QCD - introduction

lagrangian, symmetries, running coupling, Coulomb gauge

Lagrangian

Quantum Chromodynamics

we require a theory which

- has approximate chiral symmetry
- has approximate SU(3) flavour symmetry
- accounts for the parton model
- has colour
- and colour confinement
- is renormalizable

QCD

gauge $SU_c(3)$

local gauge invariance (QED):

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda \qquad \phi \to \phi - \dot{\Lambda}$$

impose local gauge symmetry:

$$\psi(x) \to \mathrm{e}^{-i\Lambda(x)}\psi(x)$$

and get an interacting field theory:

$$\mathcal{L} = \int \bar{\psi}\gamma^\mu\partial_\mu\psi \to \int \bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi \qquad A_\mu \to A_\mu + \partial_\mu\Lambda$$

Quantum Chromodynamics

local gauge invariance (QCD):

impose local gauge symmetry: $\psi(x)_a \to U_{ab}\psi(x)_b$

for invariance of L:

$$\mathcal{L} = \int \bar{\psi}_a \delta^{ab}\gamma^\mu\partial_\mu\psi_b \to \int \bar{\psi}_a \left(\delta^{ab}\gamma^\mu\partial_\mu + ig\gamma^\mu \boxed{(A_\mu)_{ab}}\right)\psi_b$$

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger$$

eight gluons

$$F_{\mu\nu} \propto [D_\mu, D_\nu] = \boxed{ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu]}$$

QCD

$$\begin{aligned}
\mathcal{L}_{QCD} &= \sum_f^{n_f} \bar{q}_f[i\gamma_\mu(\partial^\mu + igA^\mu) - m_f]q_f - \frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \\
A_\mu &= A_\mu^a \frac{\lambda^a}{2} \\
[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}] &= if^{abc}\frac{\lambda^c}{2} \\
\mathrm{Tr}(\lambda^a\lambda^b) &= 2\delta^{ab}
\end{aligned}$$

flavour, colour, Dirac indices

$$\boxed{\mathcal{L}_\theta = \theta \frac{g^2}{64\pi^2} F^{\mu\nu}\tilde{F}_{\mu\nu}}$$

Symmetries

symmetries in (classical) field theory

$$q \to f(q) \longrightarrow j_\mu \longrightarrow \partial_\mu j^\mu = 0$$

$$\frac{d}{dt}\int d^3x\, j_0 \equiv \frac{d}{dt}Q = \int d^3x\, \nabla \cdot \vec{j} = 0$$

symmetries in QCD

$U(1)_V$

$$q \to \mathrm{e}^{-i\theta} q \qquad j_\mu = \bar{q}\gamma_\mu q = \bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d \qquad Q = \int d^3x \, (u^\dagger u + d^\dagger d)$$

symmetry current charge

'baryon number conservation' $p \not\to e^+ \nu$

[violated by EW anomaly]

in full SM need 1-gamma_5, which intrduces anomaly, 't Hooft efff L has a prefactor of exp(-2 pi/alpha_2) ~ 10^-70

symmetries in QCD

$U(1)_A$ $\qquad m_u = m_d = 0$

$q \to \mathrm{e}^{-i\gamma_5\theta} q$ $\qquad j_{\mu 5} = \bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d$ $\qquad Q_5 = \int d^3x\, (u^\dagger\gamma_5 u + d^\dagger\gamma_5 d)$

symmetry current charge

this symmetry does not exist in the quantum theory

$$\partial^\mu j_{\mu 5} = \frac{3\alpha_s}{8\pi} F\tilde{F}$$

symmetries in QCD

scale invariance $m_u = m_d = 0$

$$\begin{aligned} &x \to \lambda x \\ &q \to \lambda^{3/2} q(\lambda x) \\ &A \to \lambda A(\lambda x) \end{aligned} \qquad j_\mu = x_\nu \Theta^{\mu\nu}$$

symmetry current

$$\partial^\mu j_\mu = \Theta^\mu_\mu = 0$$

this symmetry does not exist in the quantum theory

$$\Theta^\mu_\mu = m\bar{q}q + \frac{\alpha_s}{12\pi} F^2$$

symmetries in QCD

$\mathrm{SU}(3)_{\mathrm{V}}$ $\quad m_u = m_d \quad$ isospin

$q \to \mathrm{e}^{i\theta T_F^a} q$ $\quad j_\mu^a = \bar{\psi}\gamma_\mu T_F^a \psi$ $\quad Q^a = \int d^3x \psi^\dagger T_F^a \psi$

symmetry current charge

$$Q^+|\pi^-\rangle = \frac{1}{\sqrt{2}} \int d^3x \, \left(b^\dagger(\mathbf{x})\tau^+ b(\mathbf{x}) - d^\dagger(\mathbf{x})\tau^- d(\mathbf{x}) \right) |\pi^-\rangle = |\pi^0\rangle$$

$$H|\pi^-\rangle = E_{\pi^-}|\pi^-\rangle; \quad Q^+ H|\pi^-\rangle = E_{\pi^-}|\pi^0\rangle; \quad H|\pi^0\rangle = E_{\pi^-}|\pi^0\rangle$$

this symmetry is explicitly broken by quark mass and EW effects

symmetries in QCD

$\mathrm{SU(3)_A}$

$$m_u = m_d = 0$$

$$q \to \mathrm{e}^{-i\theta T^a \gamma_5} q \qquad j^a_\mu = \bar{\psi}\gamma_\mu\gamma_5 T^a \psi \qquad Q^a_5 = \int d^3x \psi^\dagger \gamma_5 T^a \psi$$

symmetry | current | charge

This symmetry is realised in the Goldstone mode.

transform the vacuum:

$$\left.\begin{aligned} \mathrm{e}^{i\theta^a Q^a_5}|0\rangle &= |0\rangle \\ Q^a_5|0\rangle &= 0 \end{aligned}\right\} \text{Wigner mode}$$

symmetries in QCD

HFM: spin direction is signalled out (NOT by an external field) it could be any direction, so small fluctutatiopns (or large) do not cost energy -> gapless dispersion relationship, E ~ k or k^2.

$SU(3)_A$

$$\left.\mathrm{e}^{i\theta^a Q_5^a}|0\rangle = |\theta\rangle \neq |0\rangle \quad\right\} \text{Goldstone mode}$$

$$H|\theta\rangle = H\mathrm{e}^{i\theta^a Q_5^a}|0\rangle = \mathrm{e}^{i\theta^a Q_5^a}H|0\rangle = E_0|\theta\rangle$$

so there is a continuum of states degenerate with the vacuum

Excitations of the vacuum may be interpreted as a particle. In this case fluctuations in theta are massless particles called Goldstone bosons.

symmetries in QCD

$SU(3)_A$

Goldstone boson quantum numbers:

$$|\delta\theta\rangle = \theta^a Q_5^a |0\rangle$$

$$\sim \theta^a \int d^3x b^\dagger(\mathbf{x}) T_F^a d^\dagger(\mathbf{x})|0\rangle$$

spin singlet, spatial singlet, flavour octet $\Rightarrow$ the pion octet

Chiral Symmetry Breaking

$$SU_L(2) \times SU_R(2) \times U_A(1) \times U_V(1)$$

Isospin Invariance

$$\psi \to \mathrm{e}^{i\theta\cdot\tau}\psi$$

equal quark masses

$$j^a_{V\mu} = \bar{\psi}\gamma_\mu\tau^a\psi$$

$$Q^a_V = \int d^3x\psi^\dagger\tau^a\psi$$

$$[H, Q^a_V] = 0$$

Isospin Invariance

$$[H, Q_V^a] = 0$$

$$H(Q_V^a|M\rangle) = E_M(Q_V^a|M\rangle)$$

$$Q_V^a = \int \frac{d^3k}{(2\pi)^3}\left[b_{\mathbf{k}}^\dagger \tau^a b_{\mathbf{k}} - d_{\mathbf{k}}^\dagger (\tau^a)^T d_{\mathbf{k}}\right]$$

$$Q_V^+|\rho^0\rangle = Q_V^+ \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - d\bar{d}\rangle)$$

$$Q_V^+|\rho^0\rangle = \frac{1}{\sqrt{2}}(-|u\bar{d}\rangle - |u\bar{d}\rangle)$$

$$Q_V^+|\rho^0\rangle \propto |\rho^+\rangle$$

Axial Symmetry

$$\psi \rightarrow \mathrm{e}^{i\gamma_5 \theta\cdot\tau}\psi$$

$$j^a_{A\mu} = \bar{\psi}\gamma_\mu\gamma_5\tau^a\psi$$

$$Q^a_A = \int d^3x\,\psi^\dagger\gamma_5\tau^a\psi$$

$$[H, Q^a_A] = 0$$

Axial Symmetry

$$Q^a_{A1} = \int \frac{d^3k}{(2\pi)^3} c_k \left[b^\dagger_{\mathbf{k}\lambda} 2\lambda b_{\mathbf{k}\lambda} - d_{-\mathbf{k}\lambda} 2\lambda d^\dagger_{-\mathbf{k}\lambda} \right]$$

$$Q^a_{A1} \left(|++\rangle + |--\rangle \right) = \left(|++\rangle - |--\rangle \right)$$

$$Q^a_{A1} \left(|+-\rangle + |-+\rangle \right) = 0$$

$$J_H^{(J)(J)} = J^{(J+1)(J)}$$

Axial Symmetry

$$Q^a_{A2} = \int \frac{d^3k}{(2\pi)^3} s_k \left[b^\dagger_{\mathbf{k}} \tau^a d^\dagger_{-\mathbf{k}} + d_{-\mathbf{k}} \tau^a b_{\mathbf{k}} \right]$$

pion RPA creation operator!

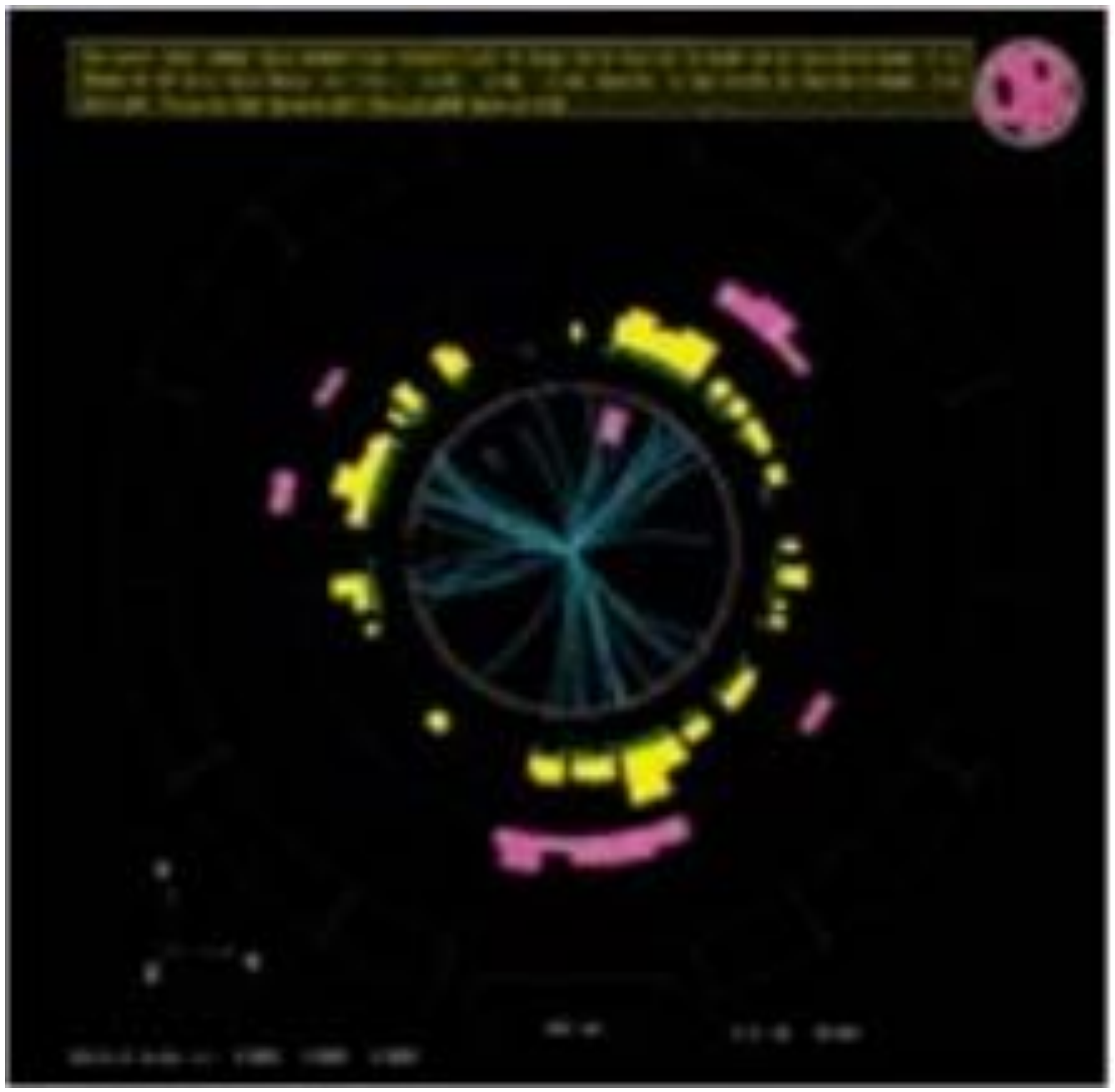
$$Q^a_{A2} |M\rangle = |M\pi^a\rangle$$

Goldstone's theorem says nothing about the excited pion spectrum.

$$\pi'(1300) \approx \rho'(1450)$$

gluons

Quantum Chromodynamics



A three jet event at DESY. August, 1979

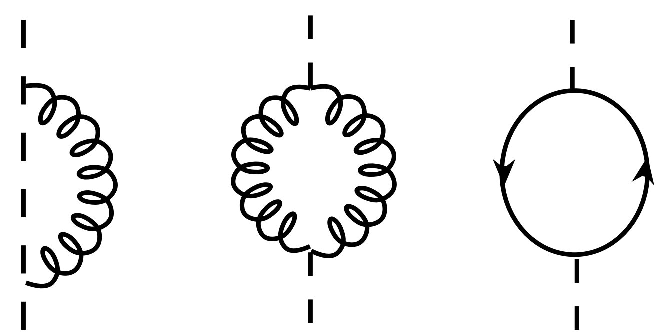
Q: are these peculiar gluons real?
There was indirect evidence from deviation from DIS scalang. Direct evidence was achieved at DESY with three jet events.
REF:http://cerncourier.com/cws/article/cern/39747

John Ellis, Mary Gaillard, Graham Ross

Running Coupling

running coupling

Khriplovich Yad. F. **10**, 409 (69)



$\frac{12}{3}N_c$ $\quad -\frac{1}{3}N_c$ $\quad -\frac{2}{3}N_f$

$$\mu\frac{dg(\mu)}{d\mu} = -\frac{\beta_0}{(4\pi)^2}g^3(\mu)$$

$$\alpha_s(\mu^2) = \frac{4\pi}{(11-\frac{2}{3}n_f)\ln\mu^2/\Lambda^2_{QCD}}$$

$$\mu_R^2 \frac{d\alpha_s}{d\mu_R^2} = \beta(\alpha_s) = -(b_0\alpha_s^2 + b_1\alpha_s^3 + b_2\alpha_s^4 + \cdots)$$

$$b_0 = (11C_A - 4n_f T_R)/(12\pi) = (33 - 2n_f)/(12\pi)$$

$$b_1 = (17C_A^2 - n_f T_R(10C_A + 6C_F))/(24\pi^2) = (153 - 19n_f)/(24\pi^2)$$

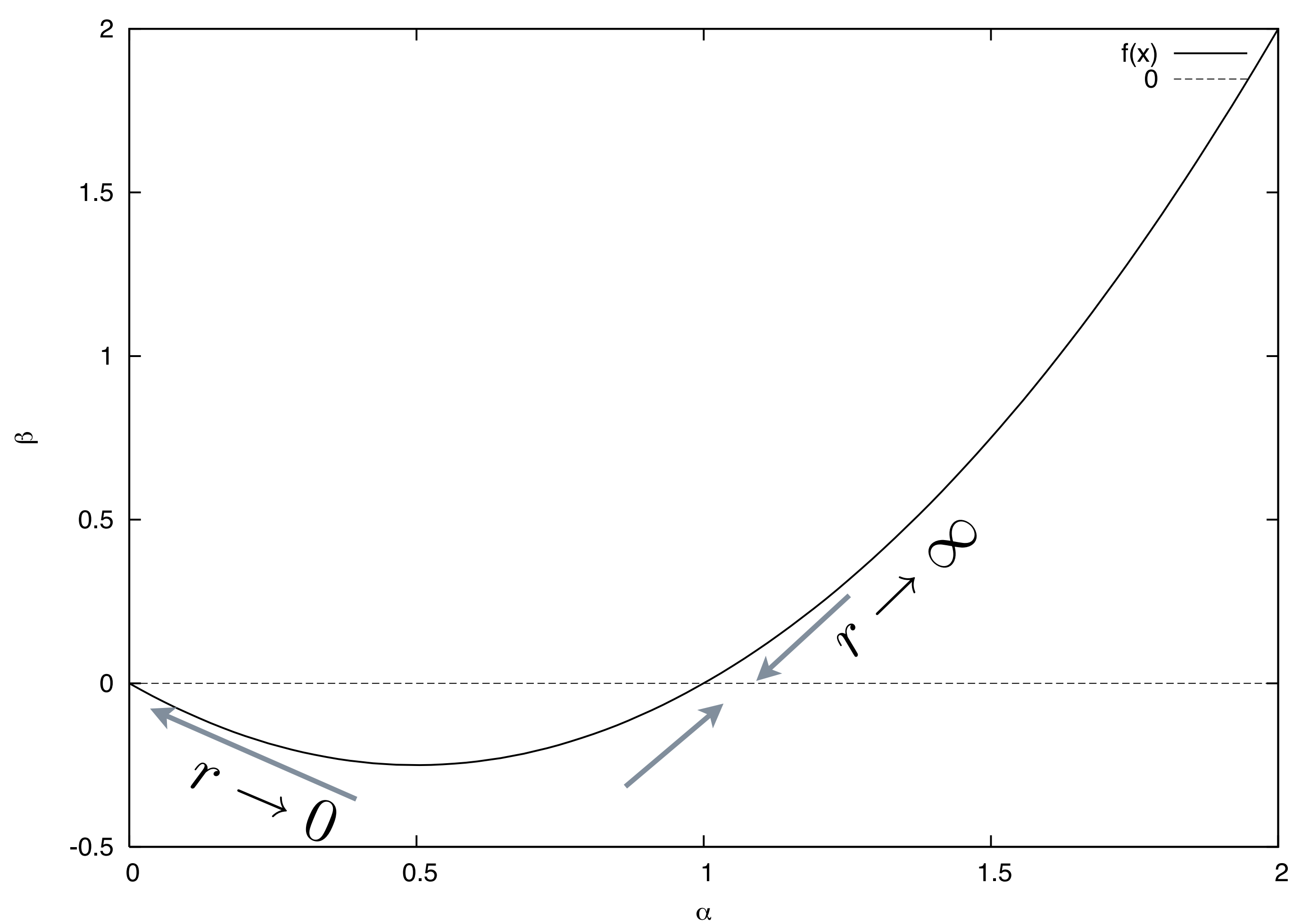
$$b_2 = (2857 - \tfrac{5033}{9}n_f + \tfrac{325}{27}n_f^2)/(128\pi^3)$$

THE RUNNING COUPLING ...

UV stable fixed point $\bar{\alpha} = 0$

IR stable fixed point $\bar{\alpha} = 1, \ \alpha \to \infty$

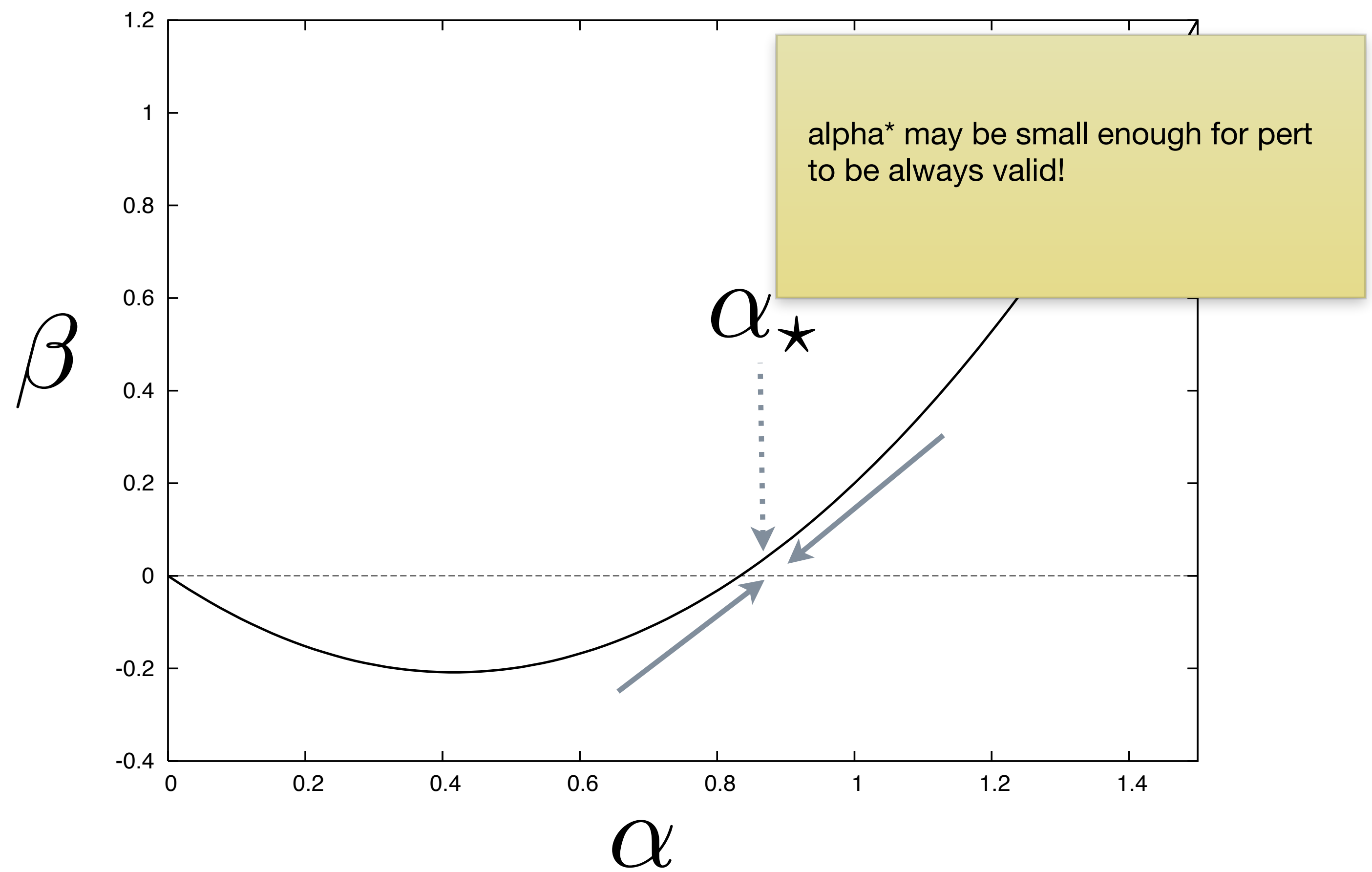
makes the IR limit stable...



Walking Technicolour

The Conformal Window

For $N_f < N^{AF}$



In the infrared limit $\alpha(\mu) \to \alpha_\star$; there is no scale dependence.

Thus the theory is *conformal* and chiral symmetry breaking and confinement are lost.

aks conformal window (of interest for walking TC models)

More specifically, suppose that we find that the beta function of a theory up to two loops has the form

$$\beta(g) = -b_0 g^3 + b_1 g^5 + \mathcal{O}(g^7)$$

where b_0 and b_1 are positive constants. Then there exists a value $g = g_*$ such that $\beta(g_*) = 0$:

$$g_*^2 = \frac{b_0}{b_1}.$$

the theory flows to this conformal pt in the IR

For QCD this happens when

$$\frac{11}{2} N_c > N_f > \frac{68 N_c^2}{(16 + 20 N_c)}$$

16.5 > Nf > 8.05

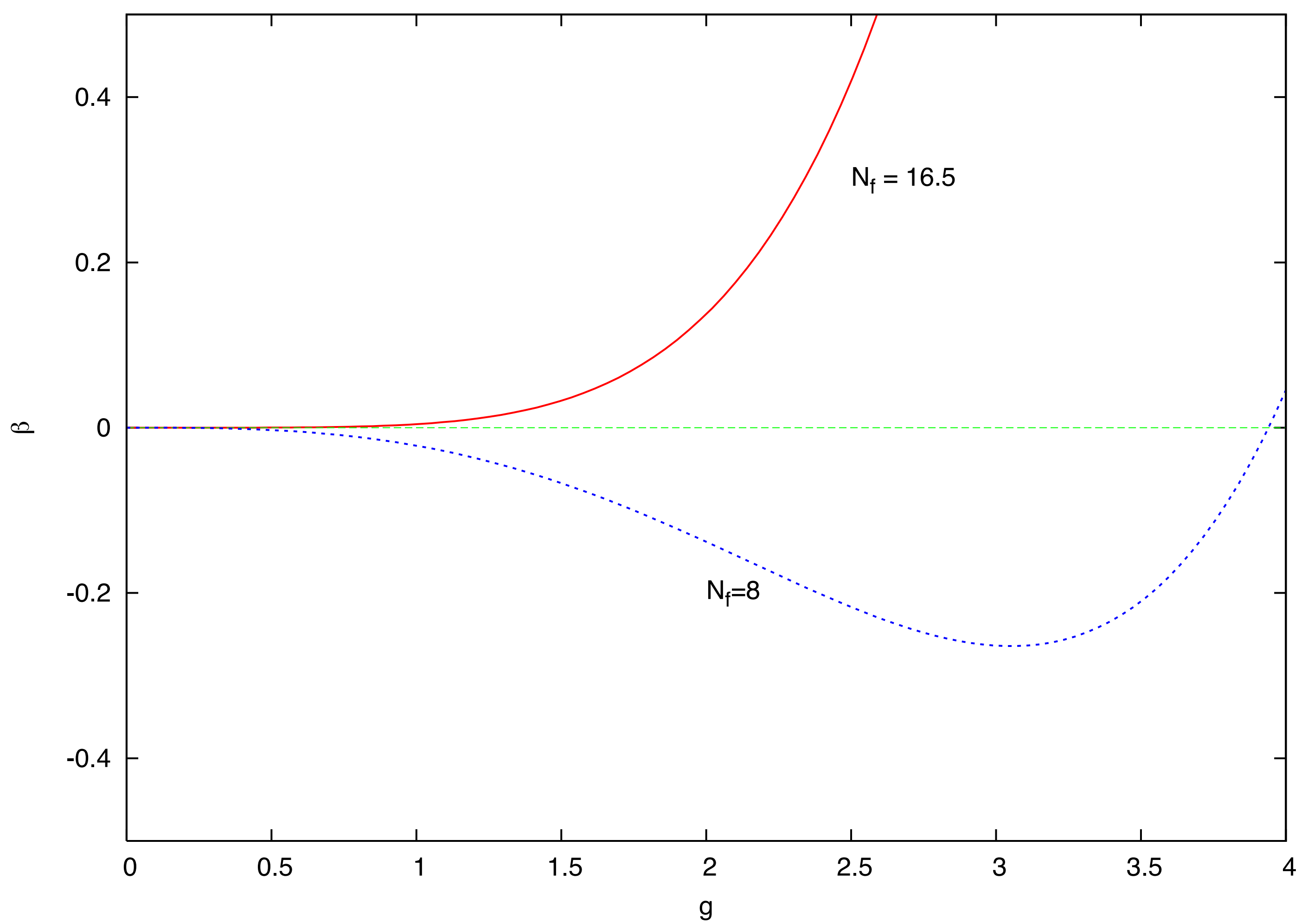
ore generally, since we really need the full form of beta.

Walking Technicolour

The Conformal Window

Banks and Zaks, NPB196, 189 (82)

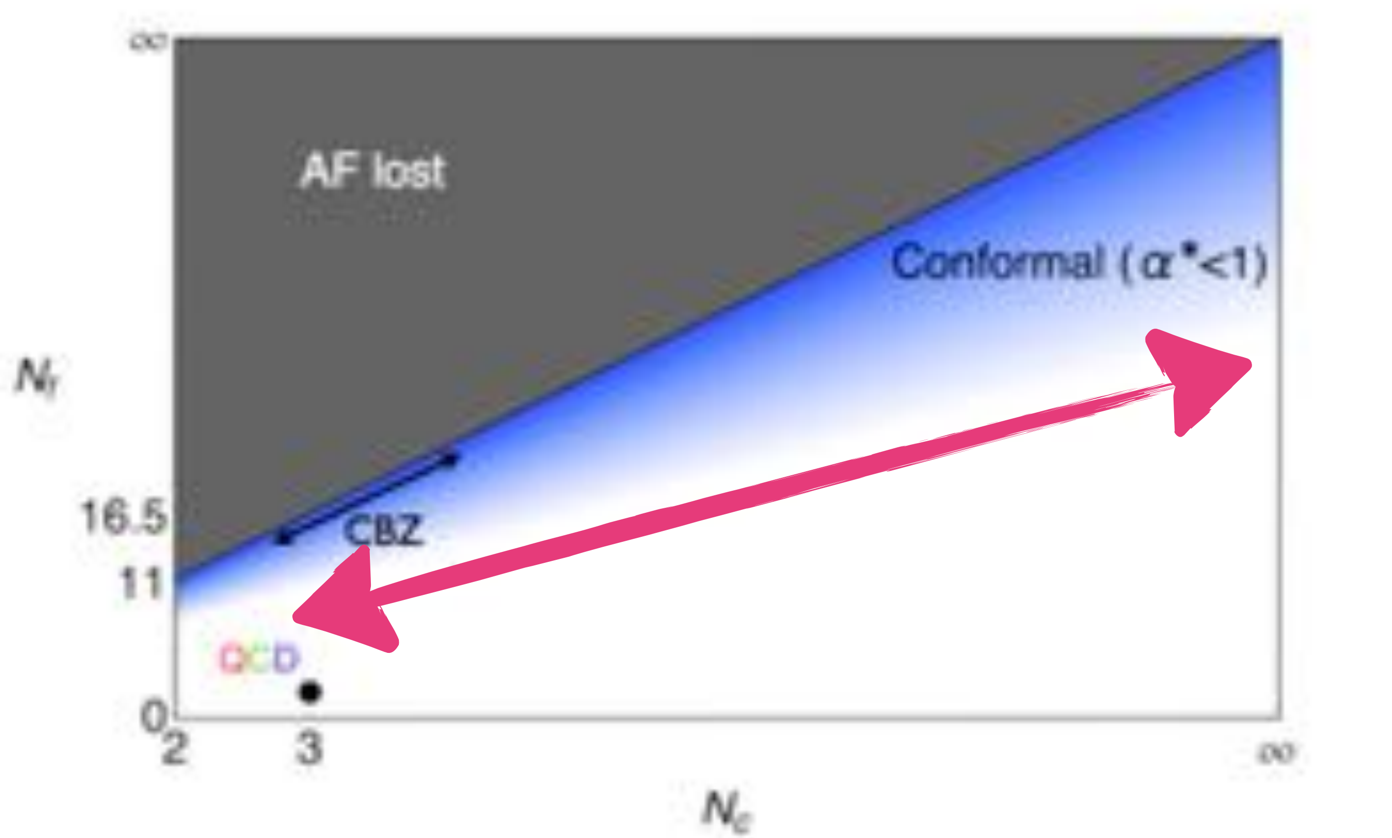
$$\text{For } \underbrace{N_f^c < N_f < N^{AF}}_{\text{the } \textit{conformal window}}$$



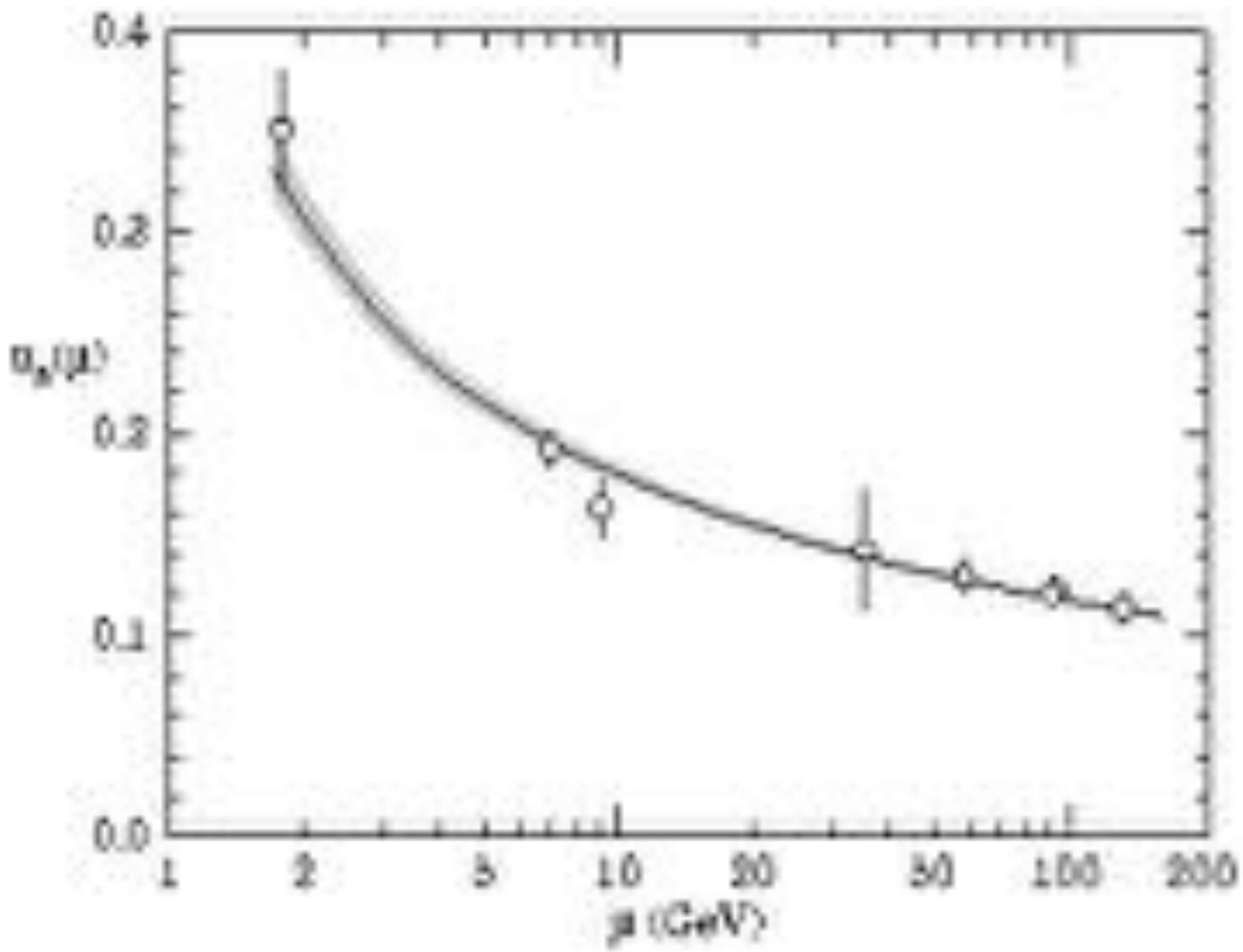
Walking Technicolour

The Conformal Window

For walking TC we want to sit just below the conformal window.

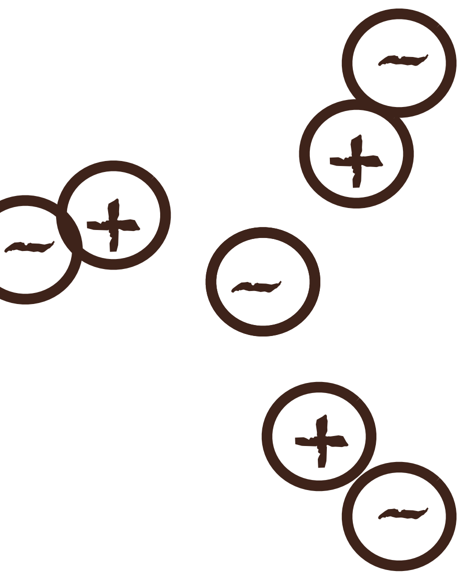


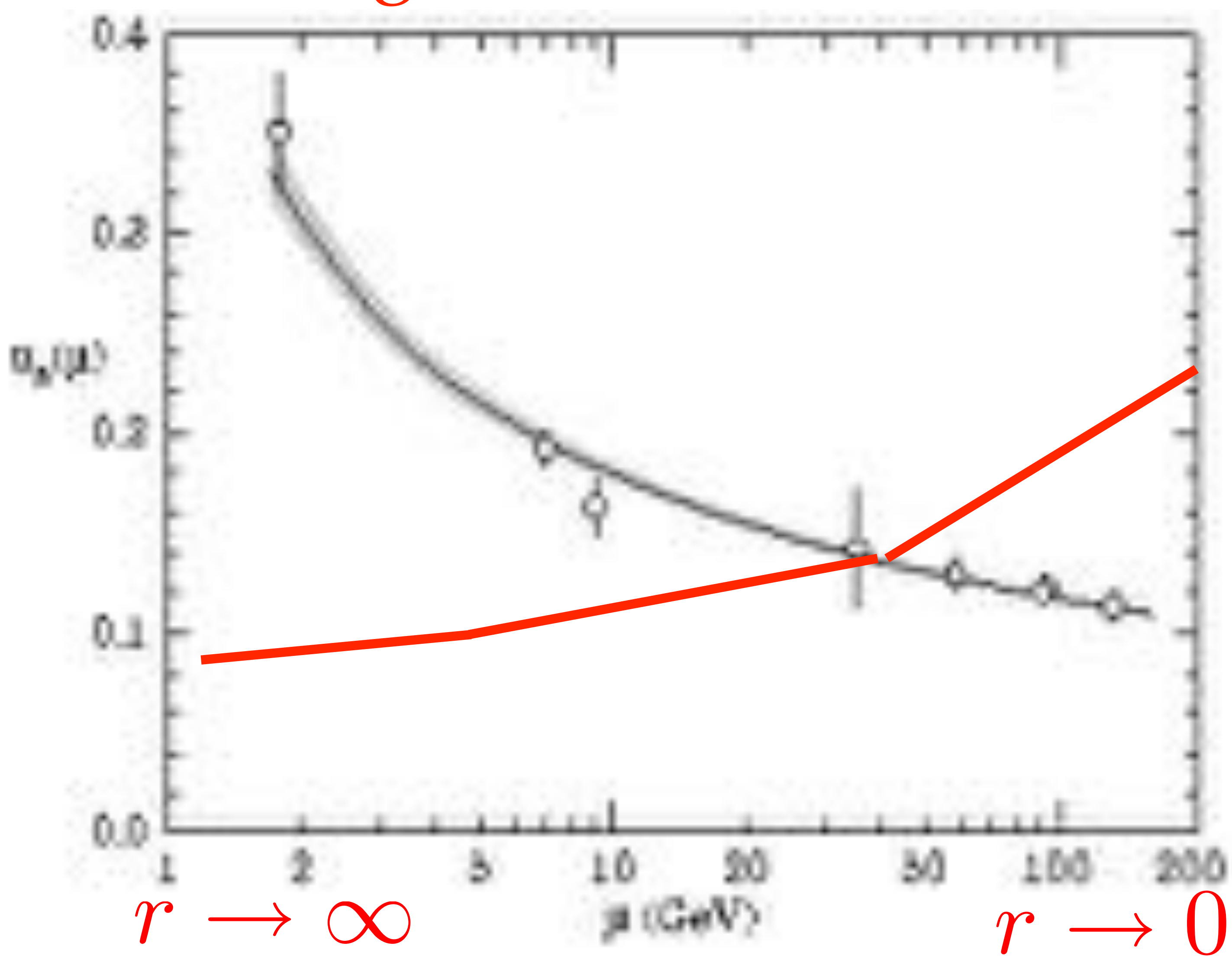
running coupling



running coupling

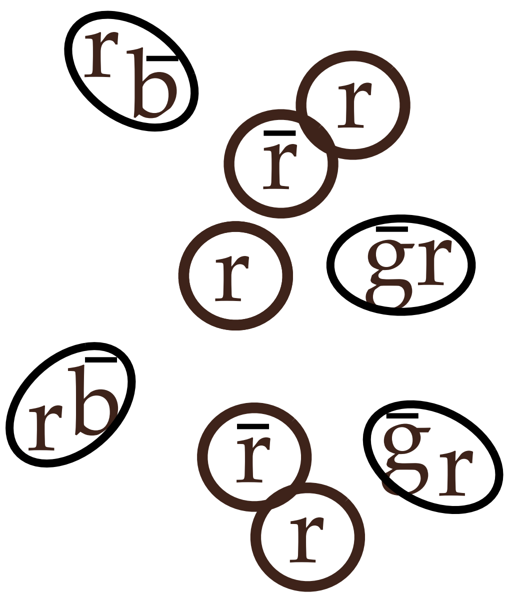
QED & charge screening

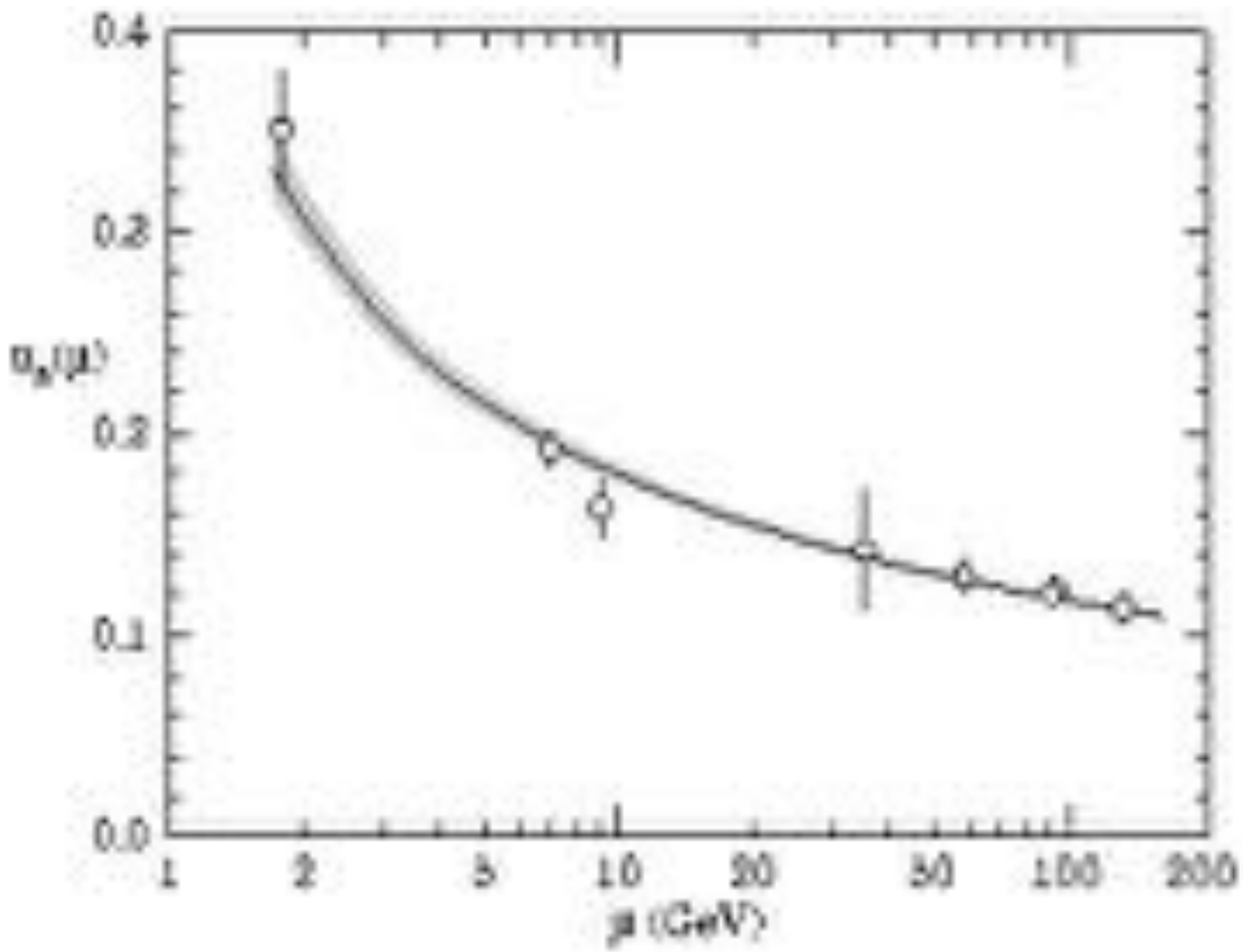




running coupling

QCD & charge antiscreening

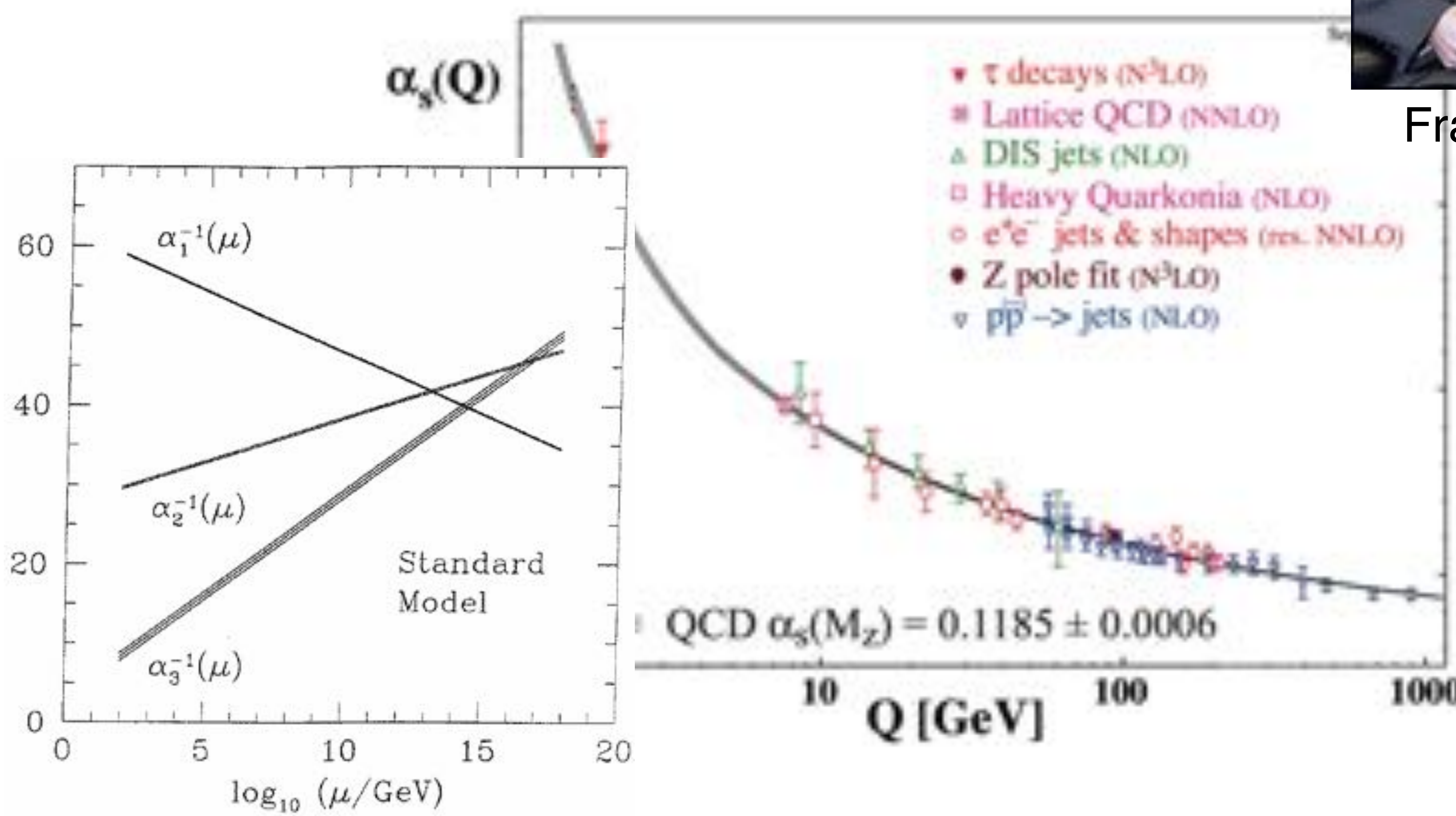




Properties: Asymptotic Freedom

QCD and anti-screening

Frank Wilczek
(1951-)



Coulomb Gauge

Why Coulomb Gauge?

- Hamiltonian approach is similar to CQM
- all degrees of freedom are physical, no constraints need be imposed
- [degree of freedom counting is important for T>0]
- T>0 chooses a special frame anyway
- no spurious retardation effects
- is renormalizable (Zwanziger)
- is ideal for the bound state problem
- very good for examining gluodynamics

Derivation

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\bar{\psi}T^a\gamma^\mu\psi A^a_\mu - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}$$

minimal coupling: $\partial_\mu \to D_\mu = \partial_\mu + igA_\mu$ $\qquad A_\mu = T^a A^a_\mu$

gauge group: $[T^a, T^b] = if^{abc}T^c$

gauge (Faraday) tensor: $T^a F^a_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu]$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$$

Coulomb Gauge

define the chromoelectric field:

$$E^i = F^{i0}$$

$$E^{ia} = -\dot{A}^{ia} - \nabla A^{0a} + g f^{abc} A^{0b} A^{ic}$$

define the chromomagnetic field:

$$B^i = -\frac{1}{2}\epsilon^{ijk} F_{jk}$$

$$\vec{B}^a = \nabla \times \vec{A}^a + \frac{1}{2} g f^{abc} \vec{A}^b \times \vec{A}^c$$

Coulomb Gauge

Equations of motion: $$\partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^{a\alpha})} = \frac{\partial \mathcal{L}}{\partial A^{a\alpha}}$$

$$\partial^\beta F^a_{\beta\alpha} = g j^a_\alpha + g f^{abc} F^b_{\alpha\mu} A^{c\mu}$$

Gauss's Law ($\alpha = 0$):

$$\nabla \cdot \vec{E}^a + g f^{abc} \vec{A}^b \cdot \vec{E}^c = g\rho^a_{(q)}$$

Introduce the adjoint covariant derivative

$$\vec{D}^{ab} = \delta^{ab}\nabla - g f^{abc} \vec{A}^c$$

$$\vec{D}^{ab} \cdot \vec{E}^b = g\rho^a_{(q)}$$

Coulomb Gauge

$$\vec{D}^{ab} \cdot \vec{E}^b = g\rho^a_{(q)}$$

resolve Gauss's Law:

$$\vec{E} = \vec{E}_{tr} - \nabla\phi \qquad \nabla \cdot \vec{E}_{tr} = 0 \quad \nabla \cdot \vec{A} = 0 \quad \nabla \cdot \vec{B} = 0$$

define the full colour charge density:

$$\rho^a = \rho^q_{(q)} + f^{abc}\vec{E}^b_{tr} \cdot \vec{A}^c$$

Use this in Gauss's Law to get:

$$-(\vec{D}^{ab} \cdot \nabla)\phi = g\rho^a$$

Coulomb Gauge

$$-(\vec{D}^{ab} \cdot \nabla)\phi = g\rho^a$$

Solve for ϕ:

$$\phi^a = -\frac{g}{\nabla \cdot \vec{D}}\rho^a$$

notice that this is a “formal” solution

We have two expressions for the divergence of E

$$\nabla \cdot \vec{E}^a = -\nabla \cdot D^{ab} A^{0b} = -\nabla^2 \phi^a$$

$$A^{0b} = \frac{1}{\nabla \cdot \vec{D}}(-\nabla^2)\frac{1}{\nabla \cdot \vec{D}} g\rho^b$$

Coulomb Gauge

$$H = \frac{1}{2}(E^2 + B^2) = \frac{1}{2}(E_{tr}^2 - \phi \underbrace{\nabla^2} \phi + B^2)$$

$$H_c = \frac{1}{2}\int d^3x d^3y \, \rho^a(x) K_{ab}(x, y; A) \rho^b(y)$$

$$K_{ab}(x, y; A) = \langle x, a | \frac{g}{\nabla \cdot \vec{D}} (-\nabla^2) \frac{g}{\nabla \cdot \vec{D}} | y, b \rangle$$

Coulomb Gauge

A complication due to the curved gauge manifold

$$\hat{H} = \frac{1}{2m} g^{-1/4}\, \hat{p}_i g^{ij} g^{1/2} \hat{p}_j g^{-1/4}$$

g = metric $\Longrightarrow$ $\mathcal{J} = \det(\nabla \cdot \vec{D})$ Faddeev-Popov determinant

$$H = \frac{1}{2} \int d^3x \, (\mathcal{J}^{-1} \vec{\Pi} \mathcal{J} \cdot \vec{\Pi} + \vec{B} \cdot \vec{B})$$

Inner product:

$$\langle \Psi | \Phi \rangle = \int DA \mathcal{J} \Psi^* \Phi$$

$$H \to \mathcal{J}^{1/2} H \mathcal{J}^{-1/2}$$

Coulomb Gauge

Schwinger, Kriplovich
Christ and Lee

$$H_q = \int d^3x\, \psi^\dagger(x)(-i\alpha\cdot\nabla + \beta m)\psi(x)$$

$$H_{YM} = \mathrm{tr}\int d^3x\,(\mathcal{J}^{-1}\vec{\Pi}\cdot\mathcal{J}\vec{\Pi} + \vec{B}\cdot\vec{B})$$

$$H_{qg} = -g\int d^3x\,\psi^\dagger(x)\alpha\cdot\vec{A}\psi(x)$$

$$H_C = \frac{1}{2}\int d^3x d^3y\,\mathcal{J}^{-1}\rho^a(x)K_{ab}(x,y;A)\mathcal{J}\rho^b(y)$$

$$\rho^a(x) = \psi^\dagger(x)T^a\psi(x) + f^{abc}\vec{\Pi}^b(x)\cdot\vec{A}^c(x)$$

$$K^{ab}(x,y;A) = \langle x,a|\frac{g}{\nabla\cdot\vec{D}}(-\nabla^2)\frac{g}{\nabla\cdot\vec{D}}|y,b\rangle$$

The Gribov Ambiguity

- $\nabla \cdot \vec{A} = 0$ does not uniquely specify the gauge field.

$$F_A[g] = \mathrm{tr} \int d^3x \, (\vec{A}^g)^2$$

Zwanziger

$$\vec{A}^g = g\vec{A}g^\dagger - g\nabla g^\dagger$$

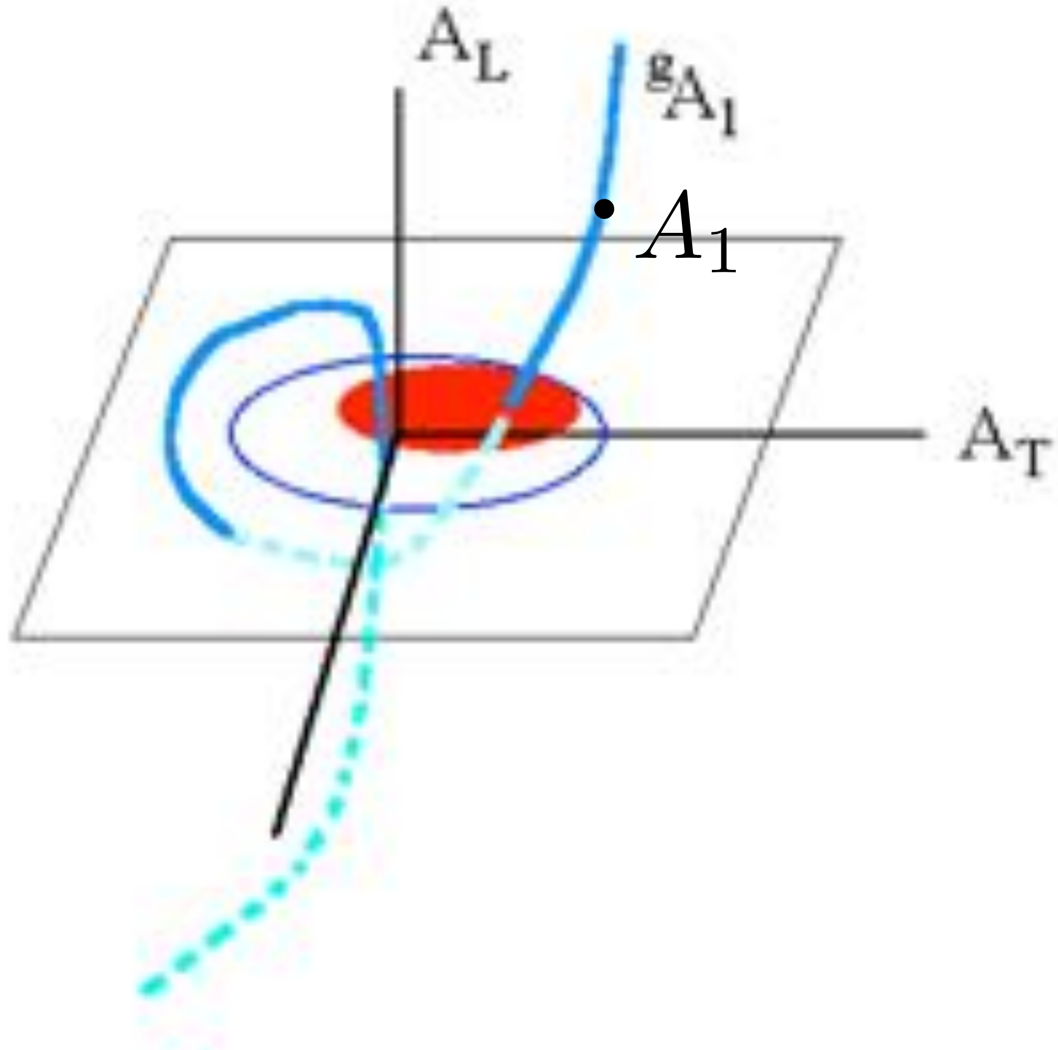
• The absolute minimum of F[g] fixes one field configuration on the gauge orbit. The set of such minima is the Fundamental Modular Region.

• The Faddeev Popov operator is positive definite in the FMR

The Gribov Ambiguity

Coulomb gauge and the Gribov problem

$$\det(\nabla \cdot D) = 0$$

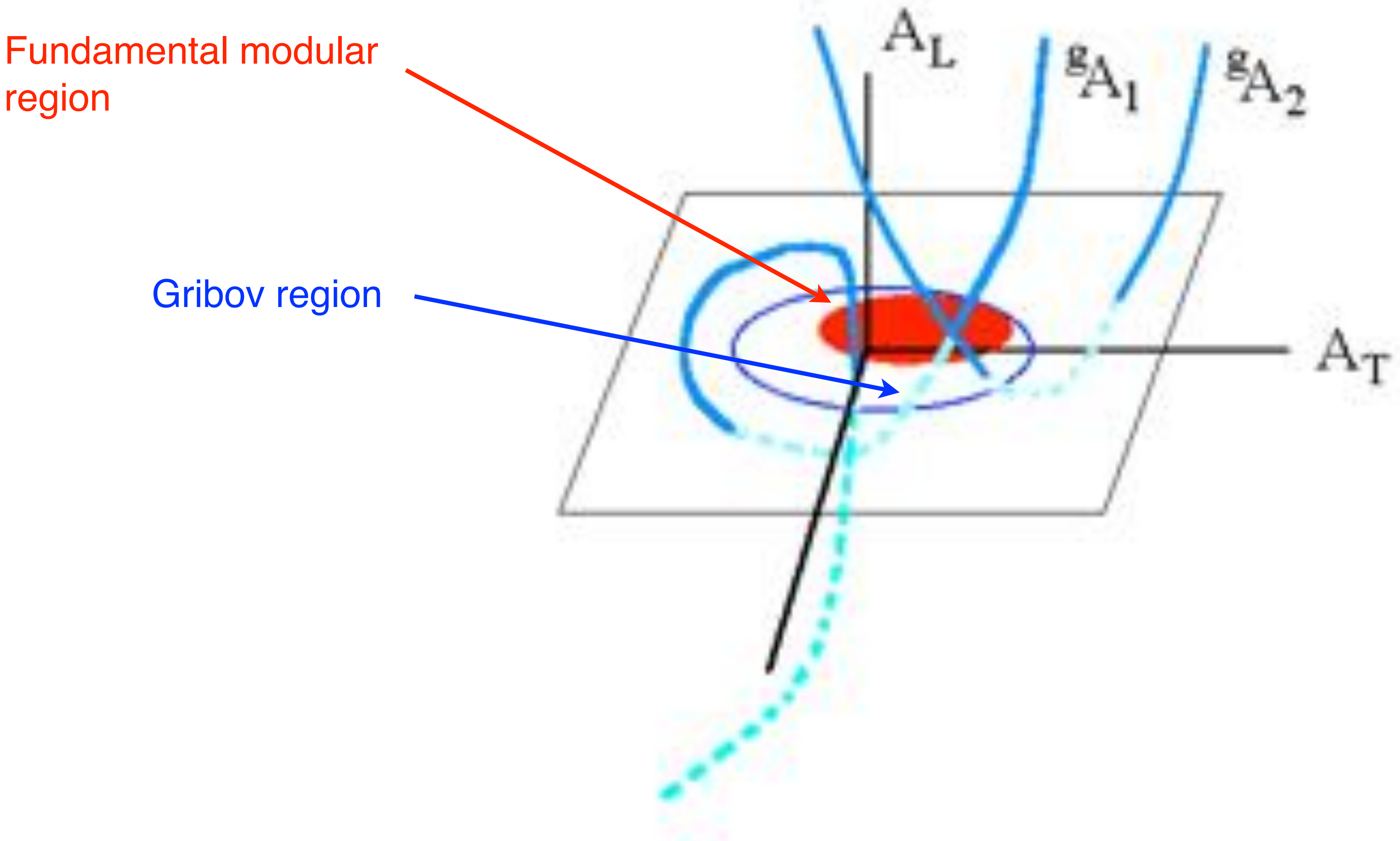


$$\nabla \cdot \vec{A}^a = 0$$

The Gribov Ambiguity

Coulomb gauge and the Gribov problem

$$\det(\nabla \cdot D) = 0$$

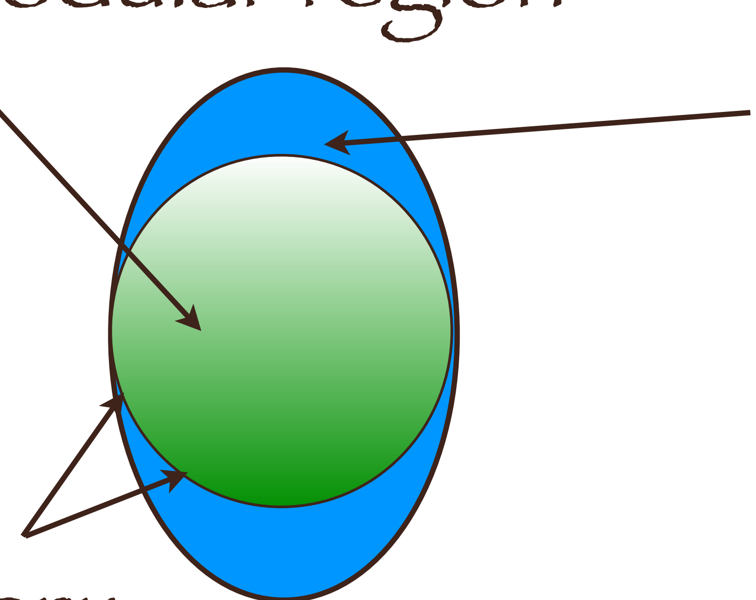


$$\nabla \cdot \vec{A}^a = 0$$

The Gribov Ambiguity

Coulomb gauge and the Gribov problem

Fundamental modular region



Gribov region $\det(\nabla \cdot D) = 0$

identify boundary configurations

FMR is convex

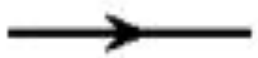
GR contains the FMR

FMR contains A=0

physics lies at the intersection of the FMR and GR

Zwanziger; van Baal

Feynman Rules

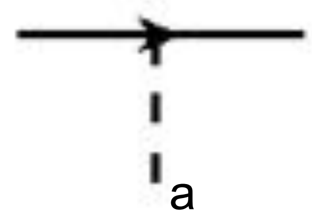


$$\frac{i}{\not{k} - m + i\epsilon}$$

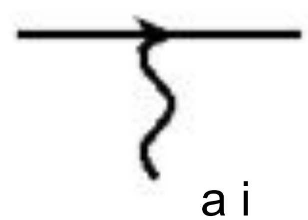
$$-i\frac{\delta^T_{ij}(k)\delta_{ab}}{k^2 + i\epsilon}$$

$$-i\frac{\mathbf{k}^2\delta^T_{ij}(k)\delta_{ab}}{k^2 + i\epsilon}$$

$$-\frac{i}{\mathbf{k}^2}\delta_{ab}$$



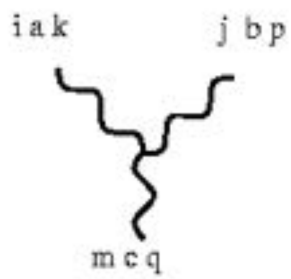
$$-ig\gamma^0 \mathrm{T}^a$$



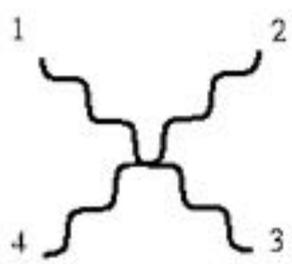
$$-ig\gamma^i \mathrm{T}^a$$

$$\boxed{\delta^{\mathbf{T}}_{\mathbf{ij}}(\mathbf{k}) = \delta_{\mathbf{ij}} - \hat{\mathbf{k}}_{\mathbf{i}}\hat{\mathbf{k}}_{\mathbf{j}}}$$

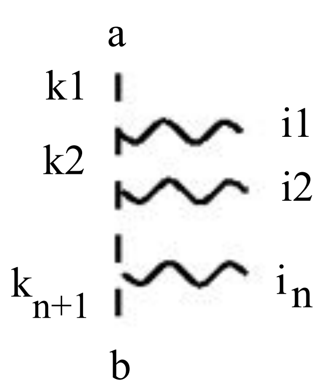
Feynman Rules



$$-g f^{abc}\left[\delta_{ij}(p-k)_m + \delta_{jm}(q-p)_i + \delta_{im}(k-q)_j\right]$$



$$-ig^2(1+P_{23}+P_{24})\left[f_{12m}f_{34m}\epsilon_{12n}\epsilon_{34n}\right]$$



$$(i)^n\,(n+1)g^n f_{ac_1a_1}f_{a_1c_2a_2}\ldots f_{a_{n-1}c_nb}\frac{k_2^{i_1}\ldots k_n^{i_n}}{\mathbf{k}_2^2\ldots\mathbf{k}_n^2}$$

Coulomb Gauge

$$H = \frac{1}{2}\int d\mathbf{x}\,(E^2+B^2) - \frac{1}{2}\int d\mathbf{x}d\mathbf{y}\, f^{abc}E^b(\mathbf{x})A^c(\mathbf{x})\langle \mathbf{x}a|\frac{g}{\nabla\cdot D}\nabla^2\frac{g}{\nabla\cdot D}|\mathbf{y}d\rangle f^{def}E^e(\mathbf{y})A^f(\mathbf{y})$$

$$K(\mathbf{x}-\mathbf{y};\mathbf{A})$$

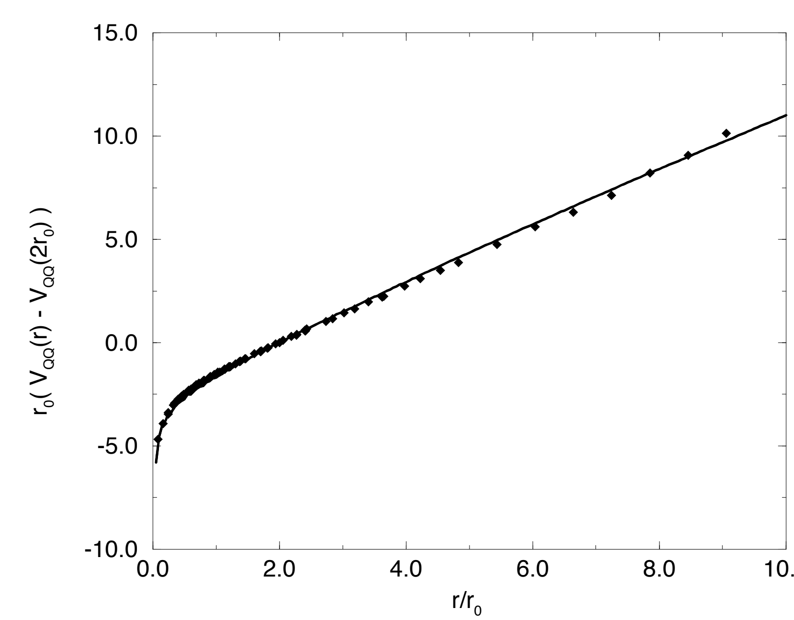
an instantaneous potential that depends on the gauge potential

K generates the beta function

K is renormalization group invariant

K is an upper limit to the Wilson loop potential

K is infrared enhanced at the Gribov boundary

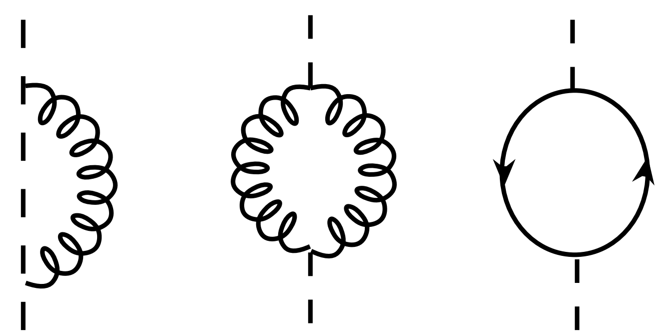


Szczepaniak & Swanson

Coulomb Gauge

Khriplovich Yad. F. **10**, 409 (69)

running coupling



$\frac{12}{3} N_c$ $\quad -\frac{1}{3} N_c$ $\quad -\frac{2}{3} N_f$

$$\mu \frac{dg(\mu)}{d\mu} = -\frac{\beta_0}{(4\pi)^2} g^3(\mu)$$

$$\alpha_s(\mu^2) = \frac{4\pi}{(11 - \frac{2}{3} n_f) \ln \mu^2 / \Lambda_{QCD}^2}$$

QCD - lattice gauge theory

lattice intro, Monte Carlo, SU(2)-Higgs

Lattice Gauge Theory — Review

Wegner (1971): gauged $Z(2)$ spin model

Wilson (1974): lattice gauge theory

Creutz (1980): numerical lattice gauge theory

F. Wegner, J. Math. Phys. 12, 2259 (1971)

K. Wilson, PRD10, 2445 (1974)

M. Creutz, PRD21, 2308 (1980)

Lattice Gauge Theory — Review

extra - in SE from d/dt^2 - D^2 -.> -(d/dtau^2 + D^2)

Euclidean field theory with a spacetime regu

$$x_0 \to -ix_4 \qquad S \to iS_E$$

$$\int D\phi\, \mathrm{e}^{iS[\phi]} \to \int D\phi_E\, \mathrm{e}^{-S_E[\phi_E]}$$

$$\phi(x_\mu) \to \phi(an_\mu) \to \phi_{n_\mu} \qquad D\phi \to \prod_{n_\mu} d\phi_{n_\mu}$$

Maps quantum field theory to a statistical mechanics problem

Lattice Gauge Theory — Review

Complex scalar field:

$$S_E = \int d^4x \, [(\partial_\mu \phi)^\dagger (\partial_\mu \phi) + m_0^2 \phi^\dagger \phi + \lambda_0 (\phi^\dagger \phi)^2] \qquad \rightarrow$$

$$\sum_n \left[\hat{\phi}_n^\dagger \hat{\phi}_n + \lambda(\hat{\phi}_n^\dagger \hat{\phi}_n - 1)^2 - \kappa \sum_\mu (\hat{\phi}_n^\dagger \hat{\phi}_{n+\mu} + \hat{\phi}_{n+\mu}^\dagger \hat{\phi}_n) \right]$$

$$\hat{\phi} = \frac{a}{\sqrt{\kappa}} \phi \qquad \lambda = \kappa^2 \lambda_0 \qquad \kappa = \frac{1 - 2\lambda}{2d + a^2 m_0^2}$$

Lattice Gauge Theory — Review

umklapp (over turn) is when the sum of two momneta in a BZ results in one outside of teh BZ (and then gets mapped back into it)

lattice symmetries

- discrete translation: momentum conservation up to $2\pi/a$

  This is ok since the IR EFT only deals with $p \ll 1/a$

- hypercubic symmetry: a subgroup of O(4).

  This is potentially a problem, but all hypercubic operators that are also not O(4) symmetric are irrelevant ➙ no problem for the IR EFT, i.e., O(4) symmetry is recovered as an accidental symmetry.

ex. hypercubic, not O(4), symmetric; dim 6: $\sum_\mu \phi^\dagger \partial_\mu^4 \phi$

Lattice Gauge Theory — Review

Gauge Fields

Consider the gauge covariant laplacian

$$\phi^\dagger D_\mu^2 \phi \to \phi^\dagger [\Delta^2 \phi - i A_\mu \frac{\phi_{n+\mu} - \phi_{n-\mu}}{a} + A^2 \phi_n]$$

discrete derivative ruins the gauge transformation

Lattice Gauge Theory — Review

Gauge Fields

one could press ahead and obtain gauge invariance up to $O(a^5)$

but terms like A^2 and $A^\mu A^\mu A^\mu A^\mu$ are induced in the IR

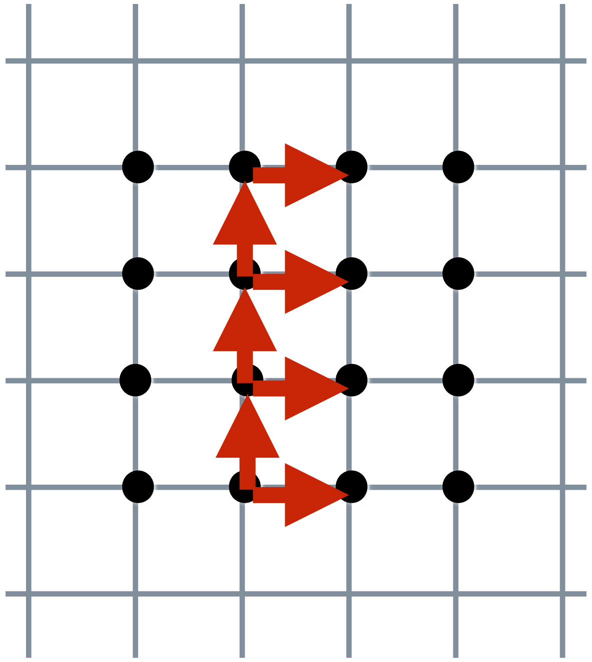
effective field theory — which is a disaster: the first is dimension 2; the second is not O(4) invariant; and tuning is required to eliminate these.

we should build in gauge invariance from the start!

Lattice Gauge Theory — Review

Gauge Fields

Think of the covariant derivative as a connection in an internal space. Then it is natural to regard the gauge field as a *link* variable.



$$U(x,\mu) \sim P\mathrm{e}^{ig\int_x^{x+\mu} A_\mu(z)dz^\mu}$$

Lattice Gauge Theory — Review

Gauge Fields

Gauge transformation:

$$\phi(x) \to \Lambda^\dagger(x)\phi(x)$$

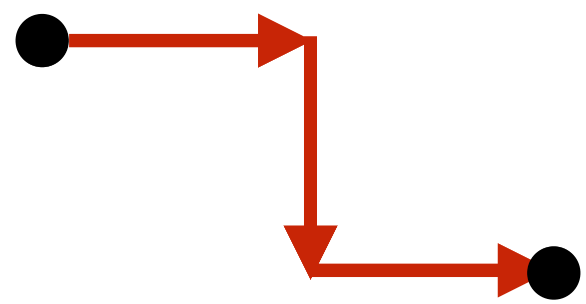
$$U(x,\mu) \to \Lambda^\dagger(x)U(x,\mu)\Lambda(x+\mu)$$

Thus the gauge covariant derivative is

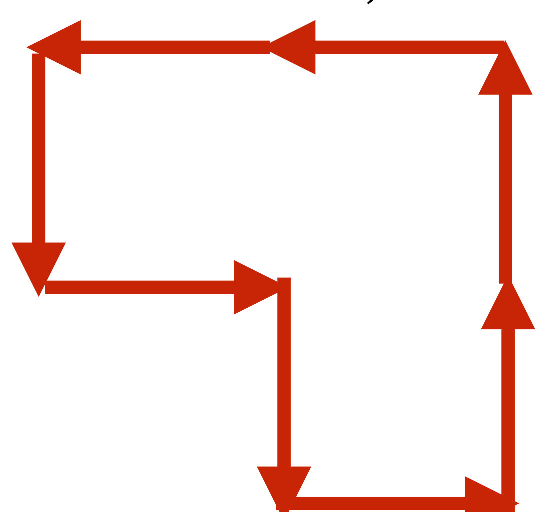
$$\frac{1}{2}(D_\mu\phi)^2 \to \frac{a^2}{2}\sum_{x,\mu}[2\phi^\dagger(x)\phi(x) - \phi^\dagger(x)\,U(x,\mu)\,\phi(x+\mu) - \phi^\dagger(x+\mu)\,U^\dagger(x,\mu)\,\phi(x)]$$

Lattice Gauge Theory — Review

Gauge Fields

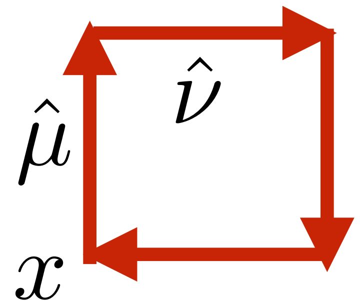


Closed loops of link variables form gauge invariant objects



The most local pure gauge object is a *plaquette*

$$\square_{\mu\nu}(x) \equiv$$



Lattice Gauge Theory — Review

Gauge Fields

$$\Box_{\mu\nu}(x) = 1 - ia^2 F^a_{\mu\nu} T^a - \frac{a^4}{2} F^a_{\mu\nu} F^b_{\mu\nu} T^a T^b + \ldots$$

$$S = \beta \sum_{x,\mu>\nu} \left(1 - \frac{1}{N} \mathrm{tr}\, \Box_{\mu\nu}(x)\right)$$

$$\beta = \frac{2N}{g_0^2}$$

Lattice Gauge Theory — Review

Fermions

- Represented as Grassmann fields, which must be explicitly integrated

$$\int DU\, D\bar{\psi}\, D\psi\, \mathrm{e}^{-S_F} = \int DU\ \det(\not{D}[U])\, \mathrm{e}^{-S_E}$$

(evaluating this determinant is extremely expensive!)

$$\int D\bar{\psi}\, D\psi\, DA_\mu \mathrm{e}^{iS[A]+i\int d^4x \bar{\psi}(i\not{\partial}-m-g\not{A})\psi + i\int \bar{\eta}\psi + i\int \bar{\psi}\eta}$$

$$= \int DA_\mu \det(i\not{\partial} - m - g\not{A})\mathrm{e}^{iS[A]-i\int \bar{\eta}(i\not{\partial}-m-g\not{A})^{-1}\eta}$$

Lattice Gauge Theory — Review

Fermions

- Attempts to simulate directly lead to the *fermion sign problem.*

(cf. flipping rows or columns in the determinant)

Lattice Gauge Theory — Review

Fermions

these zeroes are IN the Brillouin zone!

- fermion *doubler* problem

$$\bar{\psi}\gamma^\mu D_\mu \psi \to a^4 \sum_{x,\mu} \bar{\psi}(x)\gamma^\mu \left(U(x,\mu)\psi(x+\mu) - U^\dagger(x-\mu,\mu)\psi(x-\mu) \right)/(2a)$$

$$S^{-1}(p) = m + i\sum_\mu \gamma^\mu \frac{1}{a}\sin(ap_\mu)$$

NB WIlson fermionds break chiral symmetry (hence multiplicative mass renormalization) and recovering it in the IR requires careful tuning

Zeros at $p_\mu = (0,0,0,0)$ $p_\mu = (0,\pi/a,0,0)$ $\ldots$

i.e., there are 2^4 'species' of fermions.

Lattice Gauge Theory — Review

Fermions

- massless fermion problem

(removing the doublers necessarily breaks chiral symmetry)

- chiral fermion problem

Weyl spinor => 8 LH + 8 RH Weyl spinors.

(chiral gauge theories cannot be placed on the lattice)

Lattice Gauge Theory — Review

called the “Haar measure” — is gauge invariant and integrates over the gauge manifold

Gauge Fields – the measure

$$Z = \int DU \mathrm{e}^{-S_E} \qquad DU = \prod_{x,\mu} dU(x,\mu)$$

U(1)

$$U(x,\mu) = \mathrm{e}^{i\theta(x,\mu)}$$

$$dU = \frac{d\theta}{\pi}$$

SU(2)

$$U(x,\mu) = \mathrm{e}^{i\vec{a}\cdot\vec{\sigma}/2} = a_0 + i\vec{\sigma}\cdot\vec{a}$$

$$dU = \frac{1}{\pi^2}\delta(1-a^2)d^4a$$

Lattice Gauge Theory — Review

Gauge Fields – the measure

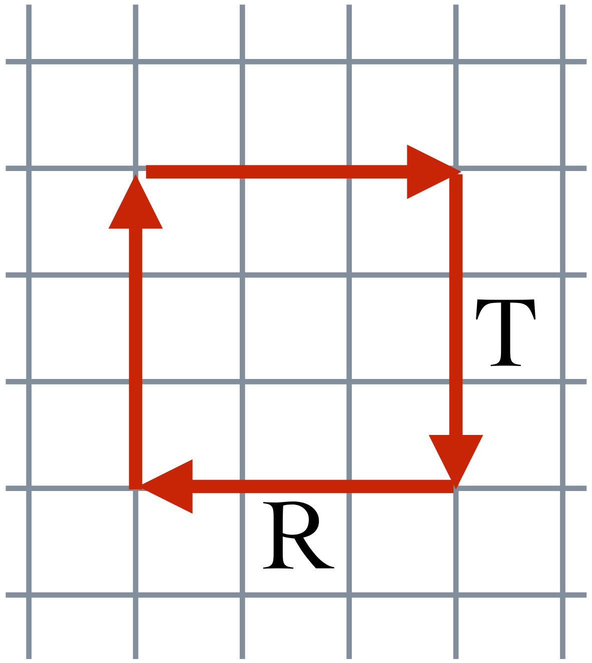
$$dU = |\mathrm{det} g|^{1/2}\, d\vec{\alpha}$$

g is the metric on the group manifold

$$g_{AB} = \mathrm{tr}[U^\dagger\,(\partial_A U)\,U^\dagger(\partial_B U)]$$

Lattice Gauge Theory — Review

Gauge Fields – the Wilson loop



Work in axial gauge:

$$A_{\hat{t}}(x) = 0,\ \ U(x,\hat{t}) = 1$$

$$\langle \mathrm{tr}\Box(R,T)\rangle = Z^{-1}\int DU\,\Psi^\dagger(R,T)\,U(0\to R,T)\,\Psi(0,T)\cdot\Psi^\dagger(R,0)\,U(0\to R,0)\,\Psi(0,0)\mathrm{e}^{-S_E}$$

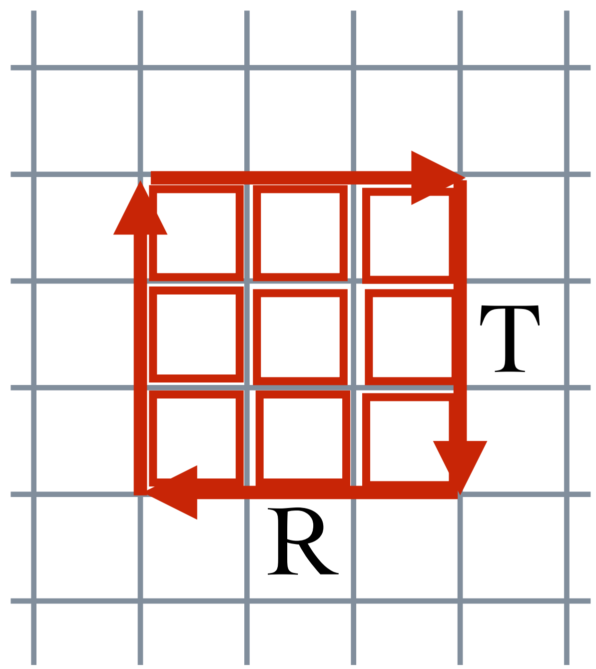
Make the spectral decomposition

$$\langle \mathrm{tr}\Box(R,T)\rangle = \sum_n |\langle M(0)|n\rangle|^2 \mathrm{e}^{-E_n(R)T}$$

$$\boxed{\langle \mathrm{tr}\,\Box(R,T)\rangle \underset{T \gg R}{\longrightarrow} \mathrm{e}^{-V(R)T}}$$

Lattice Gauge Theory — Review

Gauge Fields – the Wilson loop



Evaluate in the strong coupling (small beta) limit

- $$\int dU\, U(x,\mu) = 0$$

- $$\int dU(x,\mu)\, [U(x,\mu)]_{ij}\, [U^\dagger(y,\nu)]_{k\ell} = \frac{1}{N}\delta_{xy}\delta_{\mu\nu}\delta_{i\ell}\delta_{jk}$$

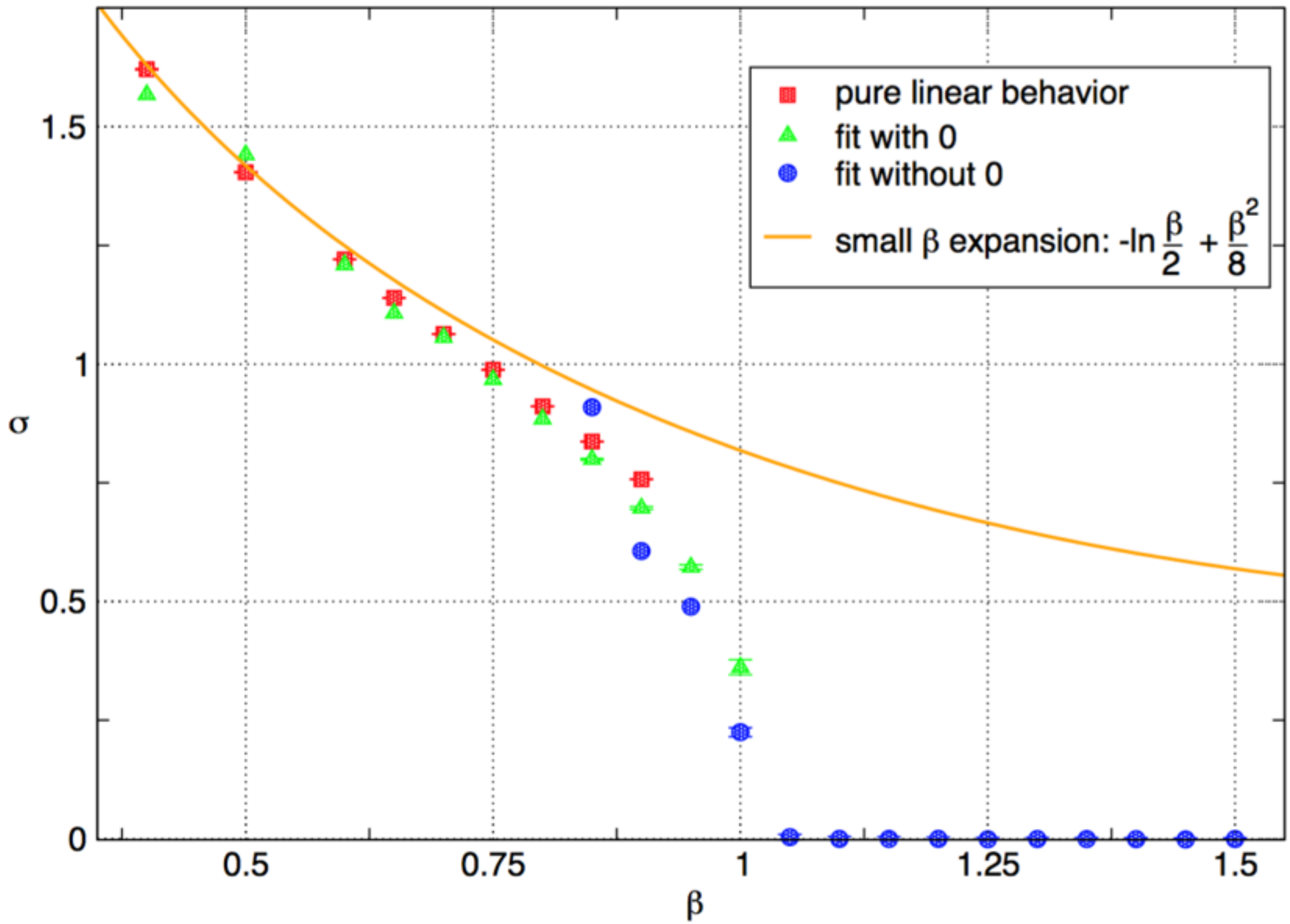
$$\langle \mathrm{tr}\,\Box(R,T)\rangle = \left(\frac{2N}{g_0^2}\right)^{RT} = \mathrm{e}^{-\log(g_0^2/2N)RT}$$

This is an *area law* – colour confinement!

Lattice Gauge Theory — Review

Note that by the same argument (compact) QED is also confining, but it is separated from the continuum by a first order phase transition (to the massless photon phase).

There is a line of phase transitions in 4d SU(N), but it ends, and the strong coupling and weak coupling regimes are smoothly connected. Thus SU(N) gauge theory is confining.



string tension U(1) 4d

Lattice Gauge Theory

Monte Carlo evaluation of the integral

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int DA_\mu \mathcal{O}[A, M^{-1}] \det(M[A]) \mathrm{e}^{-S_E[A]}$$

important shift to Euclidean space!

$$\langle \mathcal{O} \rangle = \sum_{\{A\}} \mathcal{O}[A, M^{-1}]$$

$$\{A\} \leftarrow \frac{\det(M[A]) \mathrm{e}^{-S_E[A]}}{\int DA_\mu \det(M[A]) \, \mathrm{e}^{-S_E[A]}}$$

warnings: autocorrelation, critical slowing down, determinant

Lattice Gauge Theory

compute a hadron mass...

$$C(t) = \langle 0|\mathcal{S}^\dagger(t)\mathcal{S}(0)|0\rangle \qquad \mathcal{S} = \bar{\psi}\gamma_5\psi(x,0)$$

$$C(t) = \langle 0|\mathrm{e}^{iHt}\mathcal{S}^\dagger(0)\mathrm{e}^{-iHt}\mathcal{S}(0)|0\rangle$$

$$C(t) = \langle 0|\mathrm{e}^{iHt}\mathcal{S}^\dagger(0)\mathrm{e}^{-iHt}\sum_n |n\rangle\langle n|\mathcal{S}(0)|0\rangle$$

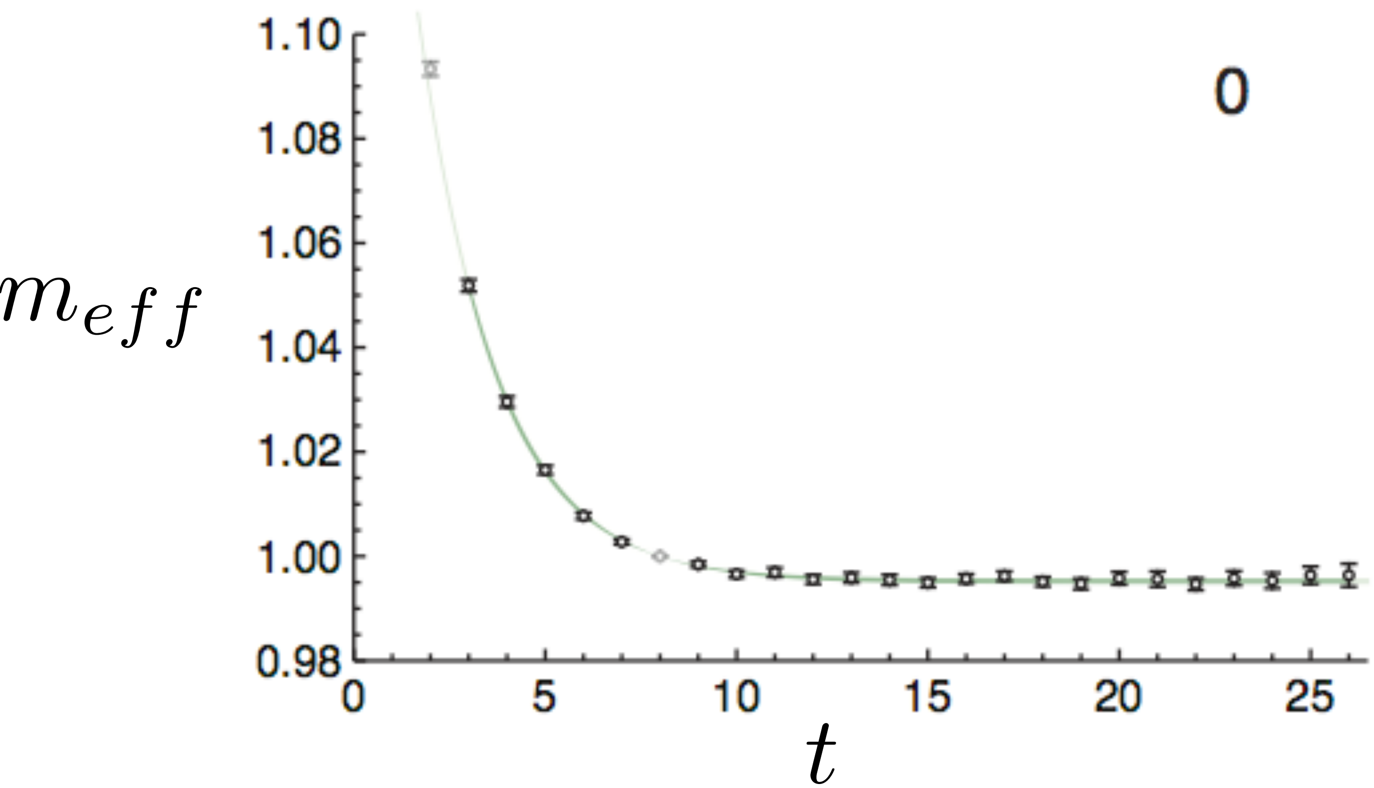
$$C(t) = \sum_n \mathrm{e}^{-i(E_n - E_0)t}|\langle n|\mathcal{S}(0)|0\rangle|^2$$

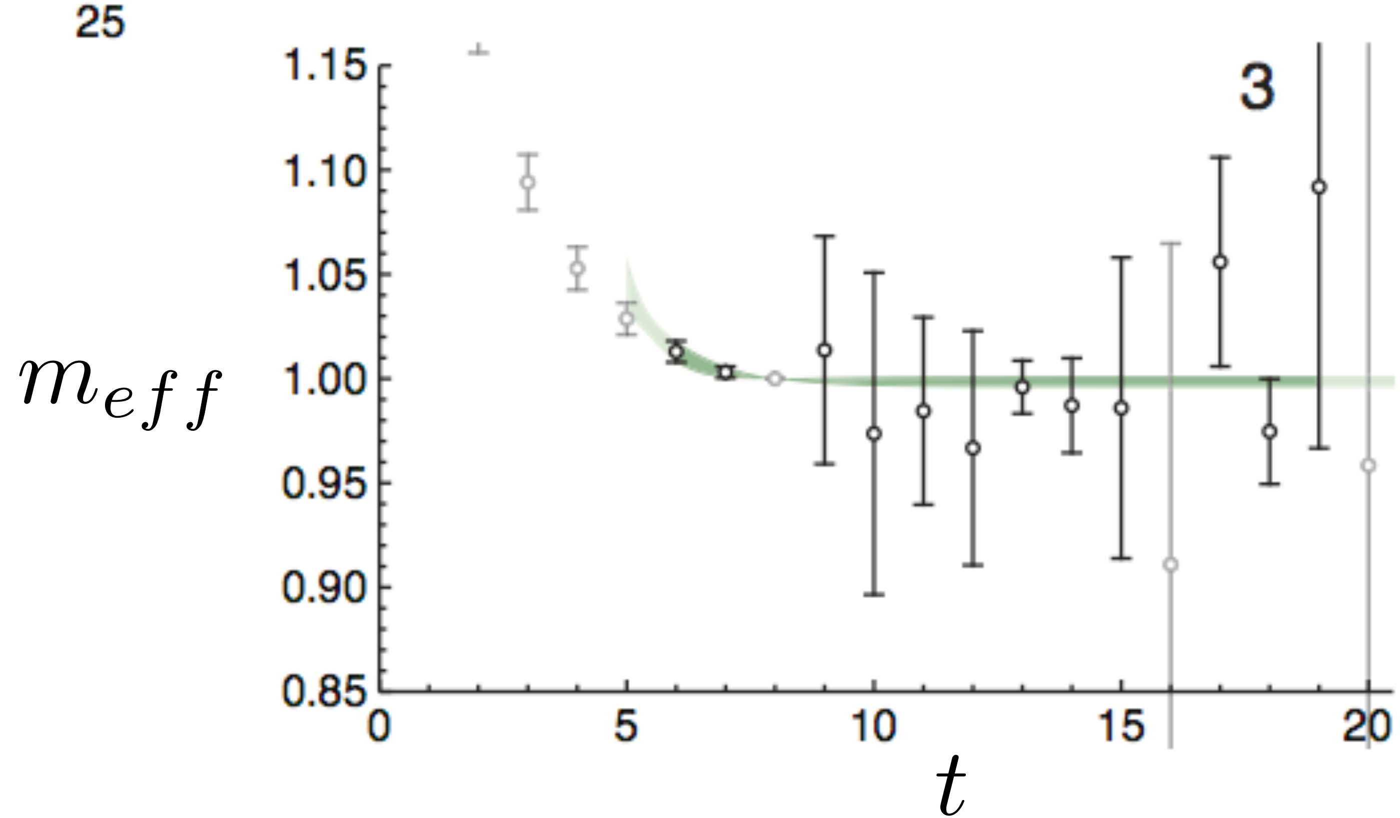
$$m_{eff} = \log\frac{C(t)}{C(t+1)}$$

Lattice Gauge Theory

one seeks 'plateaus'

warnings: how are these defined?, close levels





Lattice Gauge Theory

renormalise

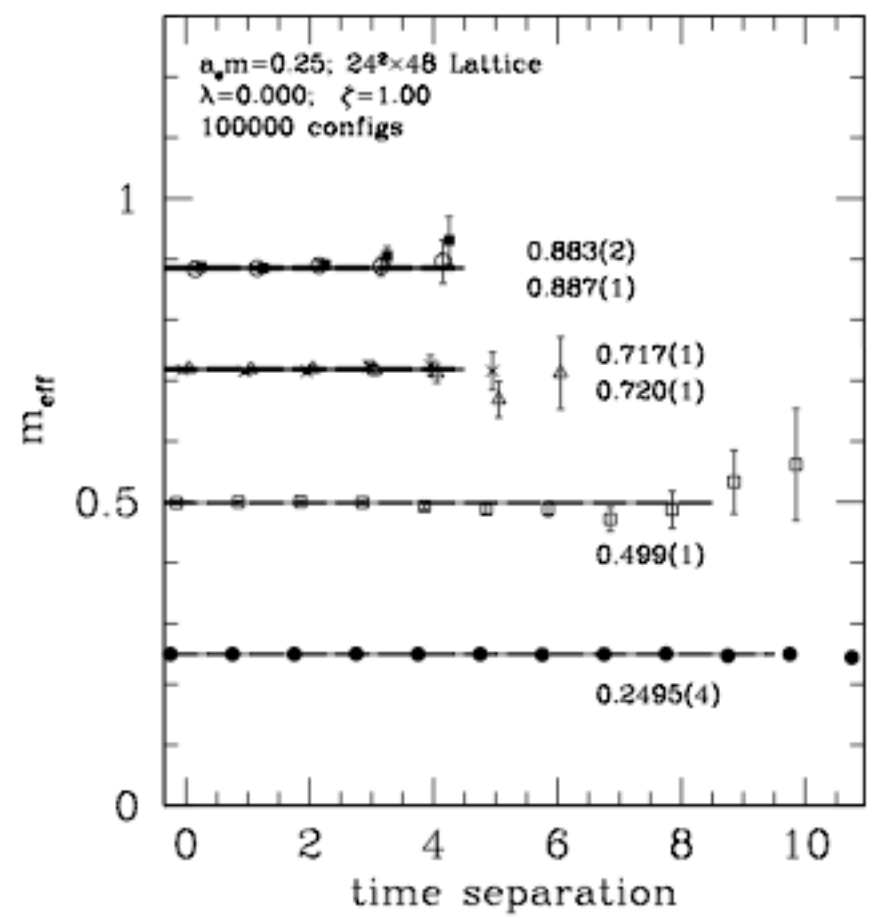
- (for m=0) select the coupling, g
- work in units of the lattice spacing, a
- compute a physical quantity, such as m_N. Set a such that $m_N = 0.94$ GeV.
- This gives $a(g)$ for one point. Repeat to trace out $a(g)$, or $g(a)$. Attempt to take limit a->0 and V->infinity.

Lattice Gauge Theory

<u>more warnings:</u>

- signal/noise: good interpolators!
- finite density
- light cone correlations
- systems with many scales (D_s)
- statistics/operators/correlators
- highly excited states
- unstable states

warnings:
how is a plateau defined?
closely spaced levels?



Lattice Gauge Theory — Review

continuum limit

umklapp (over turn) is when the sum of two momneta in a BZ results in one outside of teh BZ (and then gets mapped back into it)

- The lattice spacing has been scaled out of the problem. It is recovered upon renormalisation.

  cf. $\hat{m}(\beta) = a(\beta) m_{\rm phys} \;\rightarrow\; \beta(a)\ [g_0(a)]$

  Or can extract ratios as $\beta(a) \rightarrow \infty$

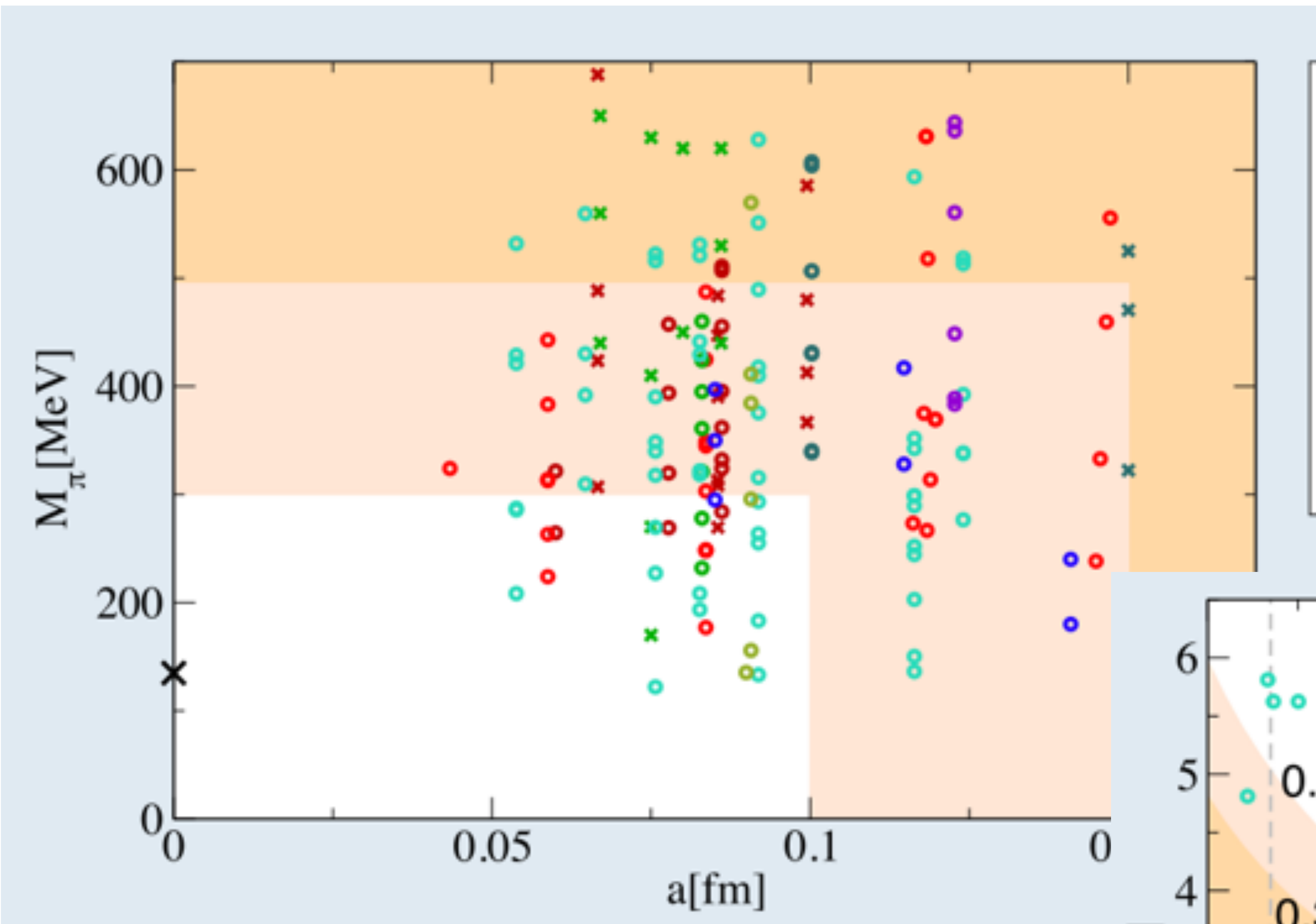
- There should be no phase transition if one wants to obtain strong coupling continuum physics.

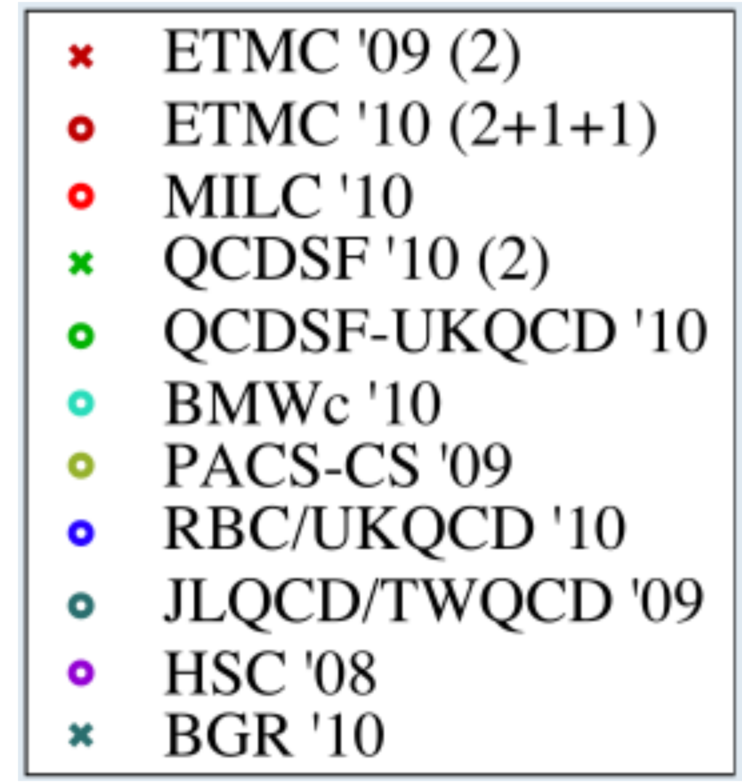
- The system correlation length should be large wrt the lattice spacing

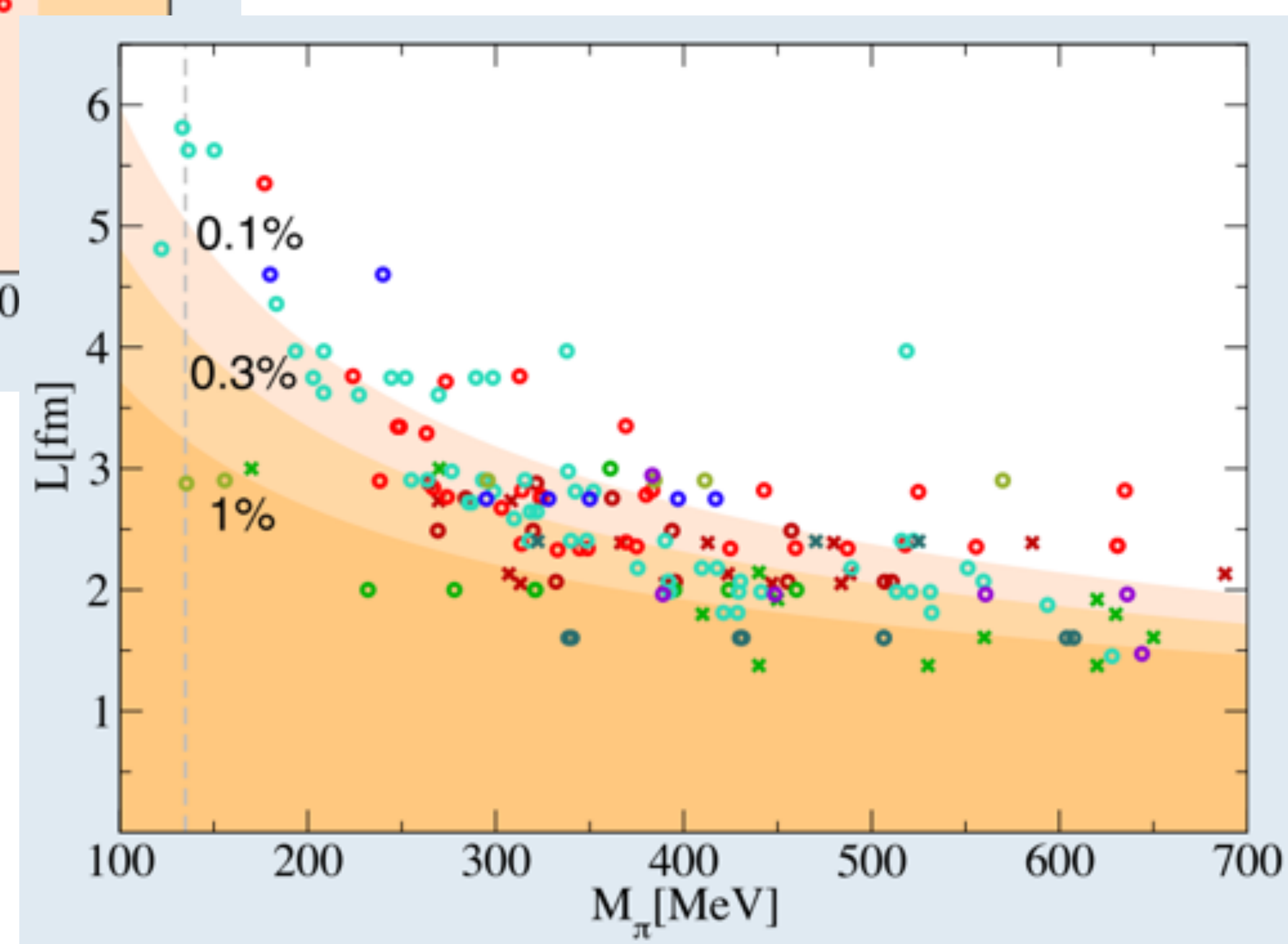
  $$\xi/a \gg 1 \qquad a m_{\rm gap} \ll 1$$

Lattice Gauge Theory — Review

lattices



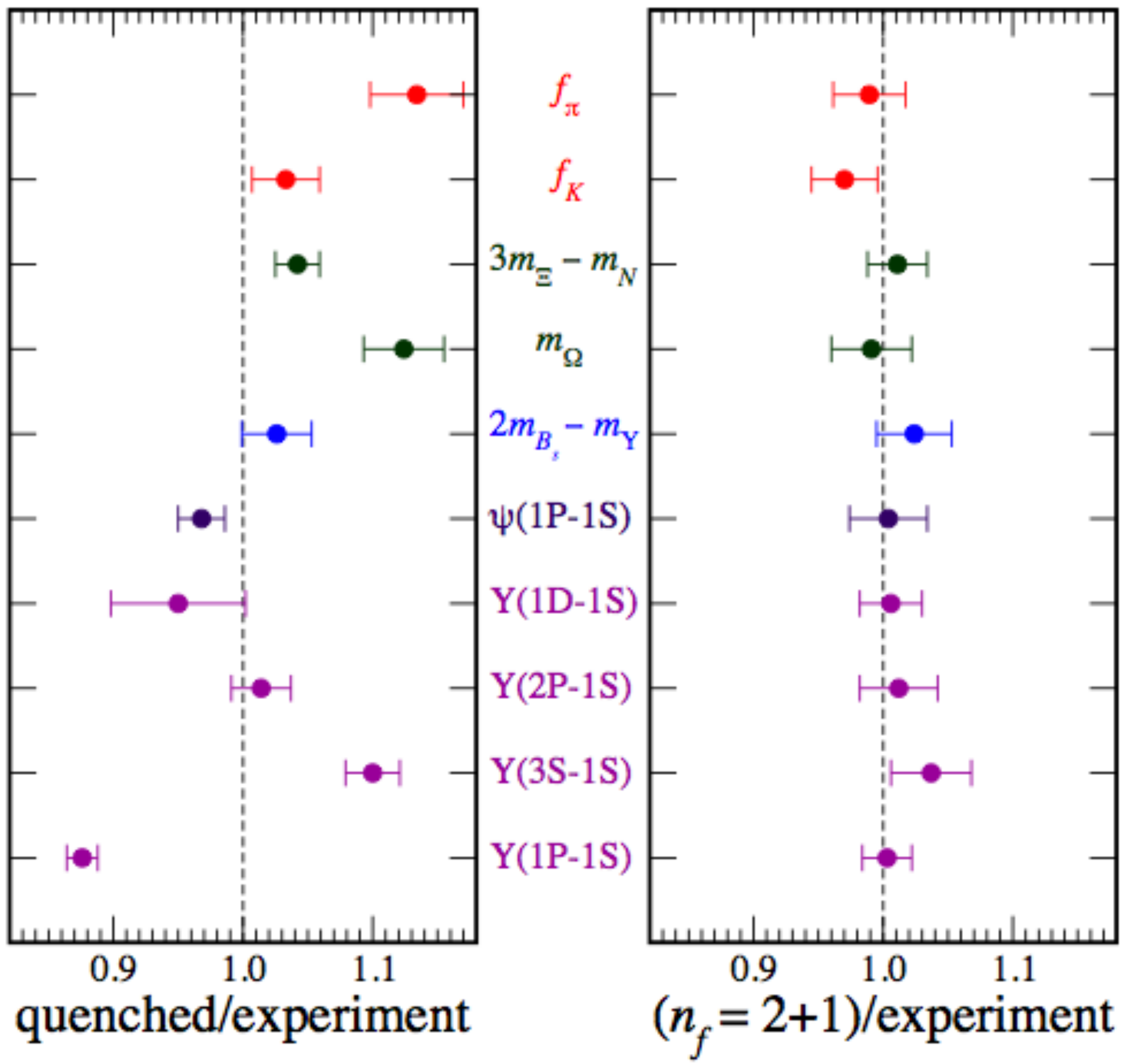




C Hoelbling,
PoS (LATTICE2010) 011,
arXiv:1102.0410.

Lattice Gauge Theory — Review

unquenching



C. T. H. Davies et al., PRL 92, 022001, (2004)

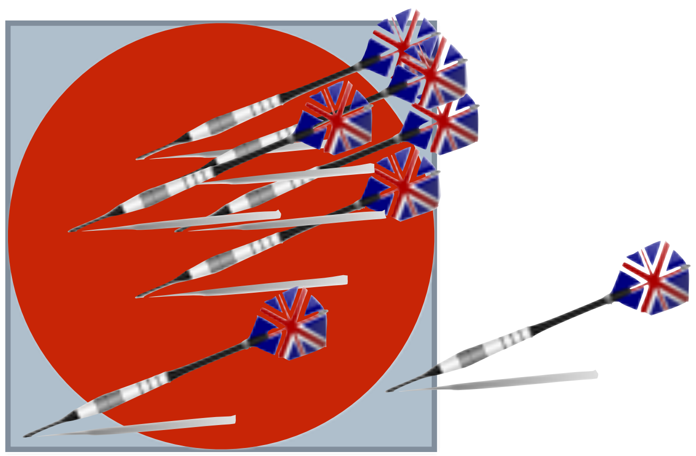
(input: $m_\pi,\ m_K,\ m_{\Upsilon(2S)} - m_{\Upsilon(1S)} \to \beta,\ (m_u + m_d)/2,\ m_s$)

SU(2)-Higgs

the importance of importance sampling!

V_n (R=1) is rapidly driven to zero!

Monte Carlo



$$V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

importance sampling

$$I = \int f(x) \qquad I = \frac{\int \rho(x)\, g(x)}{\int \rho(x)}$$

$$I = \frac{1}{N}\sum_i g(\hat{x}_i) + O\left(\frac{1}{\sqrt{N}}\right) \qquad \hat{x} \leftarrow \frac{\rho(x)}{\int \rho(x)}$$

SU(2)-Higgs

detailed balance guarantees the fiixed point exists:

P(U -> U') C(U) = P(U' -> U) C(U')

Monte Carlo

To "throw darts" one generates a *Markov chain* of field configurations:

$$\{U(x,\mu),\varphi(x)\}_1 \underset{\mathcal{P}}{\longrightarrow} \{U(x,\mu),\varphi(x)\}_2 \underset{\mathcal{P}}{\longrightarrow} \ldots$$

- $$\int DU'\,\mathcal{P}(U\to U') = 1$$ (& memoryless stochastic process)
- $$\mathcal{C}[U] = \int DU'\,\mathcal{P}(U'\to U)\,\mathcal{C}[U']$$ (a fixed point probability density exists)

$$\mathcal{C}[U] = \frac{\exp(-S[U])}{\int DU'\,\exp(-S[U'])}$$

- ergodicity

SU(2)-Higgs

b is the su(2) four-vector representing B
(actually, not normalized)

Monte Carlo — scalar field

Form $B(x) = \kappa \sum_\mu [U(x,\mu)\,\phi(x+\mu) + U^\dagger(x-\mu)\,\phi(x-\mu)]$

Then $S_\phi = \sum_x [(\phi(x) - b(x))^2 + \lambda(\phi(x)^2 - 1)^2 - b^2(x)]$

thus local actions are important!

heat bath

Seek a new configuration via $\mathcal{P} \sim dP(\phi) \sim \mathrm{e}^{-S_E(\phi)}\, d^4\phi$

Metropolis

Propose a new configuration; accept it if the action is lowered; otherwise accept it conditionally with probability $\min(1, \exp[V(\phi) - V(\phi')])$

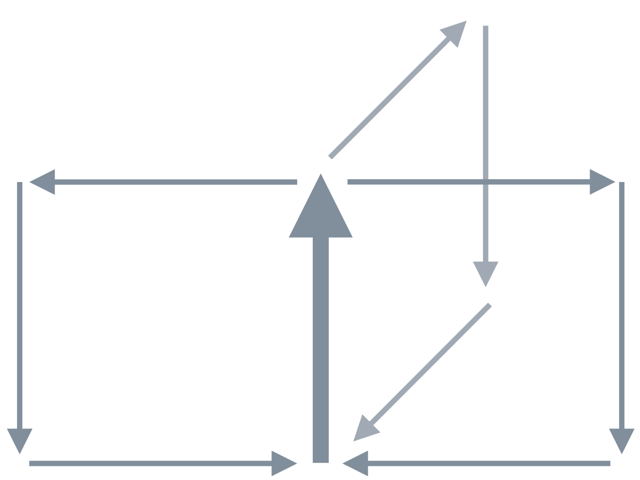
SU(2)-Higgs

Monte Carlo – gauge field

heat bath

Consider a single link $S_U|_{x\mu} = -\frac{\beta}{2}\,\mathrm{tr}\,U(x,\mu)W^\dagger(x,\mu)$

$$W(x,\mu) = \frac{2\kappa}{\beta}\phi(x)\phi^\dagger(x+\mu) + \sum_{\nu\neq\mu} U(x\,\nu)U(x+\nu,\mu), U^\dagger(x+\mu,\nu) + \sum_{\nu\neq\mu} U^\dagger(x-\nu,\nu)U(x-\nu,\mu), U^\dagger(x-\nu+\mu,\nu)$$



SU(2)-Higgs

Monte Carlo – gauge field

heat bath

And update:

$$W = r\hat{W}$$

$$\hat{W} \in SU(2)$$

$$dP(U) \sim \mathrm{e}^{\frac{\beta}{2}\,\mathrm{tr}(UW^\dagger)}\, dU$$

$$dP(U\hat{W}) \sim \mathrm{e}^{\frac{\beta r}{2}\,\mathrm{tr}(U)}\, dU$$

$$dP(a_\mu) \sim \sqrt{1 - a_0^2}\mathrm{e}^{\beta r a_0}\, \delta(1 - \vec{a}^2)\, da_0\, d\vec{a}$$

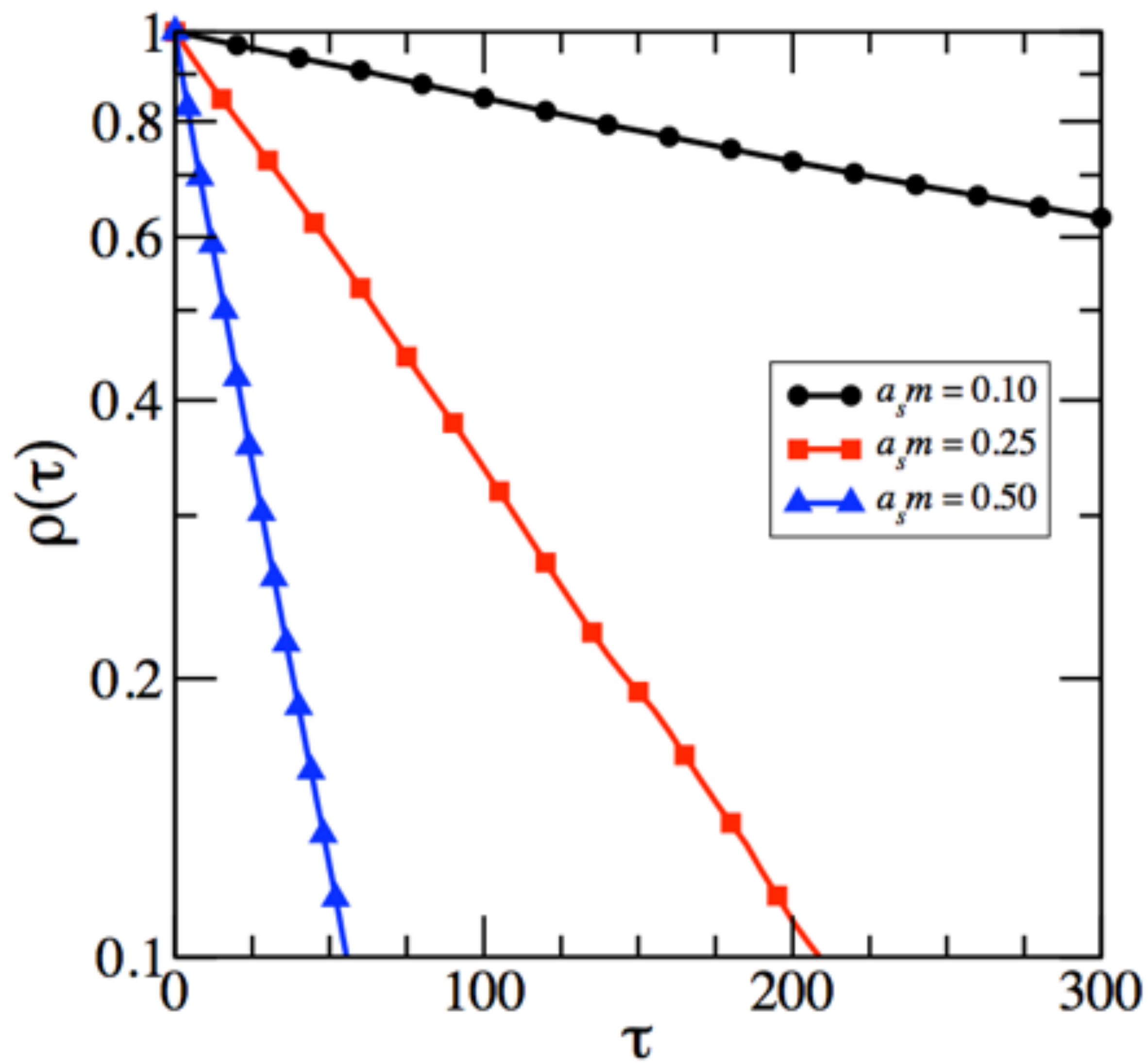
$$U \to U(a)\hat{W}$$

SU(2)-Higgs

Monte Carlo – autocorrelation

$$\rho(\tau) = \frac{\sum_t O(t+\tau)O(t)}{\sum_t O^2(t)}$$

(t and τ are "algorithmic time")



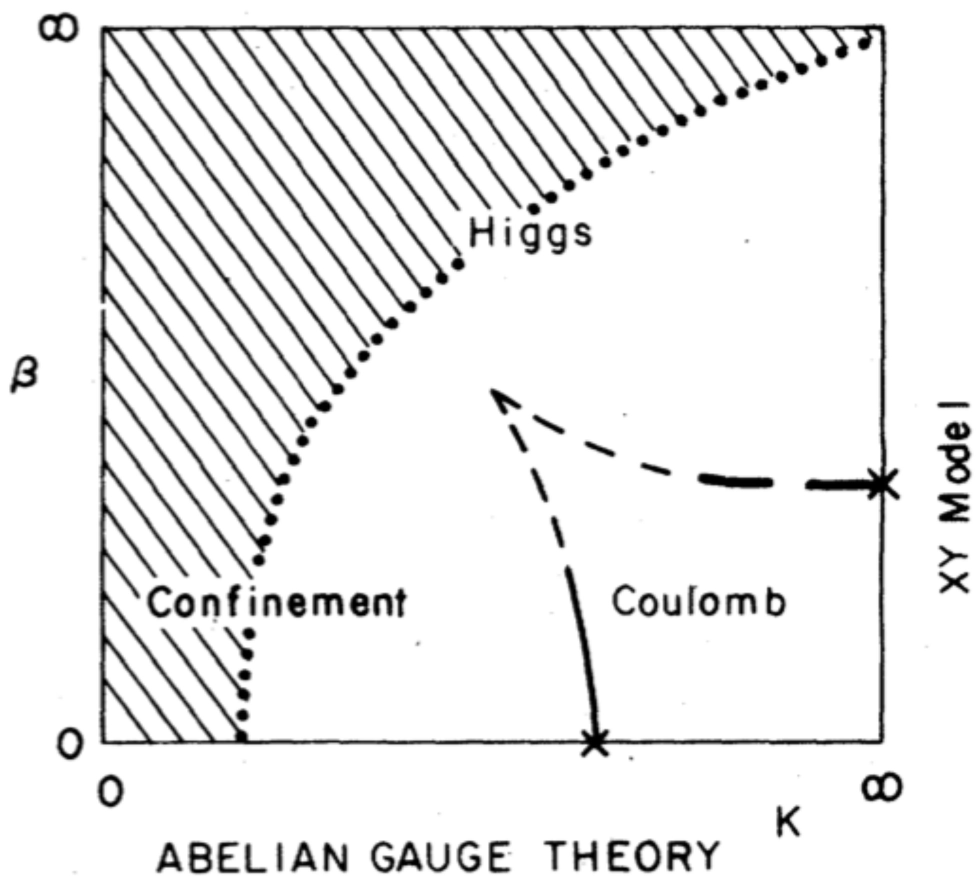
$$O = \bar{\phi}(\vec{x}, 1/(2am))\bar{\phi}(\vec{x}, 0)$$

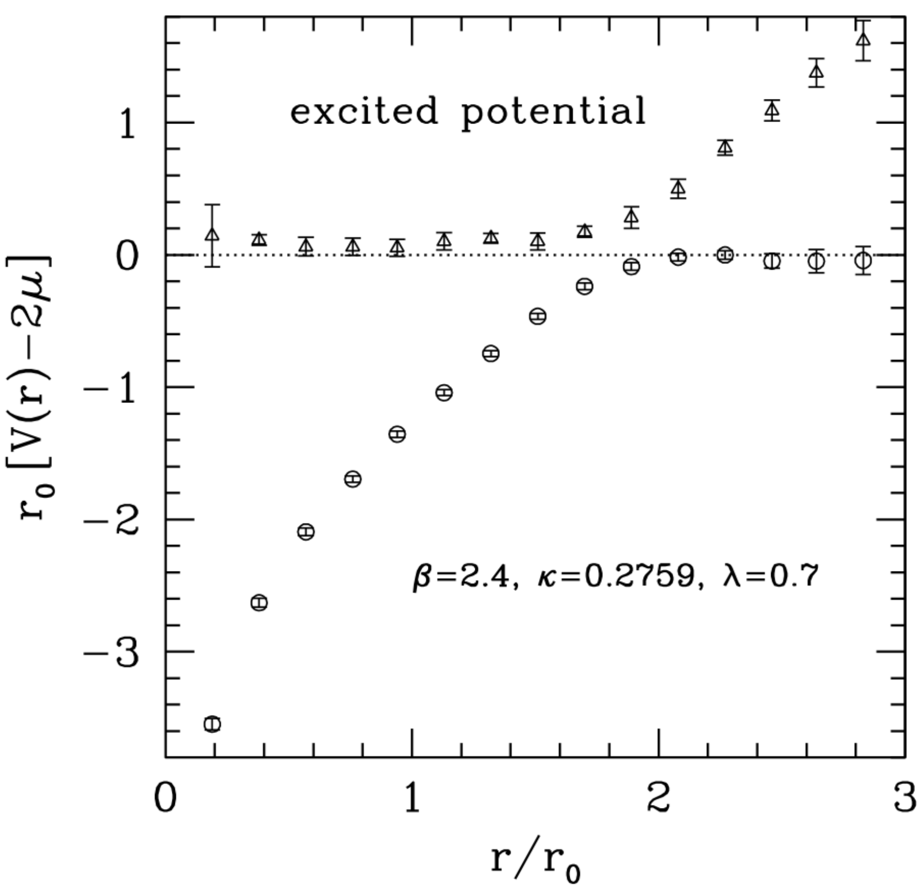
(3d phi4 theory)

SU(2)-Higgs

Results

need Higgs in the fundamental for no phase trans





E. Fradkin and S. Shenker, PRD19, 3682 (1979)

F. Knechtli, hep-lat/9910044