Supersymmetric vortex defects in two dimensions

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Plan

Part I: Supersymmetric vortex defects [1705.10623 with K. Hosomichi and S. Lee]

Part II: SUSY renormalization (Pauli-Villars and counterterms) [1705.06118 TO]

Plan for Part I (vortex defects)

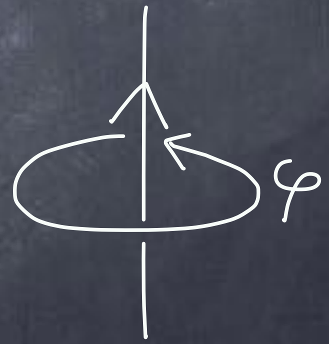
- Motivations and the set-up
- Three inequivalent definitions of defects
- Relations among definitions
- Applications
  - Twisted chiral ring relations
  - Mirror symmetry for minimal models

Motivations

Defects characterized by gauge field singularity

$$A \sim \eta d\varphi$$

- Surface operator in 4d theory
- Vortex line operator in 3d theory



- Vortex (local) operator in 2d theory

Motivations

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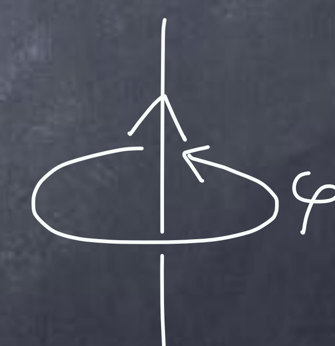


- Vortex (local) operator in 2d theory <= today

Motivations

- Sometimes, defects characterized by the **gauge field singularity** $A \sim \eta d\varphi$ are also described by the **insertion of local degrees of freedom**. (3d: Assel-Gomis,…, 4d: Gukov-Witten, Gaiotto, Nawata, …)
- What is the **mechanism** that guarantees the **equivalence of the two descriptions**?
- Will give an answer in the 2d abelian case.

More motivations

- Meaning of vortex defects in N=(2,2) GLSM for Calabi-Yau models.

- Holonomy for discrete symmetry

  ==> Twist field in orbifold theory

More motivations

- Mirror symmetry
  - Hori-Vafa mirror symmetry
  - Minimal model and its orbifold
  - Fundamental fields are mapped to defects
- Path integral description of the defects in these theories.

Set-up

- 2d N=(2,2) gauged linear sigma models.
- First focus on a single chiral multiplet coupled with charge +1 to U(1) gauge multiplet.
- Will embed to a larger theory, such as the quintic Calabi-Yau model.

- Chiral multiplet with charge +1 $\phi, \psi^{\pm}, F$
- U(1) gauge multiplet: dynamical or non-dynamical $A_\mu, \lambda^{\pm}, \bar{\lambda}^{\pm}, \Sigma, D$
- 1/2 BPS (twisted chiral) defect
- Invariant under type A supercharges
- A chiral multiplet decomposes into (ϕ, ψ^+) (ψ^-, F)
- Use SUSY as guidance to construct defects

Three inequivalent definitions of defects

1. Boundary conditions
2. Smearing regularization
3. 0d-2d couplings

Three inequivalent definitions of defects

1. Boundary conditions (~ [Drukker-TO-Passerini] in 3d)

2. Smearing regularization ([Kapustin-Willet-Yaakov] in 3d)

3. 0d-2d couplings (~ [Assel-Gomis] 3d)

Will derive relations among the definitions.

1: Defects via boundary conditions

There are two natural boundary conditions compatible with type A SUSY.

Normal boundary condition:

$$(\phi, \psi^+), D_z(\psi^-, F) : \text{finite}$$

$$(\psi^-, F) = \mathcal{O}(r^\gamma)\,, \quad -1 < \gamma \leq 0$$

Flipped boundary condition:

$$D_{\bar{z}}(\phi, \psi^+), (\psi^-, F) : \text{finite}$$

$$(\phi, \psi^+) = \mathcal{O}(r^\gamma)$$

1: Defects via boundary conditions

For multiple chiral multiplets, choose one boundary condition for each. The choice is a label of the defect.

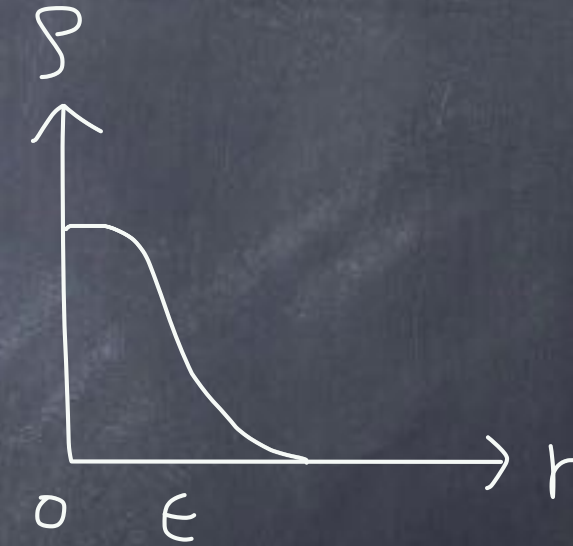
We can and did perform SUSY localization for the two-point function of defects on the sphere.

2: Defects via smearing

$$A \sim \eta d\varphi \qquad F_{12} \sim \eta \cdot \delta^2(x)$$

$$x^1 + ix^2 = re^{i\varphi}$$

- Regularize by a smooth function



$$F_{12} = \rho(x)$$

- Type A SUSY ==> D=2πi ρ

(3d: [Kapustin-Willet-Yaakov], 2d: TO)

3: Defects by 0d-2d couplings

0d SUSY with two super charges

= type A subalgebra of 2d N=(2,2) SUSY

≃ 2d N=(0,2) SUSY

Use terminology for N=(0,2)

3: Defects by 0d-2d couplings

0d Chiral multiplet (u, ζ)

$$S \sim \bar{u}\bar{\Sigma}\Sigma u + \bar{\zeta}\bar{\Sigma}\zeta$$

$$\int du d\zeta e^{-S} \sim \frac{1}{\Sigma}$$

0d Fermi multiplet (η, h)

$$S \sim \bar{\eta}\Sigma\eta + \bar{h}h$$

$$\int d\eta dh e^{-S} \sim \Sigma$$

Derivation of the relations among the definitions

Key points

- Start with the smearing definition. For some values of vorticity η, the 2d bulk fields develop **localized modes**.
- The localized modes form 0d multiplets.
- The non-localized modes obey normal/flipped boundary conditions.

Localized modes in smeared vortex background

- Recall SUSY condition D=2πi ρ . We get

$$S \sim \int \bar{\phi}(-D_z D_{\bar{z}} + \bar{\Sigma}\Sigma)\phi + \bar{\psi}\begin{pmatrix} \bar{\Sigma} & D_z \\ D_{\bar{z}} & \Sigma \end{pmatrix}\psi + \bar{F}F$$

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}.$$

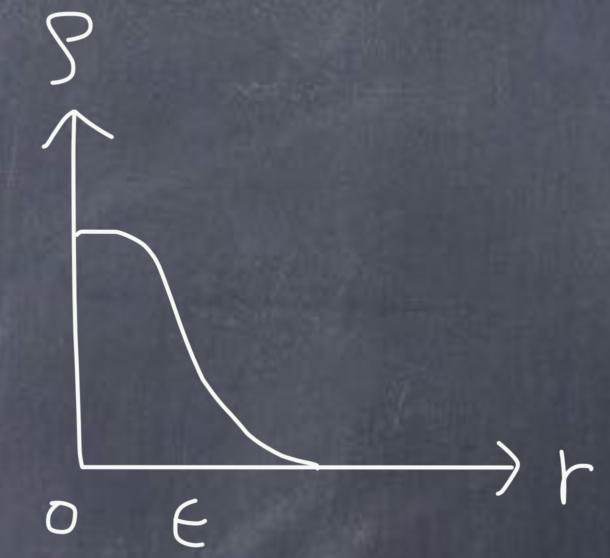
- Expand ϕ, ψ^+ in eigenmodes of $-D_z D_{\bar{z}}$. Zero-modes, if present, are annihilated by $D_{\bar{z}}$ and are localized.

- Expand ψ^-, F in eigenmodes of $-D_{\bar{z}} D_z$. Zero-modes, if present, are annihilated by D_z and are localized.

First order ODE for zero-mode

$$D_{\bar{z}}\Psi = 0 \text{ for } \Psi = \phi, \psi^+$$

$$\Psi = \hat{\Psi}(r)e^{im\varphi} \quad \hat{\Psi} \sim \begin{cases} r^m & \text{for} \quad r \ll \epsilon \\ r^{m-\eta} & \text{for} \quad r \gg \epsilon \end{cases}$$

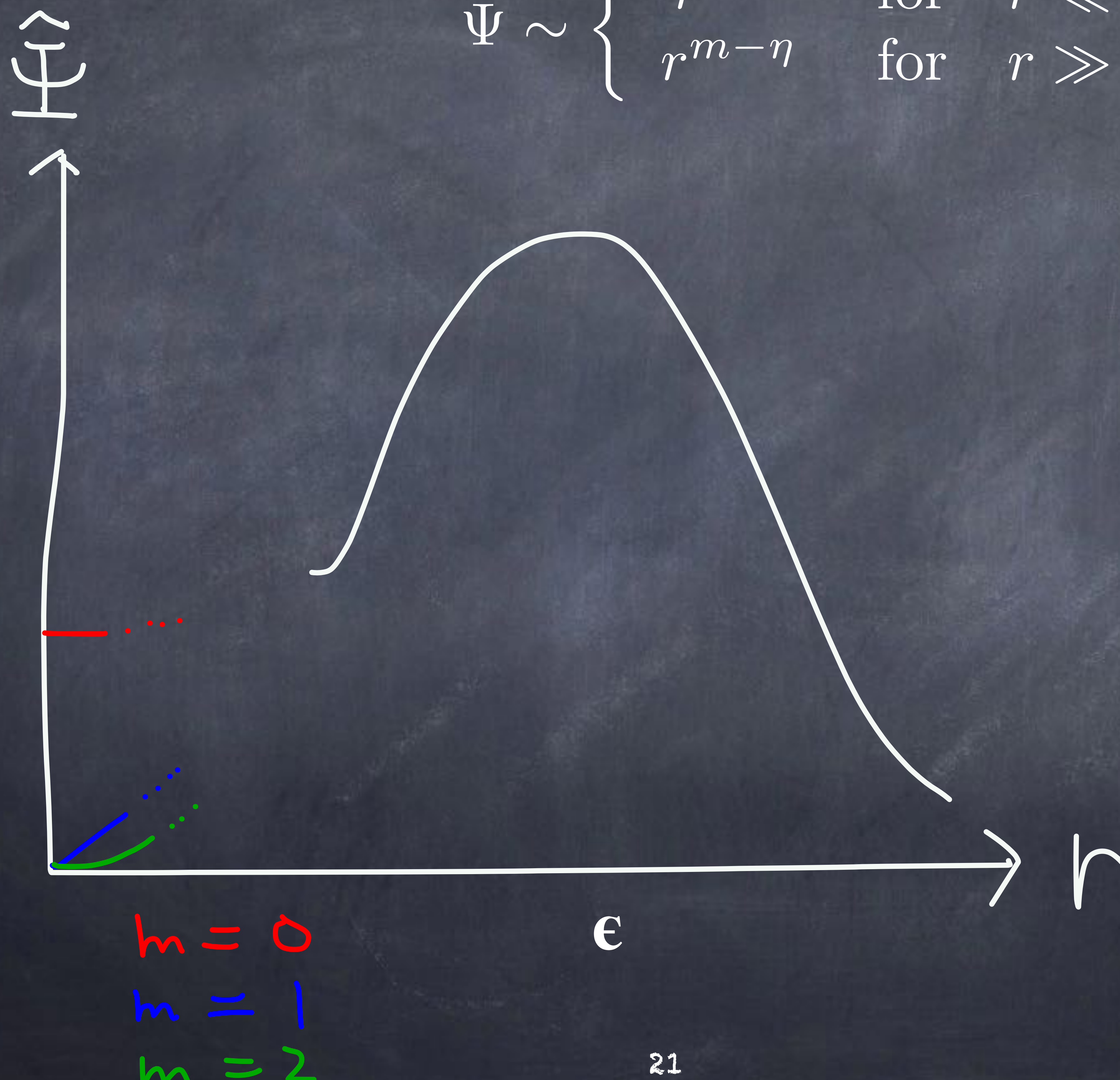


- Need m≧0 for regularity.
- Need m-η < -1 for the mode to be localized.

==> Localized modes exist for $m = 0, 1, \ldots, \lfloor \eta \rfloor - 1$

if η > 1. (Non-integer η assumed.)

$$\hat{\Psi} \sim \begin{cases} r^{-m} & \text{for} \quad r \ll \epsilon \\ r^{m-\eta} & \text{for} \quad r \gg \epsilon \end{cases}$$



Similar results for $D_z \Psi = 0\,, \quad \Psi = \psi^-, F$.

Effective boundary conditions for non-localized modes

- We performed the asymptotic analysis of the second-order ODEs as $\epsilon \to 0$.

$$-D_z D_{\bar{z}} \hat{\Psi} = \lambda \hat{\Psi} \text{ for } \hat{\Psi} = \phi, \psi^+$$

$$-D_{\bar{z}} D_z \hat{\Psi} = \lambda \hat{\Psi} \text{ for } \hat{\Psi} = \psi^-, F$$

- Non-localized modes in the bulk region behave as if they obey the normal/flipped boundary conditions.

Relations for the path integral measures

$$\mathcal{D}(\text{2d chiral})_{V_\eta^{\text{smeared}}}$$

$$= \begin{cases} \mathcal{D}(\text{2d chiral})_{V_\eta^{\text{flipped}}} \times \displaystyle\prod_{a=0}^{\lfloor \eta \rfloor -1} d(\text{0d chiral})_a & (\eta > 0) \\ \mathcal{D}(\text{2d chiral})_{V_\eta^{\text{normal}}} \times \displaystyle\prod_{\alpha=0}^{\lfloor -\eta \rfloor -1} d(\text{0d Fermi})_\alpha & (\eta < 0) \end{cases}$$

Vortex defect for gauge symmetry

- When the gauge field is dynamical, the smearing regularization gives a trivial defect because the gauge field is integrated over.
- Triviality of the smeared ``gauge vortex defect” implies the equivalence of a defect defined by boundary conditions and a defect defined by 0d-2d couplings.

Chiral ring relations and defects: CP^{N-1} model

- U(1) gauge multiplet and N chiral multiplets of charge +1.
- For $1<\eta<2$, from the relations between the measures,

$$1 = V_\eta^{\text{smeared}} = V_\eta^{\text{flipped}} \left(\int \mathcal{D}(\text{0d chiral}) e^{-S} \right)^N$$

- We can invert the 0d-2d coupling

$$V_\eta^{\text{flipped}} = \left(\int \mathcal{D}(\text{0d Fermi}) e^{-S} \right)^N = \Sigma^N$$

- For shifted vorticity,

$$V_{\eta-1}^{\text{flipped}} = 1$$

- The boundary conditions are invariant under an integer shift of η. Only the FI-theta coupling is affected. ==>

$$V_{\eta}^{\text{flipped}} = e^{-t} V_{\eta-1}^{\text{flipped}}$$

- Putting everything together, we get the chiral ring relation

$$\Sigma^N = e^{-t}$$

- On the sphere, a similar consideration leads to the Picard-Fuchs equation for the sphere partition function. [Closset-Cremonesi-Park, ...]

- From the Picard-Fuchs equation also one can read off the chiral ring relation by taking the large radius limit. [Givental]

- The same works for the quintic Calabi-Yau. Twisted chiral operators Σ^j can be realized as vortex defects V_η^{gauge} for suitable values of η.

Vortex defect for flavor symmetry

- When the gauge field is non-dynamical, the smearing regularization gives a non-trivial defect. [TO]
- Flavor vortex defect V_η^{flavor} realizes the twisted chiral operator $e^{\eta Y}$ in the Hori-Vafa mirror theory.
- For discrete symmetries, vortex defects are nothing but twist fields.

Application: N=2 Minimal model and its mirror

- Level h-2 minimal model with h=2,3,4,...

$$c = \frac{3(h-2)}{h}$$

- Its mirror is the Z_h orbifold of itself.

- N=2 Landau-Ginzburg model with superpotential

$$W = g_0 \Phi^h.$$

- Twist fields are vortex defects with vorticity

η =-p/h, p=0,1,...,h-1.

Two-point function of twist fields in the Z_h-orbifolded Minimal model

Two-point functions of twist fields can be computed by localization. Agree with known results and mirror symmetry expectations.

$$\left\langle V_{-p/h}(\mathrm{N}) V_{-p/h}(\mathrm{S}) \right\rangle_{S^2} = \frac{1}{h} \frac{\Gamma(\frac{1+p}{h})}{\Gamma(1-\frac{1+p}{h})}$$

$$= \frac{\Gamma(\frac{1+p}{h})^2}{h\pi} \sin\frac{(1+p)\pi}{h}$$

Explicit renormalization by Pauli-Villars and supergravity counterterms. [TO]

Coincides with the known and mirror results.

Summary for Part I

- Found a mechanism for the equivalence of the vortex defect defined by boundary condition and the defect defined by 0d-2d coupling.
- Gave a precise path-integral formulation of twist fields in Landau-Ginzburg realization of the minimal model.
- (In the paper) gave prescriptions for computing two-point functions of vortex defects.

Future directions for Part I

- More detailed study of the non-Abelian case.
- Higher dimensions: vortex lines, surface operators.
- Brane construction, chiral ring relations from branes? ([Assel] in 3d)
- Relation to the Higgsing construction of a surface operator [Gaiotto-Rastelli-Razamat]

Part II

How does renormalization **actually** work in a supersymmetric theory?

Will see an explicit example in 2d N=(2,2) theory

For amusement/obsession

Plan for Part II (SUSY renormalization)

- Pauli-Villars regularization in 2d N=(2,2) theory
- Supergravity counterterms
- Renormalization

SUSY Pauli-Villars

- Goal: regularize the one-loop determinant for a single physical chiral multiplet.
- Add $2N_{PV}-1$ **ghost/regulator** chiral multiplets.
- Introduce fictitious symmetry $U(1)_{PV}$

	$J=0$	$J=j\in\{1,\ldots,2N_{\rm PV}-1\}$
	physical	unphysical (PV ghosts)
statistics	$\epsilon_0=+1$	$\epsilon_j=\pm 1$
$U(1)_{\rm PV}$-charge	$a_0=0$	$a_j\in\mathbb{R}-\{0\}$
flavor/gauge charge	$b_0=+1$	$b_j\in\mathbb{Z}$
twisted mass	σ	twisted mass $a_j\Lambda+b_j\sigma$
vector R-charge	$q_0=q$	R-charge c_jq

$c_0 = 1$

Linear constraints

- Often in localization literature, the one-loop determinant is given as an infinite product after bose/fermi cancellation.
- In this case, the following linear constraints are enough for UV regularization.

$$\sum_J \epsilon_J = \sum_J \epsilon_J a_J = \sum_J \epsilon_J b_J = \sum_J \epsilon_J c_J = 0$$

Quadratic constraints

- It is possible to UV regularize the bosonic and fermionic determinants separately, by imposing quadratic constraints.

$$\sum_J \epsilon_J a_J^2 = \sum_J \epsilon_J b_J^2 = \sum_J \epsilon_J c_J^2 = \sum_J \epsilon_J a_J b_J = \sum_J \epsilon_J b_J c_J = \sum_J \epsilon_J c_J a_J = 0$$

- Can be seen by explicit enumeration of eigenvalues or the heat kernel analysis.

An example that satisfies the linear and quadratic constraints:

$$
\begin{aligned}
&N_{\mathrm{PV}} = 3\,, \\
&(\epsilon_1, \ldots, \epsilon_5) = (+1, +1, -1, -1, -1)\,, \\
&b_j = c_j = 1 \text{ for all } j, \\
&(a_1, \ldots, a_5) = (3, 3, 1, 1, 4)
\end{aligned}
$$

Combinations of parameters

$$C_0 := \prod_j |a_j|^{-\epsilon_j}\,, \qquad \Xi_1 := \sum_j \epsilon_j b_j \mathrm{sgn}(a_j)\,,$$

$$C_1 := \sum_j \epsilon_j b_j \log|a_j|\,, \qquad \Xi_2 := \sum_j \epsilon_j |a_j|\,,$$

$$C_2 := \sum_j \epsilon_j a_j \log|a_j|\,, \qquad \Xi_3 := \sum_j \epsilon_j c_j \mathrm{sgn}(a_j)\,,$$

$$C_3 := \sum_j \epsilon_j c_j \log|a_j|\,, \qquad \Xi_4 := \sum_j \epsilon_j \mathrm{sgn}(a_j)\,.$$

Pauli-Villars regularization for SUSY two-sphere

- The usual expression for the 1-loop determinant is

$$Z^{\text{SUSY}}_{\text{1-loop}} \text{ ``$=$'' } \prod_{n=0}^{\infty} \frac{n+1+\frac{1}{2}|B|-\hat{\sigma}}{n+\frac{1}{2}|B|+\hat{\sigma}}$$

$$\left(\hat{\sigma} = i\ell\sigma_1 + \frac{q}{2}\right)$$

- By Pauli-Villars we get

$$Z^{\text{SUSY}}_{\text{1-loop, reg}} = \prod_{n=0}^{\infty} \left[\frac{n+1+\frac{1}{2}|B|-\hat{\sigma}}{n+\frac{1}{2}|B|+\hat{\sigma}} \prod_j \left(\frac{n+1+\frac{1}{2}|b_j B| - M_j}{n+\frac{1}{2}|b_j B| + M_j} \right)^{\epsilon_j} \right]$$

$$M_j \equiv c_j \frac{q}{2} + i\ell(a_j \Lambda + b_j \text{Re}(\sigma))$$

- Gamma function identities allow us to remove the absolute value symbols without changing the result.

- Stirling's formula gives, for large Λ>0,

$$Z^{\rm SUSY}_{\text{1-loop, reg}} = \frac{\Gamma(\hat\sigma + \frac{B}{2})}{\Gamma(1-\hat\sigma+\frac{B}{2})}\prod_j\left(\frac{\Gamma(M_j + b_j\frac{B}{2})}{\Gamma(1-M_j+b_j\frac{B}{2})}\right)^{\epsilon_j}$$

$$= C_0 e^{i\frac{\pi}{2}\Xi_1 B} e^{(C_3-C_1)q} e^{2C_1\hat\sigma} e^{2iC_2\ell\Lambda} (\ell\Lambda)^{1-2\hat\sigma}\,\frac{\Gamma(\hat\sigma+\frac{B}{2})}{\Gamma(1-\hat\sigma+\frac{B}{2})}\left(1+\mathcal{O}(\Lambda^{-1})\right) ,$$

- This is regularized by not renormalized, because of the Λ-dependence. Also we need to deal with the ugly prefactors...

Supergravity counterterms

- Claim: the counterterms given by the following twisted superpotential renormalize the one-loop partition functions in arbitrary backgrounds. (μ: renormalization scale)

$$\widetilde{W}_{\rm ct}(\sigma,\widehat{\mathcal{H}},\Lambda) = -\frac{\widehat{\mathcal{H}}}{8\pi}\sum_j \epsilon_j \log\frac{ia_j\Lambda}{\mu} + \frac{1}{4\pi}\sum_j \epsilon_j\left(a_j\Lambda + b_j\sigma + \frac{c_j q}{2}\widehat{\mathcal{H}}\right)\log\frac{a_j\Lambda + b_j\sigma + \frac{c_j q}{2}\widehat{\mathcal{H}}}{\mu\, e}\,.$$

- $\widehat{\mathcal{H}}$:twisted chiral field constructed from the gravity multiplet/R-symmetry gauge multiplet in N=(2,2) $U(1)_V$ SUGRA.

- Similar to Witten's effective twisted superpotential.

$$\frac{\hat{c}_{\rm UV}}{8\pi}\int d^2x\sqrt{g}\,R\,\log\frac{\Lambda}{\mu}$$

For large Λ>0,

Renormalization of FI-theta terms for flavor/gauge symmetry

$$\widetilde{W}_{\rm ct}(\sigma,\widehat{\mathcal{H}},\Lambda)=\frac{1-q}{8\pi}\widehat{\mathcal{H}}\log\frac{\Lambda}{\mu}+\frac{1}{4\pi}\left(C_1-\log\frac{\Lambda}{\mu}+i\frac{\pi}{2}\Xi_1\right)\sigma$$
$$+\frac{1}{4\pi}\left(C_2+i\frac{\pi}{2}\Xi_2\right)\Lambda+\frac{q}{4\pi}\left(C_3+i\frac{\pi}{2}\Xi_3\right)\frac{\widehat{\mathcal{H}}}{2}$$
$$+\frac{1}{4\pi}\left(\log C_0-i\frac{\pi}{2}\Xi_4\right)\frac{\widehat{\mathcal{H}}}{2}+\mathcal{O}(\Lambda^{-1})\,.$$

Renormalization of FI-theta terms for $U(1)_{PV}$

Renormalization of FI-theta terms for vector R-symmetry

Renormalization of FI-theta terms for flavor/gauge symmetry

$$\log\frac{\Lambda}{\mu} - C_1 + r(\mu) = r_0(\Lambda)\,, \qquad \theta + \frac{\pi}{2}\Xi_1 = \theta_0$$

$$t = r - i\theta$$

Renomarlization for SUSY two-sphere

Combining the physical action, Pauli-Villars regularization, and supergravity counterterms, we get

$$
\begin{aligned}
Z^{\rm SUSY} &= \lim_{\Lambda\to\infty} e^{-S_{\rm ren}-S_{\rm ct}}\, Z^{\rm SUSY}_{\text{1-loop, reg}} \\
&= e^{-S_{\rm ren}}(\ell\mu)^{1-2\hat\sigma}\,\frac{\Gamma(\hat\sigma+\frac{B}{2})}{\Gamma(1-\hat\sigma+\frac{B}{2})} \\
&= e^{-iB\theta} e^{4\pi i[r(\mu)-\frac{1}{2\pi}\log(\ell\mu)]\ell{\rm Re}\,\sigma}(\ell\mu)^{1-q}\,\frac{\Gamma(\hat\sigma+\frac{B}{2})}{\Gamma(1-\hat\sigma+\frac{B}{2})}\,.
\end{aligned}
$$

- A convenience choice is to take $\mu = 1/\ell$. Then

$$Z^{\rm SUSY} = e^{4\pi i r\sigma} e^{-iB\theta} \frac{\Gamma(\hat{\sigma} + \frac{B}{2})}{\Gamma(1 - \hat{\sigma} + \frac{B}{2})}$$

- This is the formula often quoted in the literature.

Comments

- Zeta function regularization is equivalent to a specialization (limit) of parameters.

- Our scheme works uniformly for different SUSY backgrounds, such as A-twist with/without omega deformation on two-sphere. (See paper.) It is meaningful to compare partition functions in different backgrounds.

Comments

- For vortex defects, we can read off the scaling dimension from μ or l dependence.
- With boundary, we also need boundary counterterms. One has to choose different counterterms depending on which the symmetry (gauge or charge conjugation symmetry) to preserve (unpublished).

Thank you!