weights, and 5d Nekrasov functions $Z_{N,k} = Z_{\text{cl}}^{N,k} \cdot Z_{1-\text{loop}}^{N} \cdot Z_{\text{inst}}^{N,k}$ are defined by (we use them here for $q_1q_2 = 1$)

$$
\begin{aligned}
Z_{\mathsf{cl}}^{N,k} &= \exp\left(\log z \frac{\sum (\log u_i)^2}{-2\log q_1 \log q_2} + k \frac{\sum (\log u_i)^3}{-6\log q_1 \log q_2}\right), \\
Z_{1-\mathsf{loop}}^{N} &= \prod_{1\le i\neq j\le N} (u_i/u_j; q_1, q_2)_\infty\,, \\
Z_{\mathsf{inst}}^{N,k} &= \sum_{\vec{\lambda}} \frac{z^{|\vec{\lambda}|}\prod_{i=1}^{N} \mathsf{T}_{\lambda^{(i)}}(u;q_1,q_2)^k}{\prod_{i,j=1}^{N} \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(u_i/u_j;q_1,q_2)}
\end{aligned}
\tag{13}
$$

with

$$
\mathsf{N}_{\lambda,\mu}(u,q_1,q_2) = \prod_{s\in\lambda}(1-uq_2^{-a_\mu(s)-1}q_1^{\ell_\lambda(s)})\prod_{s\in\mu}(1-uq_2^{a_\lambda(s)}q_1^{-\ell_\mu(s)-1})
$$

$$
\mathsf{T}_\lambda(u;q_1,q_2) = u^{|\lambda|}q_1^{\frac{1}{2}(\|\lambda^t\|-|\lambda^t|)}q_2^{\frac{1}{2}(\|\lambda\|-|\lambda|)} = \prod_{(i,j)\in\lambda} uq_1^{i-1}q_2^{j-1},
$$

and $\vec{\lambda} = (\lambda^{(1)},\dots,\lambda^{(N)})$, $|\vec{\lambda}| = \sum|\lambda^{(i)}|$, $|\lambda| = \sum\lambda_j$, $\|\lambda\| = \sum\lambda_j^2$.

Solutions:

- Given in terms of 5d Nekrasov functions for the $SU(N)$ theory with CS-term at level $|k| \leq N$;

- Depend on the vacuum condensates $u = e^{Ra}$, dual parameters s ($\sim e^{Ra_D}$) and $q = q_2 = q_1^{-1}$ for the parameters $\{q_i = e^{R\epsilon_i}\}$ of Ω-background (*non*-refined case);

- Substitution lead to bilinear equations for q-deformed conformal blocks, which resemble the blow-up equations;

- Turn at $q \to 1$ to the Θ-function solutions of autonomous Hirota equations.

Refined case $q_1 q_2 = p \neq 1$ corresponds to the *quantization* of cluster variety.

Quantum q-difference Painlevé equation

$$\begin{cases} G^{1/2}(q^{-1}z)\, G^{1/2}(qz) = \dfrac{G(z)+pz}{G(z)+p}, \\ G(z)G(q^{-1}z) = p^4 G(q^{-1}z)G(z) \end{cases} \tag{14}$$

now with two different (q and p!) parameters.

Instead of functions $G(z)$ are now elements of a non-commutative algebra, equation depends on the quantum parameter p.

The corresponding quantum tau-functions $G(z) = pz^{1/2}\mathcal{T}_1^2\mathcal{T}_3^{-2}$, $G(qz) = pq^{1/2}z^{1/2}\mathcal{T}_2^2\mathcal{T}_4^{-2}$ satisfy

$$\begin{aligned} \mathcal{T}_1(q^{-1}z)\mathcal{T}_1(qz) &= \mathcal{T}_1(z)^2 + p^2z^{1/2}\mathcal{T}_3(z)^2 \\ \mathcal{T}_3(q^{-1}z)\mathcal{T}_3(qz) &= \mathcal{T}_3(z)^2 + p^2z^{1/2}\mathcal{T}_1(z)^2, \end{aligned} \tag{15}$$

and are still given by Kiev formulas ($q_2 = q^{1/2}$, $q_1 = q_2^{-1}p^2$)

$$\begin{aligned} \mathcal{T}_1 &= a\sum_{m\in\mathbb{Z}} s^m Z(uq_2^{4m}|z), \quad & \mathcal{T}_2 &= ab\sum_{m\in\mathbb{Z}} s^m Z(uq_2^{4m}|q_2^2z), \\ \mathcal{T}_3 &= ia\sum_{m\in\frac{1}{2}+\mathbb{Z}} s^m Z(uq_2^{4m}|z), \quad & \mathcal{T}_4 &= iab\sum_{m\in\mathbb{Z}+\frac{1}{2}} s^m Z(uq_2^{4m}|q_2^2z). \end{aligned} \tag{16}$$

but with the *non-commutative* parameters

$$\begin{aligned} q_2^2a &= p^{-2}aq_2^2 \\ us = p^4su, &\quad zb = p^2bz \end{aligned} \tag{17}$$

Main conclusions:

- For 5d SUSY gauge theories the non-perturbative partition functions satisfy q-difference equations of the Painlevé type;

- These equations are generated by mutations of corresponding cluster varieties, whose quantization gives rise to refined topological strings.

Thank you!