# The high-multiplicity frontier for two-loop QCD

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# Outline

- Background

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- Numerical unitarity for 2-loop amplitudes

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- Differential equations at high multiplicity

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- Differential equations at high multiplicity
- Future outlook

# Main references

Phys. Rev. Lett. 119, 142001, arXiv:1703.05273,
S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page, MZ

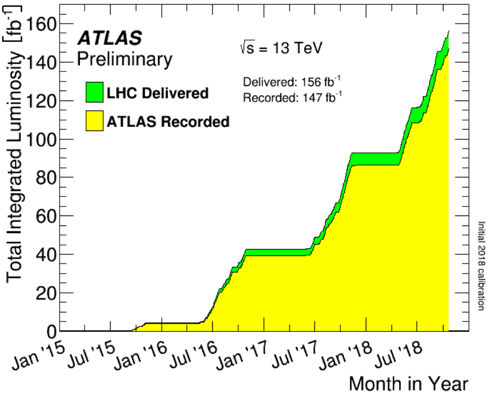
Phys. Rev. D. 97, 116014, arXiv:1712.03946,
S. Abreu, F. Febres Cordero, H. Ita, B. Page, MZ

arXiv:1807.11522, Samuel Abreu, Ben Page, MZ

# Background

# Precision QCD

- $\sim$140 fb$^{-1}$ of data from LHC Run 2.
  $\Longrightarrow$ Precision measurements and BSM searches.



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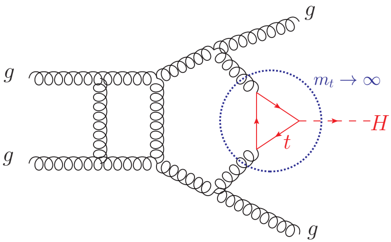
  $\rightarrow$ explosion of $2 \rightarrow 2$ calculations.
  (amplitudes + subtractions)
- Beginning to break the $2 \rightarrow 3$ barrier!

# NNLO $2 \to 3$ processes

- $pp \to 3j$: constrains strong coupling constant $\alpha_s$.
- $pp \to H + 2j$: gluon-fusion background for VBF Higgs production.




- Many more: $V + 2j$, $V + V' + j$, $t\bar{t} + j$ ...

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- Master integrals: analytic / numerical evaluation
- Phenomenology: need sophisticated IR subtraction.

# Numerical unitarity for 2-loop amplitudes

# Numerical unitarity: one loop

Hugely successful at one loop, "NLO revolution".

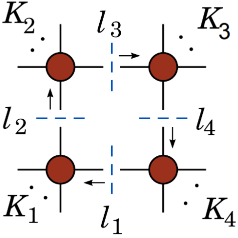


Figure 1: arXiv:0803.4180

Ossola, Papadopoulos, Pittau, 2006
Ellis, Giele, Kunszt, 2007
Giele, Kunszt, Melnikov, 2008
Berger, Bern, Dixon, Febres Cordero,
Forde, Ita, Kosower, Maitre, 2008 ...

BlackHat, GoSam, HELAC-1Loop/CutTools,
Madgraph, NJet, OpenLoops, Recola ...

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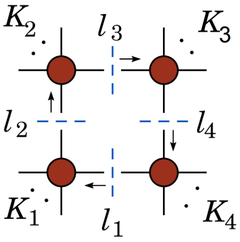

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Example: NLO $pp \to W + 5j \to l\bar{\nu} + 5j$ (BlackHat & Sherpa).
[Bern, Dixon, Febres Cordero, Hoeche, Ita, Kosower, Maitre, Ozeren, 2013]

Polynomial complexity, faster than analytic results in high-multiplicity limit!

# Overview of one-loop numerical unitarity

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  (Berends-Giele recursion)

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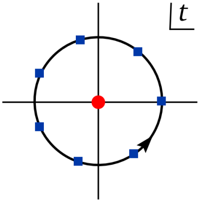


Figure 2: arXiv:0803.4180

Fix $n$ coefficients from $n$ sample points.
Inversion of linear system from discrete Fourier transform

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   - High-precision floating point for direct calculation
   - Finite-field arithmetic for functional reconstruction

# Two-loop integrand decomposition

Mastrolia, Ossola, 2011; Badger, Frelllesvig, Zhang, 2012; Zhang, 2012; Mastrolia, Mirabella, Ossola, Peraro, 2012; Mastrolia, Peraro, Primo, 2016

**Milestone I**: non-redundant parametrization of integrand

- In *d* dimensions, ISPs or Baikov representation
- In 4 dimensions, Groebner basis and polynomial division

# Two-loop integrand decomposition

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- In $d$ dimensions, ISPs or Baikov representation
- In 4 dimensions, Groebner basis and polynomial division

**Milestone II**: isolate spurious terms from transverse space

- E.g. numerator $(l_1 \cdot n)$ with $n \perp p_i$.

# Two-loop integrand decomposition (cont.)

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**Milestone III**: unitarity-compatible IBP relations as surface terms, no need for extra IBP reduction

Gluza, Kajda, Kosower, 2010; Ita, 2015; Larsen, Zhang, 2015

$$0 = \int d^d l \frac{\partial}{\partial \ell^\mu} \frac{v^\mu}{\prod_j D_j}$$

Chetyrkin, Tkachov, 1981

# Two-loop integrand decomposition (cont.)

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$$0 = \int d^d l \frac{\partial}{\partial \ell^\mu} \frac{v^\mu}{\prod_j D_j}$$ Chetyrkin, Tkachov, 1981

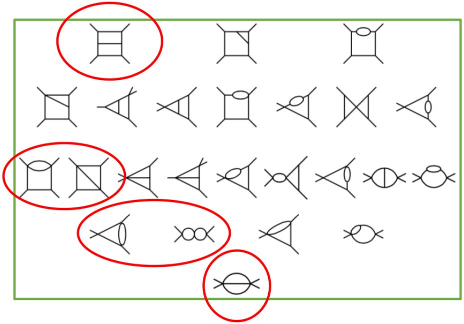
No doubled propagators if *IBP-generating vector* $v^\mu$ satisfies

$$v^\mu \frac{\partial}{\partial \ell_\mu} D_j = f_j D_j$$

with polynomials $f_j$. "Syzygy equations".

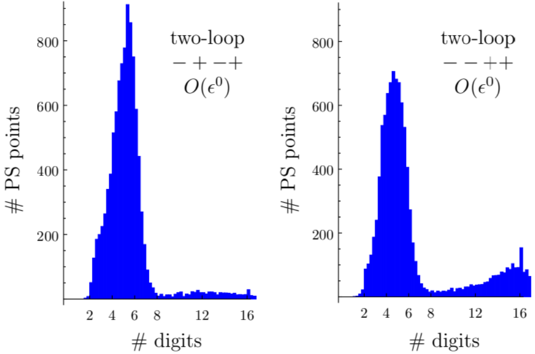
# Proof of principle: 2-loop 4-gluon amplitudes

[Abreu, Febres Cordero, Ita, Jacquier, Page, MZ, 2017]



- SINGULAR finds IBP-generating vectors.
- Random sampling & numerical solution of linear systems of size $\sim 100$ (LAPACK).

# Results: 2-loop 4-gluon amplitudes




- Double precision + quad precision rescue. Agrees with Glover, Oleari, Tejeda-Yeomans, 2001; Bern, De Freitas, Dixon, 2002
- Quad-double precision for reconstructing analytic result.

# Frontier: 2-loop 5-point amplitude

## All-plus gluon integrand (planar & nonplanar)

Badger, Frellesvig, Zhang, 2013

Badger, Mogull, Ochirov, O'Connell, 2015,

Dunbar, Perkins, 2016

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## Arbitrary helicities (planar) - see next slides!

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## Extension to quarks

(proceeding) Badger, Brønnum-Hansen, Gehrmann, Hartanto, Henn, Lo Presti, 2018
Abreu, Febres Cordero, Ita, Page, Sotnikov, 2018

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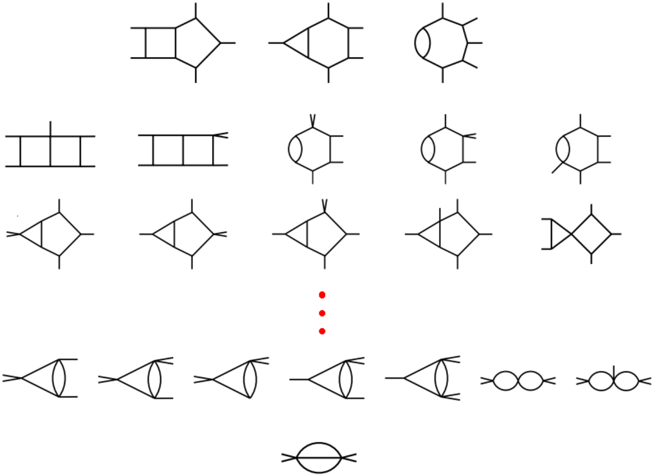
## Master integrals in dimensional regularization

Gehrmann, Henn, Lo Presti, 2015
Tommasini, Papadopoulos, Wever, 2015
Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser, 2018
(Talk) Papadopoulos, Wever, 2018

# Topology hierarchy for 2-loop 5-gluon amplitudes

[Abreu, Febres Cordero, Ita, Page, MZ, 2017]



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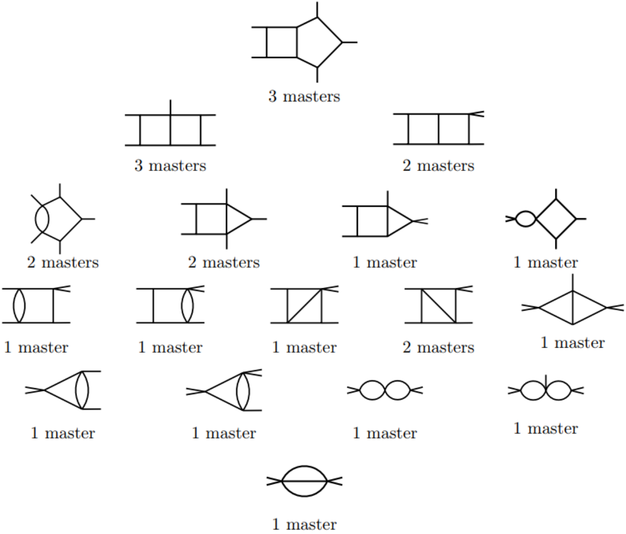

# Implementation for 2-loop 5-gluon amplitudes

- **Improved algorithm** finds IBP-generating vectors in under 1 second for every sector

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- Arbitrary precision from **exact finite field** computation

  [von Manteuffel, Schabinger, 2014; Peraro, 2016]

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- Arbitrary precision from **exact finite field** computation
  [von Manteuffel, Schabinger, 2014; Peraro, 2016]
- **Quad precision floating point** also under testing
  Preliminary: uniform performance across phase space

# Results: 2-loop 5-gluon amplitudes

## Euclidean point

$$p_1 = \left(\frac{1}{2}, \frac{1}{16}, \frac{i}{16}, \frac{1}{2}\right), \quad p_2 = \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right), \quad p_3 = \left(\frac{9}{2}, -\frac{9}{2}, \frac{7i}{2}, \frac{7}{2}\right),$$
$$p_4 = \left(-\frac{23}{4}, \frac{61}{16}, -\frac{131i}{16}, -\frac{37}{4}\right), \quad p_5 = \left(\frac{5}{4}, \frac{5}{8}, \frac{37i}{8}, \frac{19}{4}\right).$$

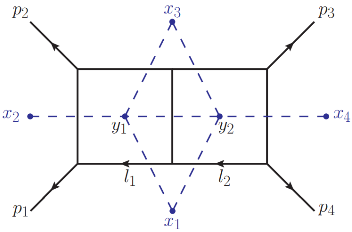
| $\mathcal{A}^{(2)}/\mathcal{A}^{\text{norm}}$ | $\epsilon^{-4}$ | $\epsilon^{-3}$ | $\epsilon^{-2}$ | $\epsilon^{-1}$ | $\epsilon^{0}$ |
|---|---|---|---|---|---|
| $(1^+,2^+,3^+,4^+,5^+)$ | | | -5.0000000 | -3.89317903 | 5.98108858 |
| $(1^-,2^+,3^+,4^+,5^+)$ | | | -5.0000000 | -16.3220021 | -10.3838132 |
| $(1^-,2^-,3^+,4^+,5^+)$ | 12.50000 | 25.462469 | -1152.8431 | -4072.9383 | -3637.2496 |
| $(1^-,2^+,3^-,4^+,5^+)$ | 12.50000 | 25.462469 | -6.1216296 | -90.221842 | -115.78367 |

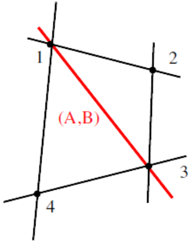
Table 1: $\mathcal{A}^{\text{norm}}$ is $\mathcal{A}^{\text{tree}}$ if amplitude exists at tree level, otherwise $\mathcal{A}^{\text{1-loop}}$.

Perfect agreement with universal IR poles [Catani, 1998] and results in [Badger, Brønnum-Hansen, Hartanto, Peraro, 2017]

# Connection with dual conformal symmetry

Dual coordinates: cut propagator mapped to null-seperated points.




Conformal transformation preserves null separation, and generates unitarity-compatible IBP & differential equations.

This connection also motivated **nonplanar generalization** of DCS

Z. Bern, M. Enciso, H. Ita, MZ, 2017
Z. Bern, C. Shen, M. Enciso, MZ, 2018
D. Chicherin, J. Henn, E. Sokatchev, 2018

# Differential equations at high multiplicity

# Master integrals from differential equations

- Many methods for evaluating master integrals
  Schwinger / Feynman $\alpha$ parameters, Mellin-Barnes representation, Differential equations ...

- Differential equations method:
  [Kotikov, 1991; Bern, Dixon, Kosower, 1993; Remiddi, 1997; Gehrmann, Remiddi, 1999; Argeri, Mastrolia, 2007]

$$\frac{\partial}{\partial x} I_i \overset{\mathrm{IBP}}{=} (M_x)_{ij}\, I_j$$

- A breakthrough: canonical form of DEs: [J. Henn, 2013, 2014]

$$\frac{\partial}{\partial x} I_i = \Big[ \epsilon \sum_{\alpha} \frac{\partial \log r_\alpha}{\partial x} \underbrace{(M_\alpha)_{ij}}_{\text{rational numbers!}} \Big] I_j$$

# Numerical construction of DEs

Pure integrals $\mathrm{I} = (I_1, I_2, \ldots, I_n)$, with *m symbol letters* $r_\alpha$.

$$d\mathrm{I} = \mathbb{M} \cdot \mathrm{I} = \epsilon \sum_{\alpha=1}^{m} (d \log r_\alpha)\, \mathsf{M}_\alpha \cdot \mathrm{I},$$

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**Inspired by numerical unitarity:** exploit canonical form to simplify *construction* of DEs [Samuel Abreu, Ben Page, MZ, 2018]

See also: construction in generic basis: [Tiziano Peraro talk, 2018]

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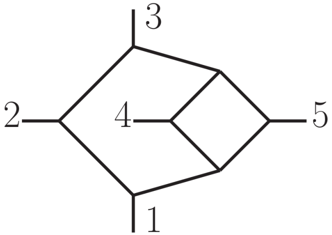
See also: construction in generic basis: [Tiziano Peraro talk, 2018]

Fit $(m \times n \times n)$ matrix entries: computing the $(n \times n)$ matrix $\mathbb{M}$ at $m$ points in phase space.

Use finite fields to speed up calculation

# DEs for nonplanar hexabox




Five kinematic scales.
Extremely difficult using conventional IBP techniques!

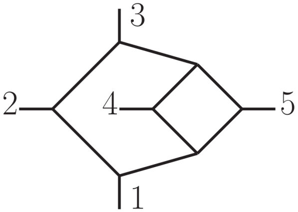
Very recent progress on IBPs:

- Max.-cut IBPs / DEs: MZ, 1702.02355; Chawdhry, Lim, Mitov, 1805.09182
- Rank-4 w/o dot: Boehm, Georgoudis, Larsen, Schoenemann, Zhang, 1805.01873

- Rank 3 + 1 dot $\implies$ Canonical DEs: Abreu, Page, MZ, 1807.11522
  Canonical DEs + solutions: Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser, 1809.06240

# Nonplanar hexabox: pure basis

Evidence for nonplanar amplituhedron, 1512.08591
Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz, J. Trnka



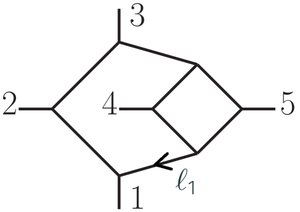

$$\mathcal{N}_1 = [13]\left(\ell_1 + \frac{P_{45}\cdot\tilde\lambda_3\tilde\lambda_1}{[13]}\right)^2 \langle 15\rangle[54]\langle 43\rangle \times (\ell_1 + k_4)^2, \quad \mathcal{N}_2 = \mathcal{N}_1\big|_{4\leftrightarrow 5}$$

A 3rd pure numerator $\mathcal{N}_3$ found by leading singularities, with poles at $\infty$.

Two loop master integrals for $\gamma^* \to 3$ jets: The nonplanar topologies, hep-ph/0101124, T. Gehrmann, E. Remiddi

for 4-point one-mass pure integrals in sub-topologies

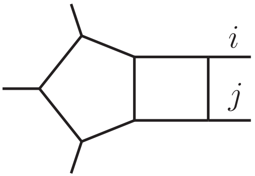
# Pure integrals: 4 versus $D$ dimensions




$$\mathcal{N}_3 = s_{12}s_{23}\langle 4\not{\ell}_1 5]\langle 5\not{\ell}_1 4]$$
$$= s_{12}s_{23}\left(\frac{4(\ell_1\cdot p_4)(\ell_1\cdot p_5)}{s_{45}} - (\ell_1^2)_{4D}\right)$$

fails $\epsilon$ factorization of DEs!

simple fix: $(\ell_1^2)_{4D} \to \ell_1^2$




$\mathcal{N}_1 =$ nonvanishing in 4D

$$\mathcal{N}_2 = [\mu_{12}]\, s_{ij}\sqrt{\det G}$$

$$\mathcal{N}_3 = [\mu_{12}^2 - \mu_{11}\mu_{22}]\,\frac{d-3}{d-5}\sqrt{\det G}$$

# Results: DEs for nonplanar hexabox

- Pure integrals from 4D leading singularities and "$\mu$-terms". Symbol alphabet with 31 letters, from permuting planar ones, conjectured by [Chicherin, Henn, Mitev, 2017]
- Only 30 phase space points used to reconstruct analytic DEs.

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$$(M)_{1,1} = 2, \quad (M)_{1,16} = 2, \quad (M)_{2,2} = 2, \quad (M)_{2,16} = -2,$$
$$(M)_{5,5} = 2, \quad (M)_{5,16} = -4, \quad (M)_{12,12} = 2, \quad (M)_{12,16} = -4,$$
$$(M)_{16,16} = -4, \quad (M)_{17,17} = 2, \quad (M)_{19,19} = 2, \quad (M)_{24,24} = 2,$$
$$(M)_{26,26} = 2, \quad (M)_{28,28} = 2, \quad (M)_{30,30} = 2\,.$$

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$$(M)_{1,1} = 2, \quad (M)_{1,16} = 2, \quad (M)_{2,2} = 2, \quad (M)_{2,16} = -2,$$
$$(M)_{5,5} = 2, \quad (M)_{5,16} = -4, \quad (M)_{12,12} = 2, \quad (M)_{12,16} = -4,$$
$$(M)_{16,16} = -4, \quad (M)_{17,17} = 2, \quad (M)_{19,19} = 2, \quad (M)_{24,24} = 2,$$
$$(M)_{26,26} = 2, \quad (M)_{28,28} = 2, \quad (M)_{30,30} = 2\,.$$

- **Ongoing**: DEs + first-entry condition fixes symbols for all pure integrals. Confirmed conjectured 2nd entry condition [Chicherin, Henn, Mitev, 2017; Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser, 2018]

# Future outlook

- Numerical unitarity for high-multiplicity QCD processes: "NLO revolution" being upgraded to NNLO!
  - **Open question**: better control over stability of linear systems. Analog of discrete Fourier transform?
- Contact with phenomenology in coming years. Physics opportunity for amplitudes, IR subtraction, resummation.
- Differential equations constructed by similar methods. **Avoids IBP obstables** at higher multiplicity.
  - **Open question**: better understanding of pure integrals outside 4 dimensions.