TBA
(and beyond: Amplitudes/WLs, N=2 partition functions)

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Yassen ad memoriam

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series of paper with M. Rossi, S.Piscaglia, A. Bonini;JE Bourgine

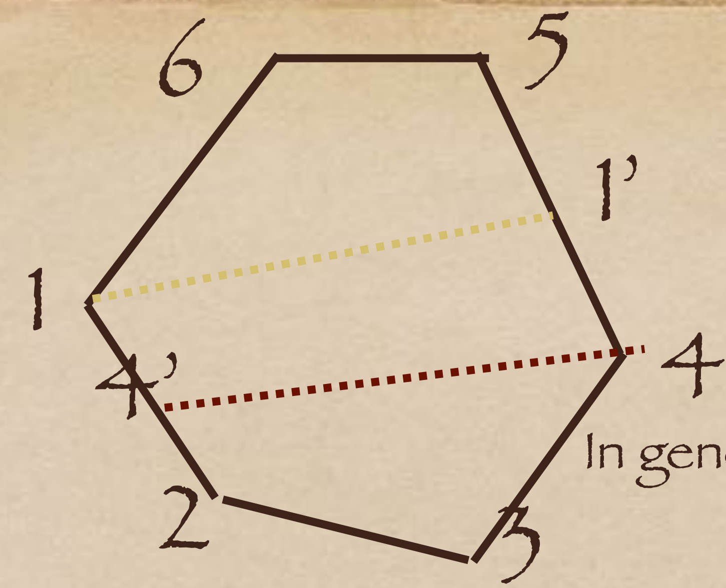
- Duality: null polygonal WL= gluon scattering amplitudes. Inspired by the common dual string (Alday-Maldacena) which describes also local sector of gauge theory (WL non-local) (Drummond,Korchemsky,Sokatchev,...)
- An itegrability perspective. Benefit for exchange of ideas between these fields!
- Sketch of a <u>PLAN</u> in integrabile words :
- <u>Form Factor (FF) Series for null polygonal WLs</u>;
- (to gain the states) Nested Bethe Ansatz;
- (to sum the FF series) <u>Thermodynamic Bethe Ansatz</u> (string theory);
- (beyond classical string theory) FFs again: scalar <u>additional contribution; fermions revised towards 1-loop</u> (bit more technical explanations);
- Parallel with N=2 partition function and beyond the NS limit

OPE for polygonal WLs

- Theory: N=4 SYM in planar limit $\lambda = N_c g^2_{YM},\ N_c \to \infty$
- Dual to quantum area of II B string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$
- Light-like polygons can be decomposed into light-like Pentagons (and Squares): an OPE (Alday, Maldacena, Basso, Sever, Vieira)
- Prototype: Hexagon into <u>two</u> Pentagons P
- The same as <u>two</u>-point correlation function <PP> into FFs
- But WL non-local: local method, i.e. insertion of identity

- In a picture:

hexagon



$=P(12341')\ P(14'456)$

In general: E-5 shared squares, E-4 pentagons

- Which mathematically means:

  - $W=\sum \exp(-rE)\langle 0|P|n\rangle\langle n|P|0\rangle$

Multi-P correlation function:general m,n transition

- =<PP>: the same as <u>2D Form Factor (FF) decomposition</u>

- FFs obey axioms with the S-matrix (Karowski,Weisz,Al. Zam,Smirnov,...): 1) Watson eqs., 2) Monodromy (q-KZ), etc.

- Eigen-states |n>? 2D excitations over the GKP folded string (of length=2 ln s+....) which stretches from the boundary to boundary (for large s).

- The quantum GKP string can be represented by the quantum spin chain vacuum (gauge, Korchemsky et al.)

$$\Omega_{GKP} = Tr\, Z D_+^s Z + \dots$$

- 2D particles: 6 scalars, 2 gluons, 4+4 (anti)fermions Bethe states (Basso):

$$\mathcal{O}_{1-particle} = Tr\, Z D_+^{s-s'} \varphi D_+^{s'} Z + \dots$$

$$\varphi = Z, W, X, F_{+\perp}, \bar{F}_{+\perp}, \Psi_+, \bar{\Psi}_+$$

Correspondence(s) and Integrability

- String/Gauge duality: N=4 Super Yang-Mills $SU(N_c)$ equivalent to II B string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$g_{YM}^2 N_c = \lambda \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'} = \frac{1}{g_{ws}^2} \qquad g_s = g_{YM}^2 \sim \frac{1}{N_c}$$

- Last equality: miracle of MULTICOLOUR

$$\lambda = N_c g_{YM}^2,\ N_c \to \infty$$

- free string (sigma model) with Planck constant

An important example: the spectrum

- Dimensions of gauge operators=Energy of the quantum string

$$DO = \Delta O \quad i.e. \quad < O(x)O(0) >= |x|^{-2\Delta}$$

- A particular string configuration shall correspond to the gauge operator (for any operator). D is the string Hamiltonian

- Multicolor: correspondence with INTEGRABLE SYSTEM, better Bethe-Yang (asympotic=large size) Beisert-Staudacher Eqs

- Important Excursus: Exact Equations form TBA Bombardelli,DF,Tateo Gromov,Kazakov,Vieira Arutyunov, Frolov.....

- $H = (\gamma_{ab})$ is a non-trivial information, namely the renormalisation of the fields

$$O_a^{bare} = \sum_b Z_{ab} O_b^{ren} \qquad \gamma_{ab} = \frac{d}{d \ln \mu} Z_{ab}$$

- No microscopic model, but, for large size (quantum numbers) Asymptotic Bethe Ansatz: 1,2,3,4,5,6,7 eqs, symmetric w.r.t. the central node 4 (seven rapidities: u1,u2,u3,u4,u5,u6,u7).

TWIST OPERATORS

- Idea: fill in the eqs. only with s u4 rapidities: covar. Deriv.
- Then s will become very large: Fermi sea, ANTI-FERROMAGNETIC vacuum
- But consistency imposes TWO HOLES in this Fermi sea.
- No novelty: the same as sl(2) spin chain, e.g. studied for QCD at one loop (Belitsky,Korchemsky,Manashov,...).
- In fact N=4 SYM at one loop gauge is almost the same.

A TRIALITY

- Gauge/String/Integrable Systems:

$$\Omega_{GKP} = Tr\, Z D_+^s Z + \dots$$

- Fast (folded) spinning string in AdS (angular momentum s): Gubser-Klebanov-Polyakov. Folded string simulates an open string which ends on AdS with the two scalars Z.
- ABA solution with s u4 and two holes Z: Fermi sea or GKP vacuum.
- Understood:we started from zero roots=BMN vacuum=
- $$TrZ^L$$

Triality: twist operators

- Gauge/String/Integrable System
- Scalar (QCD:quarks) twist operators (not only at the ends)

$$tr D_+^s Z^L + \ldots$$

- Fast spinning (s on AdS) and rotating (L on S5) (folded) string
- Two large u4-holes, L-2 small u4-holes.

A quick excursus on QCD

- Motivation for N=4 as laboratory: twist operators were born in QCD (with quarks)
- Large spin behaviour

$$\gamma(g, s, L) = f(g) \ln s + f_{sl}(g, L) + O\left(\frac{1}{\ln s}\right)$$

- gives (the cusp (Polyakov)) = f/2 (light-like WL (Korchemsky)) and the virtual scaling function (WL and amplitudes)
- Highest transcendental part is N=4
- Reciprocity is the same property:
- 1) parity: $\tilde{P}(s) = f(C^2), \;\; C^2 = \left(s + \frac{L}{2} - 1\right)\left(s + \frac{L}{2}\right)$ 2) self-tuning: $\gamma(g, L, s) = \tilde{P}\left(s + \frac{1}{2}\gamma(g, L, s)\right)$
- Of course, N=4 conformal: no mass scale, no asymptotic freedom, no confinement

All the Excitations

- 7 (class of) Bethe-Yang Equations (Beisert-Staudacher's) describe the states over the ferromagnetic (half-BPS) state of L fixed spins TrZ^L.
- Now, we find the gauge excitations over the sea of u4 Bethe roots=antiferromagnetic state= $\Omega_{GKP} = Tr\, ZD_+^s Z + \ldots$
- SCALARS are HOLES as in the non compact sl(2) spin (-1/2) chain (inversion of the l.h.s. w.r.t. the spin=1/2)
- We convert the equations into non-linear integral equations by Cauchy circulating the u4 roots (DF, Rossi).

- GLUONS: two polarisations F, $\bar{F}$ correspond to stacks of roots

$$u_{2,j} = u_j^g \,, \quad u_{3,j} = u_j^g \pm i/2 \,, \quad j = 1, ..., N_g$$

- and respectively (2—>6, 3—>5)

$$u_{6,j} = u_j^{\bar{g}} \,, \quad u_{5,j} = u_j^{\bar{g}} \pm i/2 \,, \quad j = 1, ..., N_{\bar{g}}$$

- They are isospin (SU(4)) SINGLETS.

- FERMIONS (Gauginos) :they leave on the two sheets of the Zukowsky map:

$$x(u) = \frac{u}{2}\left[1 + \sqrt{1 - \frac{2g^2}{u^2}}\right] \qquad u^2 \geq 2g^2$$

- small rapidity

$$x_f(u_1) = \frac{u_1}{2}\left[1 - \sqrt{1 - \frac{2g^2}{u_1^2}}\right] \qquad |x_f| \leq g/\sqrt{2}$$

- large rapidity

$$x_F(u_3) = \frac{u_3}{2}\left[1 + \sqrt{1 - \frac{2g^2}{u_3^2}}\right] \qquad |x_F| \geq g/\sqrt{2}$$

- Anti-fermions: 1—>7, 3—>5 (upper half into lower half):

- small

$$x_f(u_7) = \frac{u_7}{2}\left[1 - \sqrt{1 - \frac{2g^2}{u_7^2}}\right]$$

- large

$$x_F(u_5) = \frac{u_5}{2}\left[1 + \sqrt{1 - \frac{2g^2}{u_5^2}}\right]$$

- Isotopic or nesting structure of GKP Bethe Ansatz: Ka roots (linked to fermions)

$$u_{2,j} = u_{a,j}, \quad j = 1, ..., K_a$$

- Kc roots (linked to antifermions: 2->6)

$$u_{6,j} = u_{c,j}, \quad j = 1, ..., K_c$$

- Kb stacks (linked to scalars)

$$u_{b,j} = u_{3,j} = u_{5,j}, \quad u_{4,j,\pm} = u_{b,j} \pm \frac{i}{2} \quad j = 1, ..., K_b$$

- Fermions: 4 representation (fundamental)

$$\prod_{j=1}^{N_F}\left(\frac{u_{a,k}-u_{F,j}+\frac{i}{2}}{u_{a,k}-u_{F,j}-\frac{i}{2}}\right)=\prod_{j\neq k}^{K_a}\frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i}\prod_{j=1}^{K_b}\frac{u_{a,k}-u_{b,j}-\frac{i}{2}}{u_{a,k}-u_{b,j}+\frac{i}{2}}$$

$$1=\prod_{j=1}^{K_b}\frac{u_{b,k}-u_{b,j}+i}{u_{b,k}-u_{b,j}-i}\prod_{j=1}^{K_a}\frac{u_{b,k}-u_{a,j}-\frac{i}{2}}{u_{b,k}-u_{a,j}+\frac{i}{2}}\prod_{j=1}^{K_c}\frac{u_{b,k}-u_{c,j}-\frac{i}{2}}{u_{b,k}-u_{c,j}+\frac{i}{2}}$$

$$1=\prod_{j\neq k}^{K_c}\frac{u_{c,k}-u_{c,j}+i}{u_{c,k}-u_{c,j}-i}\prod_{j=1}^{K_b}\frac{u_{c,k}-u_{b,j}-\frac{i}{2}}{u_{c,k}-u_{b,j}+\frac{i}{2}}$$

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- Anti-Fermions: bar 4 representation (anti-fund.)

$$1 = \prod_{j\neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}}$$

$$\prod_{j=1}^{N_{\bar{F}}} \left(\frac{u_{c,k} - u_{\bar{F},j} + \frac{i}{2}}{u_{c,k} - u_{\bar{F},j} - \frac{i}{2}} \right) = \prod_{j\neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}}$$

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- Scalars: 6 representation (vector)

$$1 = \prod_{j\neq k}^{K_a} \frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i} \prod_{j=1}^{K_b} \frac{u_{a,k}-u_{b,j}-\frac{i}{2}}{u_{a,k}-u_{b,j}+\frac{i}{2}}$$

$$\prod_{h=2}^{L-1}\left(\frac{u_{b,k}-u_h+\frac{i}{2}}{u_{b,k}-u_h-\frac{i}{2}}\right) = \prod_{j=1}^{K_b} \frac{u_{b,k}-u_{b,j}+i}{u_{b,k}-u_{b,j}-i} \prod_{j=1}^{K_a} \frac{u_{b,k}-u_{a,j}-\frac{i}{2}}{u_{b,k}-u_{a,j}+\frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k}-u_{c,j}-\frac{i}{2}}{u_{b,k}-u_{c,j}+\frac{i}{2}}$$

$$1 = \prod_{j\neq k}^{K_c} \frac{u_{c,k}-u_{c,j}+i}{u_{c,k}-u_{c,j}-i} \prod_{j=1}^{K_b} \frac{u_{c,k}-u_{b,j}-\frac{i}{2}}{u_{c,k}-u_{b,j}+\frac{i}{2}}$$

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- We derive isotopic part of the SU(4) spin chain

$$\prod_{p=1}^{N_p}\left(\frac{u_{a,k}-u_p+i\vec{\alpha}_1\cdot\vec{w}_R}{u_{a,k}-u_p-i\vec{\alpha}_1\cdot\vec{w}_R}\right)^N = \prod_{j\neq k}^{K_a}\frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i}\prod_{j=1}^{K_b}\frac{u_{a,k}-u_{b,j}-i/2}{u_{a,k}-u_{b,j}+i/2}$$

$$\prod_{p=1}^{N_p}\left(\frac{u_{b,k}-u_p+i\vec{\alpha}_2\cdot\vec{w}_R}{u_{b,k}-u_p-i\vec{\alpha}_2\cdot\vec{w}_R}\right)^N = \prod_{j\neq k}^{K_b}\frac{u_{b,k}-u_{b,j}+i}{u_{b,k}-u_{b,j}-i}\prod_{j=1}^{K_a}\frac{u_{b,k}-u_{a,j}-i/2}{u_{b,k}-u_{a,j}+i/2}\prod_{j=1}^{K_c}\frac{u_{b,k}-u_{c,j}-i/2}{u_{b,k}-u_{c,j}+i/2}$$

$$\prod_{p=1}^{N_p}\left(\frac{u_{c,k}-u_p+i\vec{\alpha}_3\cdot\vec{w}_R}{u_{c,k}-u_p-i\vec{\alpha}_3\cdot\vec{w}_R}\right)^N = \prod_{j\neq k}^{K_c}\frac{u_{c,k}-u_{c,j}+i}{u_{c,k}-u_{c,j}-i}\prod_{j=1}^{K_b}\frac{u_{c,k}-u_{b,j}-i/2}{u_{c,k}-u_{b,j}+i/2}$$

- with the h.w. w=(1,0,0), (0,1,0), (0,0,1) respectively in the three case. Gluons are singlets.

- The physical rapidity enter as inhomogeneities, as should be.

Scattering of Physical Particles

- Scalars

$$1 = e^{iRP^{(s)}(u_h)+2iD^{(s)}(u_h)} \prod_{j=1}^{K_b} \frac{u_h - u_{b,j} + \frac{i}{2}}{u_h - u_{b,j} - \frac{i}{2}} \prod_{\substack{h'=1 \\ h' \neq h}}^{H} S^{(ss)}(u_h, u_{h'}) \prod_{j=1}^{N_g} S^{(sg)}(u_h, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(s\bar{g})}(u_h, u_j^{\bar{g}}) \cdot$$

$$\cdot \prod_{j=1}^{N_F} S^{(sF)}(u_h, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(s\bar{F})}(u_h, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(sf)}(u_h, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(s\bar{f})}(u_h, u_{\bar{f},j})$$

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- Fermions

$$1 = e^{iRP^{(F)}(u_{F,k})+2iD^{(F)}(u_{F,k})} \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + i/2}{u_{F,k} - u_{a,j} - i/2} \prod_{j=1}^{N_F} S^{(FF)}(u_{F,k}, u_{F,j})(\dots)$$

- Anti-fermions

$$1 = e^{iRP^{(F)}(u_{\bar{F},k})+2iD^{(F)}(u_{\bar{F},k})} \prod_{j=1}^{K_c} \frac{u_{\bar{F},k} - u_{c,j} + i/2}{u_{\bar{F},k} - u_{c,j} - i/2} \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{F}\bar{F})}(u_{\bar{F},k}, u_{\bar{F},j})(\dots)$$

- Also: other sheet F—>f small fermions (two-sheet Riemann surface)

- Gluons

$$1 = e^{iRP^{(g)}(u_k^g)+2iD^{(g)}(u_k^g)} \prod_{j=1,j\neq k}^{N_g} S^{(gg)}(u_k^g, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(g\bar{g})}(u_k^g, u_j^{\bar{g}}) \prod_{h=1}^{H} S^{(gs)}(u_k^g, u_h) \cdot$$

$$\cdot \prod_{j=1}^{N_F} S^{(gF)}(u_k^g, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(g\bar{F})}(u_k^g, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(gf)}(u_k^g, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(g\bar{f})}(u_k^g, u_{\bar{f},j})$$

$$1 = e^{iRP^{(g)}(u_k^{\bar{g}})+2iD^{(g)}(u_k^{\bar{g}})} \prod_{j=1}^{N_g} S^{(\bar{g}g)}(u_k^{\bar{g}}, u_j^g) \prod_{j=1,j\neq k}^{N_{\bar{g}}} S^{(\bar{g}\bar{g})}(u_k^{\bar{g}}, u_j^{\bar{g}}) \prod_{h=1}^{H} S^{(\bar{g}s)}(u_k^{\bar{g}}, u_h) \cdot$$

$$\cdot \prod_{j=1}^{N_F} S^{(\bar{g}F)}(u_k^{\bar{g}}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{g}\bar{F})}(u_k^{\bar{g}}, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(\bar{g}f)}(u_k^{\bar{g}}, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{g}\bar{f})}(u_k^{\bar{g}}, u_{\bar{f},j})$$

- Gluons form bound states as well

Interpretation

- SU(4) spin chain with representations 6, 4, 1
- GKP vacuum breaks SUSY except R-symmetry: good news?
- Particles: bosons:6+1+1=8, fermions: 4+4=8
- 16 ‘spinons’ while we started from 16 magnons (BS eqs.)
- We read off the momentum (and the energy)
- Dynamically generated length R=2 ln s (different from the original one L), the rest (coupling depending) into D
- Two coupling depending defects D (purely transmitting)

- Most importantly: we derived the (INTEGRABLE) S-matrix for the GKP dynamics

- Many quantities can be determined exactly out of the S-matrix: in present theory all the Form Factors Basso,Sever,Vieira; Belitsky;DF,Rossi.... $|\langle 0|O|n\rangle|^2 = G^{(n)}(\theta_1, \cdots, \theta_n; \lambda)$ exactly. Not always true.

- Can we use matrix part (for some operator) to general theory with the same symmetry (R-symmetry: SU(4))?

- Even more problematic: re-summation of the FF^2 series: not possible, by now, even here, except.....

- ....at strong coupling minimal area string computation (Alday-Gaiotto-Maldacena) gives rise to the A3 TBA (Al. Zamolodchikov).
- We reproduced TBA with only gluons and 'mesons' (meson is a 2D fermion-antifermion bound state only at strong coupling, other particle contribution is superficially 1-loop) (DF,Rossi,Piscaglia)
- We also reproduced the general E-gon: A3x(E-5 columns) (+Sever,Vieira) (delicate determination of the convolution integration contours) (+Bonini)
- New way to consider: 1) TBA from spectral series which gives rise to a Yang-Yang functional (=area) (similar to how it arises in N=2 SYM (Nekrasov-Shatashvili)); 2) classical Lax/quantum IS.
- Weak coupling (gauge) results: tree level and 1-loop (Basso,Sever,Vieira+Perimeter). 2-loops (Dixon,Drummond et al.) by using field theory methods.

FFs series summing to TBA

- Quite unique example (two-body product) though we may expect something similar in the UV limit of 'any' FF series (cf. infra scalars), but it does not happen

- The key idea: Hubbard-Stratonovich transformation replaces the infinite sums with a path integral

$$W^{(g)}_{hex} = Z^{(g)}[X^g] = \int \mathcal{D}X^g e^{-S^{(g)}[X^g]}$$

$$S^{(g)}[X^g] = \frac{1}{2}\int d\theta\, d\theta'\, X^g(\theta) T^g(\theta,\theta') X^g(\theta') + \\ + \int \frac{d\theta'}{2\pi}\,\mu^g(\theta')\left[\mathrm{Li}_2(-e^{-E(\theta')+i\phi}\,e^{X^g(\theta')}) + \mathrm{Li}_2(-e^{-E(\theta')-i\phi}\,e^{X^g(\theta')})\right]$$

- $S^{(g)}[X^g] \sim \sqrt{\lambda} \to \infty$: saddle point eqs. are TBA eqs.

$$X^g(\theta) - \int \frac{d\theta'}{2\pi} G^g(\theta,\theta')\mu^g(\theta')\,\log\left[(1+e^{X^g(\theta')}e^{-E(\theta')+i\phi})(1+e^{X^g(\theta')}e^{-E(\theta')-i\phi})\right] = 0$$

$$\int d\theta'\, G^g(\theta,\theta') T^g(\theta',\theta'') = \delta(\theta-\theta'')$$

- Crucially due to two-body product form of the multi particle FF (which did NOT happen before in FF theory):

$$W_{hex} = \sum_{N=0}^{+\infty} \frac{1}{N!} \sum_{a_1} \cdots \sum_{a_N} \int \prod_{k=1}^{N} \left[\frac{du_k}{2\pi} \mu_{a_k}(u_k) e^{-\tau E_{a_k}(u_k) + i\sigma p_{a_k}(u_k) + i m_{a_k} \phi} \right] \prod_{i<j}^{N} \frac{1}{P_{a_i,a_j}(u_i|u_j)\, P_{a_j,a_i}(u_j|u_i)}$$

- Gaussian fields Xs

$$\prod_{i<j}^{N} e^{\langle X_{(a_i)}(u_i)\, X_{(a_j)}(u_j) \rangle} = \langle e^{X_{(a_1)}(u_1)} \cdots e^{X_{(a_N)}(u_N)} \rangle \qquad \frac{1}{P_{a,b}(u|v)\, P_{b,a}(v|u)} = e^{\langle X_{(a)}(u)\, X_{(b)}(v) \rangle}$$

$$W_{hex} = \langle \exp \left\{ \int \frac{du}{2\pi} \sum_a \left[\mu_a(u)\, e^{-\tau E_a(u) + i\sigma p_a(u) + i m_a \phi}\, e^{X_{(a)}(u)} \right] \right\} \rangle$$

-

- Crucial simplification of strong coupling: the gluon bound states are additive $X^g_{(a)} = a\, X^g_{(1)}$; their measure=1/n^2 produces the dilogarithm

- Fermion-antifermion bound state in the 2d (GKP) S-matrix analytic structure at infinite 't Hooft coupling: new particle, 2d meson.

- New FFs or pentagonal amplitudes (DF,Piscaglia,Rossi)

- Anew, bound states of mesons: they are additive; measure=1/n^2 entails dilog.

- It add a third pseudoenergy X^M with its equation coupled to the two previous ones: A_3 Dynkin diagram (new kernel).

Spectral OPE

- Insertion of a orthonormal basis of asymptotic (free) Hamiltonian for scalars

$$W = \sum_{n=0}^{\infty} W^{(2n)} \qquad\qquad W^{(2n)} = \frac{1}{(2n)!}\int \prod_{i=1}^{2n} \frac{d\theta_i}{2\pi}\, G^{(2n)}(\theta_1, \cdots, \theta_{2n})\, e^{-z\sum\limits_{i=1}^{2n}\cosh\theta_i}$$

- Strong coupling regime: relativistic O(6) NLSM,

$$z = m_{gap}\sqrt{\tau^2+\sigma^2} \qquad\qquad m_{gap}(\lambda) = \frac{2^{1/4}}{\Gamma(5/4)}\lambda^{1/8} e^{-\sqrt{\lambda}/4}(1+O(1/\sqrt{\lambda}))$$

- Exponentially small mass in the exponent is subtle

- In general: $z\cosh\theta_i \to \tau E(\theta_i) + i\sigma p(\theta_i)$

$$G^{(2n)}(\theta_1, \cdots, \theta_{2n}) = \Pi^{(2n)}_{dyn}(\theta_1, \cdots, \theta_{2n})\, \Pi^{(2n)}_{mat}(\theta_1, \cdots, \theta_{2n})$$

- The dynamic part is two-body

$$\Pi^{(2n)}_{dyn}(\theta_1, \cdots, \theta_{2n}) = \prod_{i<j}^{2n} \Pi(\theta_i, \theta_j)$$

- The matrix part depends on differences, is group-theoretical (residual R-symmetry) and coupling independent, but cumbersome

$$\Pi^{(2n)}_{mat}(\theta_1, \ldots, \theta_{2n}) = \frac{1}{(2n)!(n!)^2} \int_{-\infty}^{+\infty} \prod_{k=1}^{n} \frac{da_k}{2\pi} \prod_{k=1}^{2n} \frac{db_k}{2\pi} \prod_{k=1}^{n} \frac{dc_k}{2\pi} \times$$

$$\times \frac{\prod_{i<j}^{n} g(a_i - a_j) \prod_{i<j}^{2n} g(b_i - b_j) \prod_{i<j}^{n} g(c_i - c_j)}{\prod_{j=1}^{2n} \left(\prod_{i=1}^{n} f(a_i - b_j) \prod_{k=1}^{n} f(c_k - b_j) \prod_{l=1}^{2n} f\left(\frac{2\theta_l}{\pi} - b_j \right) \right)}$$

- BUT......

$$f(x) = x^2 + \frac{1}{4}, \; g(x) = x^2(x^2+1)$$

- ....integrals similar to N=2 SYM partition function: sum over (symmetrised arrays of) Young tableaux

- We want the exponent=free energy

$$\mathcal{F} = \ln W = \sum_{n=1}^{\infty} \mathcal{F}^{(2n)} = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \prod_{i=1}^{2n} \frac{d\theta_i}{2\pi} g^{(2n)}(\theta_1, \cdots, \theta_{2n}) e^{-z \sum_{i=1}^{2n} \cosh \theta_i}$$

- and gs are the connected functions! Easily computable from G

Excursus on fermions

- n fermions u, n anti-fermions v

$$\Pi^{(n)}_{mat}(\{u_i\},\{v_j\}) = \frac{1}{(n!)^3}\int \prod_{k=1}^{n}\left(\frac{da_k db_k dc_k}{(2\pi)^3}\right)\cdot$$

$$\cdot\frac{\prod_{i<j}^{n} g(a_i-a_j)g(b_i-b_j)g(c_i-c_j)}{\prod_{i,j}^{n} f(a_i-b_j)f(c_i-b_j)\prod_{i,j}^{n} f(u_i-a_j)f(v_i-c_j)},$$

- These can be generalised to all particles and all these (matrix part) G=FF can be read off from Bethe Ansatz equations for SU(4)

Factorisation

- Known property for usual FFs works here too

$$G^{(2n)}(u_1+\Lambda,\cdots,u_{2k}+\Lambda,u_{2k+1},\cdots,u_{2n}) \stackrel{\Lambda\to\infty}{\longrightarrow} G^{(2k)}(u_1,\cdots,u_{2k})\,G^{(2n-2k)}(u_{2k+1},\cdots,u_{2n})+O(\Lambda^{-2})$$

- with the novelty of the milder and more subtle power-like decay (instead of the exponential one), thanks to the balance

$$\Pi^{(2n)}_{dyn}(u_1+\Lambda,\cdots,u_m+\Lambda,u_{m+1},\cdots,u_{2n}) \longrightarrow \Lambda^{2m(2n-m)}\Pi^{(m)}_{dyn}(u_1,\cdots,u_m)\Pi^{(2n-m)}_{dyn}(u_{m+1},\cdots,u_{2n})$$

$$\Pi^{(2n)}_{mat}(u_1+\Lambda,\cdots,u_{2k}+\Lambda,u_{2k+1},\cdots,u_{2n}) \longrightarrow \Lambda^{-2m(2n-m)}\Pi^{(2k)}_{mat}(u_1,\cdots,u_{2k})\Pi^{(2n-2k)}_{mat}(u_{2k+1},\cdot\cdot,u_{2n})$$

- Therefore the soft, but integrable decay

$$\lim_{\Lambda\to\infty} g^{(2n)}(\theta_1+\Lambda,\cdots,\theta_m+\Lambda,\theta_{m+1},\cdots,\theta_{2n}) \simeq \frac{1}{\Lambda^2} \to 0$$

-

- Residue integration on a and c produces

$$\Pi_{mat}^{(2n)}(u_1, \cdots, u_{2n}) = \frac{4n^2}{(2n)!(n!)^2} \int \prod_{i=1}^{2n} \frac{db_i}{2\pi} \frac{[\delta_{2n}(b_1, \ldots, b_{2n})]^2}{\prod_{i,j}^{2n} f(u_i - b_j)} \prod_{i<j} \frac{b_{ij}^2}{(b_{ij}^2 + 1)}$$

- Young Tableaux, fully encoding residues on b,

$$\Pi_{mat}^{(2n)}(u_1, \cdots, u_{2n}) = \sum_{l_1+\cdots+l_{2n}=2n, l_i<3, l_i \geq l_{i+1}} (l_1, \cdots, l_{2n})_s = \sum_{|Y|=2n, l_i<3} (Y)_s$$

- made us guess the structure

$$\Pi_{mat}^{(2n)} = \frac{P_{2n}(u_1, \cdots, u_{2n})}{\prod_{i<j}^{2n} (u_{ij}^2 + 1)(u_{ij}^2 + 4)}$$

- which we proved by factorisation.

Strong coupling expansion

- The dynamical part takes a relativistic simple form at strong coupling

$$\Pi^{(2n)}_{dyn}(\theta_1,\cdots,\theta_{2n}) \propto \prod_{i<j}^{2n} \Pi(\theta_i-\theta_j) \qquad \Pi(\theta)=\frac{8\theta\tanh\left(\frac{\theta}{2}\right)\Gamma\left(\frac{3}{4}+\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{3}{4}-\frac{i\theta}{2\pi}\right)}{\pi\Gamma\left(\frac{1}{4}+\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{4}-\frac{i\theta}{2\pi}\right)}$$

- Thanks to relativistic invariance, θ_1 integral

$$\alpha_i=\theta_{i+1}-\theta_1,\ i=1,\ldots,2n-1 \qquad \mathcal{F}^{(2n)} \propto 2\int\prod_{i=1}^{2n-1} d\alpha_i g^{(2n)}(\alpha_1,\ldots,\alpha_{2n-1})K_0(z\xi)$$

$$K_0(z\xi)=-\ln z-\ln\xi+(\ln 2-\gamma)+O(z^2\ln z) \qquad z=m_{gap}r\sim\exp(-\sqrt{\lambda})$$

$$\xi^2=2n+2\sum_{i=2}^{2n}\cosh\alpha_{i-1}+2\sum_{i=2}^{2n}\sum_{j=i+1}^{2n}\cosh(\alpha_{i-1}-\alpha_{j-1})$$

This contribution scales as

$$\ln W = \frac{\sqrt{\lambda}}{\pi} \sum_{n=1}^{+\infty} \frac{1}{(2n)!} \int \prod_{i=1}^{2n-1} \frac{d\alpha_i}{2\pi} g^{(2n)}(\alpha_1, \ldots, \alpha_{2n-1}) + O(\ln \sqrt{\lambda})$$

- the same as the classical minimal area: $-\frac{\sqrt{\lambda}}{2\pi} A_E$
- Check with Knizhnik twist field dimension $\Delta_\alpha = \frac{c}{12}(k - 1/k),\ \alpha = 2\pi k - 2\pi = \pi/2, c = 5$
  Castro-Alvaredo, Doyon, DF
- and we can also compute further: new feature is divergency (asymptotic freedom).
- Besides: hope of computing the building blocks FF=G (not F) in integrability theory as computation uses Young tableaux (N=2 SYM) systematically.

- Actually, Bethe Ansatz structure of F, but also of G=FF which is simpler.
- More precisely, Matrix Part can be expressed via BAEs, Scalar Part is a scalar problem indeed.
- G=FF means integration over the auxiliary roots a, b, c of previous transparencies.
- Connexion with q-KZ, and then N=2 SYM (work in progress).

Fermions and mesons (1-loop)

- Before meson from bootstrap of S-matrix
- From the OPE series at strong coupling for n fermions and n anti-fermions

$$W_f = \sum_{n=0}^{\infty} W_f^{(n)}$$

$$W_f^{(n)} = \frac{1}{n!n!} \int_C \prod_{k=1}^{n} \left[\frac{du_k}{2\pi} \frac{dv_k}{2\pi} \mu_f(u_k) \mu_f(v_k) \, e^{-\tau E_f(u_k) + i\sigma p_f(u_k)} \cdot e^{-\tau E_f(v_k) + i\sigma p_f(v_k)} \right] \Pi_{dyn}^{(n)}(\{u_i\}, \{v_j\}) \, \Pi_{mat}^{(n)}(\{u_i\}$$

- Dynamical part is two-body as for scalars

$$\Pi_{dyn}^{(n)}(\{u_i\}, \{v_j\}) = \prod_{i<j}^{n} \frac{1}{P(u_i|u_j)P(u_j|u_i)} \frac{1}{P(v_i|v_j)P(v_j|v_i)} \prod_{i,j=1}^{n} \frac{1}{\bar{P}(u_i|v_j)\bar{P}(v_j|u_i)}$$

- Recall the matrix factor argued from BA

$$\Pi^{(n)}_{mat}(\{u_i\},\{v_j\}) = \frac{1}{(n!)^3}\int \prod_{k=1}^{n}\left(\frac{da_k db_k dc_k}{(2\pi)^3}\right)\cdot$$

$$\cdot\frac{\prod_{i<j}^{n} g(a_i-a_j)g(b_i-b_j)g(c_i-c_j)}{\prod_{i,j}^{n} f(a_i-b_j)f(c_i-b_j)\prod_{i,j}^{n} f(u_i-a_j)f(v_i-c_j)}, \qquad f(u)=u^2+\frac{1}{4}, \quad g(u)=u^2(u^2+1)$$

- But it is not suitable: <u>we must integrate</u> to obtain the polar structure

$$\Pi^{(n)}_{mat}(\{u_i\},\{v_j\}) = \frac{P^{(n)}(u_1,\ldots,u_n,v_1,\ldots,v_n)}{\prod_{i<j}^{n}[(u_i-u_j)^2+1]\prod_{i<j}^{n}[(v_i-v_j)^2+1]\prod_{i,j=1}^{n}[(u_i-v_j)^2+4]}$$

- by Young tableaux and factorisation method as for scalars (P=polynomial)

- This is suitable to see
- 1) coalescence of fermion-antifermion=meson, which means OPE becomes sum over mesons;
- 2) short range or pinching of many mesons like Nekrasov instantons

- Privileging u_i, fermionic contribution reads

$$W_f^{(n)} = \frac{1}{n!}\int_C \prod_{i=1}^{n} \frac{du_i}{2\pi} I_n(u_1, \cdots, u_n) \prod_{i<j}^{n} p(u_{ij})$$

- with meson-meson short range potential

$$p(u_{ij}) = \frac{u_{ij}^2}{u_{ij}^2+1}\,, \quad u_{ij} = u_i - u_j\,,$$

- as

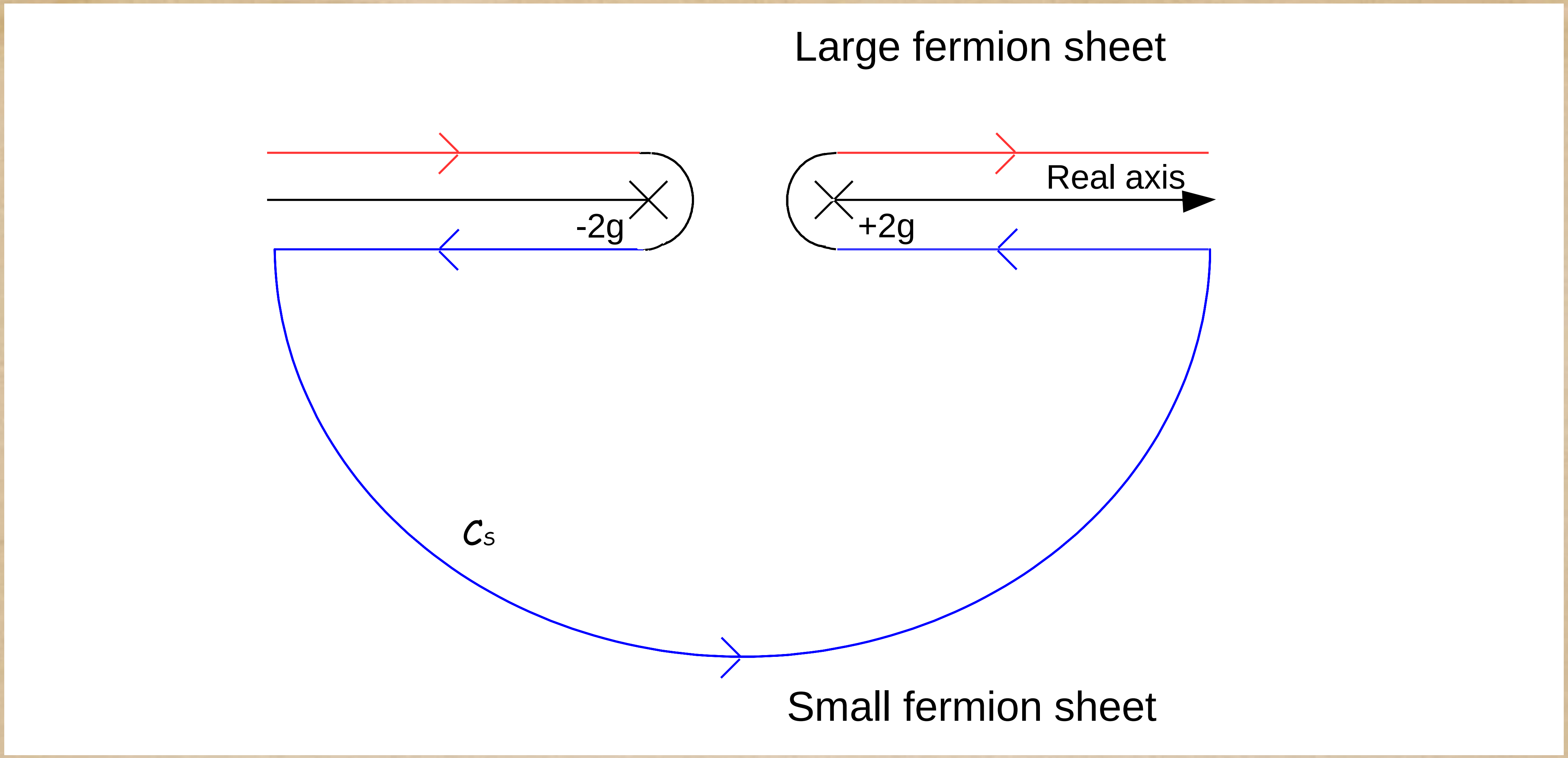
$$\lambda \to \infty \quad \bar{u}_i = u_i/2g, \bar{v}_i = v_i/2g\, finite \qquad \epsilon_2 \sim 1/g$$

- And the v-integrals

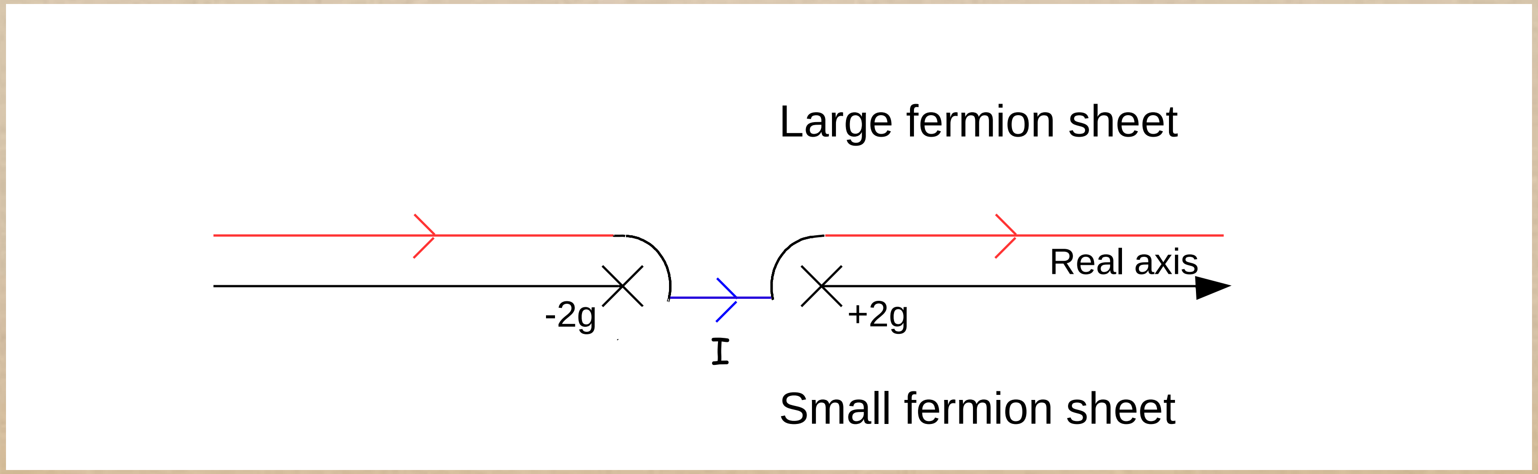
$$I_n(u_1, \cdots, u_n) \equiv \frac{1}{n!}\int_C \prod_{i=1}^{n} \frac{dv_i}{2\pi} R_n(\{u_i\},\{v_j\}) P^{(n)}(\{u_i\},\{v_j\}) \prod_{i,j=1}^{n} h(u_i - v_j) \prod_{i<j}^{n} p(v_{ij})$$

- fermion-antifermion short range potential (responsible for the meson formation)

$$h(u_i - v_j) = \frac{1}{(u_i - v_j)^2 + 4}$$



The open contour C



Its closure I: there is a cut!

- Regular function

$$R_n(\{u_i\},\{v_j\})\prod_{i<j}^{n} u_{ij}^2 v_{ij}^2 = \Pi_{dyn}^{(n)}(\{u_i\},\{v_j\})\prod_{i=1}^{n}\hat{\mu}_f(u_i)\hat{\mu}_f(v_i)$$

- Only now, we use strong coupling to close the contour C and compute by residues on lower half plane on poles

- $v_j = u_j - 2i, \quad I_n^{closed}(u_1,\cdots,u_n) = (-1)^n R_n(u_1,\cdots,u_n,u_1-2i,\cdots,u_n-2i)$

- In conclusion, new effective 2D bound state particle of energy and momentum

$$E_M(u) \equiv E_f(u+i) + E_f(u-i), \quad p_M(u) \equiv p_f(u+i) + p_f(u-i)$$

- And form-factor or pentagonal amplitude

$$P^{MM}(u|v) = -(u-v)(u-v+i)P(u+i|v+i)P(u-i|v-i)|\bar{P}(u-i|v+i)\bar{P}(u+i|v-i)$$

- Technical definitions

$$P^{MM}_{reg}(u|v) = P^{MM}(u|v)\frac{u-v}{u-v+I}, \ \hat{\mu}_M(u) = \mu_M(u)e^{-\tau E_M(u)+i\sigma p_M(u)} = -\frac{\hat{\mu}_f(u+i)\hat{\mu}_f(u-i)}{\bar{P}(u+i|u-i)\bar{P}(u-i|u+i)}$$

-

- To introduce the sum over mesons

$$W_f \simeq W^{(M)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C \prod_{i=1}^{n} \frac{du_i}{2\pi} \hat{\mu}_M(u_i - i) \cdot \prod_{i<j}^{n} \frac{1}{P_{reg}^{MM}(u_i - i|u_j - i) P_{reg}^{MM}(u_j - i|u_i - i)} \prod_{i<j}^{n} p(u_{ij})$$

- + 1-loop corrections due to the integration on the interval I, which makes C closed. Caveat: C above is not exactly closed: difference w.r.t. Nekrasov. Yet, C closed is the leading approximation (+1 loop) for large

$$g \sim 1/\epsilon_2$$

- We can do even better: average over a gaussian field X

$$e^{\langle X(u_i)X(u_j)\rangle} \equiv \frac{1}{P^{MM}_{reg}(u_i - i|u_j - i)P^{MM}_{reg}(u_j - i|u_i - i)}$$

- of a Fredholm determinant (true also for Nekrasov partition function) $\prod_{i<j}^{n} p(u_{ij}) = \prod_{i<j}^{n} \frac{u_{ij}^2}{u_{ij}^2+1} = \frac{1}{i^n}\det\left(\frac{1}{u_i - u_j - i}\right)$

$$W^{(M)} = \langle \det(1+M)\rangle = \left\langle \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} TrM^n\right]\right\rangle$$

- With power traces

- $$TrM^n = \int_C \prod_{i=1}^{n} \frac{du_i}{2\pi i} \hat{\mu}_M(u_i - i) e^{X(u_i)} \prod_{i=1}^{n} \frac{1}{u_i - u_{i+1} - i}, \quad u_{n+1} \equiv u_1$$

- At leading order, we can close C

$$TrM^n \simeq \frac{(-1)^{n-1}}{n}\int_C \frac{du}{2\pi}\hat{\mu}^n_M(u-i)e^{nX(u)} \simeq \frac{(-1)^{n-1}}{n}\int_C \frac{du}{2\pi}\hat{\mu}^n_M(u)e^{nX(u)}$$

- And hence obtain the usual dilog

$$W^{(M)} \simeq \left\langle \exp\left[-\int_C \frac{du}{2\pi}\mu_M(u)Li_2\left[-e^{-\tau E_M(u)+i\sigma p_M(u)}e^{X(u)}\right]\right]\right\rangle$$

- +1-loop corrections which are under control via the exact power trace formula.

- N.B. 1-loop corrections are easier here than above

- Very general and universal problem of correcting TBA

1-loop corrections in N=2 (Bourgine,DF)

- Nekrasov partition function or more general long-range potential of form

$$Z=\sum_{N=0}^{\infty}\frac{\Lambda^N}{N!}\left(\frac{\epsilon_+}{\epsilon_1\epsilon_2}\right)^N\int\prod_{i=1}^{N}Q(\phi_i)\frac{d\phi_i}{2i\pi}\prod_{i<j}^{N}K(\phi_i-\phi_j)$$

- For instance

$$Q(x)=\frac{\prod_{f=1}^{N_f}(x-m_f)}{\prod_{l=1}^{N_c}(x-a_l)(x+\epsilon_+-a_l)}\qquad K(x)=\frac{x^2(x^2-\epsilon_+^2)}{(x^2-\epsilon_1^2)(x^2-\epsilon_2^2)}$$

- More general K, but we have the problem of short range

$$K(x)=p(x)e^{\epsilon_2 k(x)},\quad \text{with short range}\quad p(x)=\frac{x^2}{x^2-\epsilon_2^2}$$

- For long range we can write gaussian integration

$$S_{long}[X]=\frac{1}{2\epsilon_2}\int\frac{dxdy}{(2i\pi)^2}t(x-y)X(x)X(y),\qquad \int t(x-z)k(z-y)\frac{dz}{2i\pi}=2i\pi\delta(x-y)$$

- Which means

$$\langle X(x)X(y)\rangle = \epsilon_2 k(x-y)$$

- And allows us an Hubbard-Stratonovich transformation to write exactly the partition function

$$Z = \langle Z_{short}[X]\rangle\,, \quad \text{with} \quad Z_{short}[X] = \sum_{N=0}^{\infty} \frac{q^N \epsilon_2^{-N}}{N!} \int \prod_{i=1}^{N} Q(\phi_i) e^{X(\phi_i)} \frac{d\phi_i}{2i\pi} \prod_{i<j}^{N} p(\phi_{ij})$$

- Of course, this method is not effective for the short-range p as epsilon_2—>0

- This is why we elaborated by Meyer expansion the short partition function (now trace formula)

$$Z = \left\langle \exp\left(\frac{1}{\epsilon_2} \int Li_2\left(qQ(x)e^{X(x)}\right) \frac{dx}{2i\pi} + \frac{1}{4} \int \log\left(1 - qQ(x)e^{X(x)}\right) \nabla \log\left(1 - qQ(x)e^{X(x)}\right) \frac{dx}{2i\pi} + O(\epsilon_2) \right) \right\rangle$$

- First time correction to TBA (general problem) by 'universal' nabla operator

$$\nabla U(x) = U'_{\text{reg.}}(x) - U'_{\text{sing.}}(x)$$

- provided this decomposition

$$U(x) = U_{\text{reg.}}(x) + U_{\text{sing.}}(x)$$

- reg.=analytic inside the closed integration contour (upper half plane), sing.=analytic outside (Milne)

- Problem: no AGT correspondence, so far, but complicated 2d massive scattering theory
- Still we would like PDE or ODE from some null-vector condition, e.g. Phi_21 which is surface or defect operator
- Subtlety: at the very end we wish to integrate on the rapidities u_i: playing the role of vevs?

Conclusions and Perspectives

- Modified TBA so to include this contribution which does not depend on ratios: common origin, the spectral series. String one-loop corrections?
- Universal problem of quantise/correct TBA (quantum dilog?): string, N=2, etc.
- New way to consider: 1) TBA from spectral series which gives rise to a Yang-Yang functional (=area) (similar to how it arises in N=2 SYM (Nekrasov-Shatashvili)); 2) classical Lax/quantum IS.
- Proofs of the form of the G and F from BA Eqs. and all the transitions (Belitski).
- Explicit computation of polynomials (over simple quadratic polynomials) of the matrix parts.
- Young Tableaux descriptions and computations for all the matrix parts