Systematic approximation of multi-scale Feynman integrals

arXiv:1804.06824

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29.10.2018

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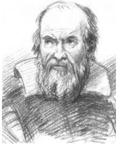


Motivation

- Feynman integrals are often the bottleneck for multi-scale multi-loop calculations, especially with massive propagators!
- Analytic evaluation of very complicated Feynman integrals not generally understood, although progress is being made!
- Numerical evaluation often the only option.
- Producing accurate numerical results over full phase space in an automated way is difficult - divergent nature of loop integrals.
- An algorithm to analytically approximate Feynman integrals - TayInt!

Philosophy

Three aims: to produce an algebraic integral library with full phase space validity for any kind of integral

- The idea - the integrand has to be Taylor expanded in the Feynman parameters - otherwise you can't integrate it - the kinematics are not to be touched.

- The algorithm brings an integral into a form optimised for an accurate Taylor expansion with validity in all kinematic regions.

- Divide and rule - take all the nastiness in the Feynman integral - distribute it so that it does not hinder a TAYLOR EXPANSION IN THE FEYNMAN PARAMETERS.

Setup I

- A generic Feynman loop integral G in an arbitrary number of dimensions D at loop level L with N propagators, wherein the propagators P_j with mass m_j can be raised to arbitrary powers ν_j,

$$G^{\mu_1\dots\mu_R}_{\alpha_1\dots\alpha_R}(\{p\},\{m\}) = \left(\prod_{\alpha=1}^{L}\int \mathrm{d}^D\kappa_\alpha\right) \frac{k_{\alpha_1}^{\mu_1}\cdots k_{\alpha_R}^{\mu_R}}{\prod_{j=1}^{N} P_j^{\nu_j}(\{k\},\{p\},m_j^2)}$$

$$\mathrm{d}^D\kappa_\alpha = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}}}\,\mathrm{d}^D k_\alpha\ ,\ \ P_j(\{k\},\{p\},m_j^2) = q_j^2 - m_j^2 + i\delta\ ,$$

- The q_j are linear combinations of external momenta p_i and loop momenta k_α.

- Henceforth scalar integrals considered.

Setup II

- Rewrite scalar integrals in terms of Feynman parameters t_j, $j = 1...N$. Integrate the loop momenta to give,

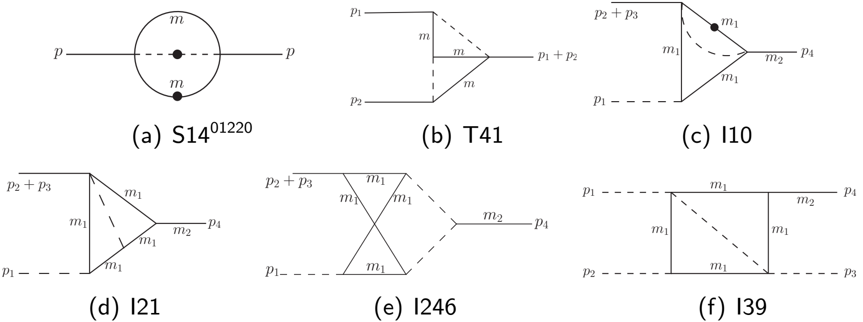
$$G = \frac{(-1)^{N_\nu}}{\prod_{j=1}^{N} \Gamma(\nu_j)} \prod_{j=1}^{N} \int_0^\infty dt_j \; t_j^{\nu_j - 1} \, \delta(1 - \sum_{l=1}^{N} t_l) \, \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}} \, ,$$

- $\mathcal{U}$ and $\mathcal{F}$ are the first and second Symanzik polynomials.

Summary of the method

U1: reduce the Feynman Integral to a quasi-finite basis	
U2: perform an iterated sector decomposition	
below threshold	above threshold
BT1: $t_j \to y_j$	OT1: $t_j \to \theta_j$, generate $\mathcal{K}$
BT2: Taylor expand and integrate	OT2: find $\Theta_{o(0),\ldots o(J-1)}$
	OT3: perform one fold integrations
	OT4: find θ_j^* and the optimum $\Theta_{o(0),\ldots o(J-2)}$
	OT5: determine $\mathcal{P}_j$
	OT6: Taylor expand and integrate

Diagrams



(a) $S14^{01220}$ (b) T41 (c) I10

(d) I21 (e) I246 (f) I39

- The finite sunrise $S14_{01220}$ and the triangle T41 are used to illustrate TayInt. Integrals I10, I21, I246 and I39 enter the Higgs+jet calculation - are computed with TayInt.

Method - U1

Universal step 1 (U1) - Feynman integral G expressed as a superposition of finite Feynman integrals G^{F} multiplying poles in ϵ

- Quasi-finite basis: (von Manteuffel, Panzer, Schabinger, arXiv:1411.7392),(Panzer, arXiv:1401.4361) these integrals contain no divergences from integration of Feynman parameters - all divergent parts restricted to prefactors.

- The G^{F} have a shifted number of dimensions or dotted propagators or both.

- Automated shell script used to direct all required Reduze (von Manteuffel, Panzer, Schabinger, arXiv:1411.7392),(von Manteuffel, Studerus, arXiv:1201.4330) jobs towards generating the quasi-finite basis.

Illustration of Method - U1

- The divergent sunrise $S14^{01110}$ in terms of the finite integrals $S14^{01220}$, $S14^{01320}$ and the tadpole $S6^{30300}$,

$$
\begin{aligned}
S14^{01110} &= \frac{8m^2\,(p^2-4\,m^2)(p^2+2\,m^2)}{(-3+D)(-8+3D)(-10+3D)}\cdot S14^{01320} \\
&+ \frac{((4-D)p^4+(-5+D)8\,m^4+(18-5D)4\,p^2m^2)}{(-3+D)(-8+3D)(-10+3D)}\cdot S14^{01220} \\
&- \frac{16m^4\,((-4+D)\,p^2+2\,(-24+7D)\,m^2)}{(-3+D)(-4+D)^2(-8+3D)(-10+3D)}\cdot S5^{30300}\ ,
\end{aligned}
$$

- Poles in ϵ: $(-4+D)^{-1}$ terms.

Method - U2

Universal step 2 (U2) - decompose integrals G^{F} to iterated sectors using version 3 of SecDec (Borowka, Heinrich et al., arXiv:1703.09692)

- Iterated sectors:

$$G_l^{\mathrm{F}} = \prod_{j=2}^{N} \int_0^1 \mathrm{d}t_j \; t_j^{A_{l_j} - B_{l_j}\epsilon} \; \frac{\mathcal{U}_l^{N_\nu - (L+1)D/2}(\vec{t_j})}{\mathcal{F}_l^{N_\nu - LD/2}},$$

  where $l = 1, \ldots, r$, and r is the number of iterated sector integrals. A_k and B_k are numbers independent of the regulator ϵ.

- Remap so that j runs from 0 to $J-1$.

- Iterated sector integrals G_l^{F} - building blocks of TayInt.

Illustration of the method - U2

- $\mathcal{O}(\epsilon^0)$ coefficient of an I10 iterated sector,

$$\text{I10}_1 = \prod_{j=0}^{2} \int_0^1 dt_j \frac{1}{(1 + t_0 + t_1 + t_2 + t_1 t_2)} \cdot [t_0(-u - m_2 t_1) + m_1^2(1 + t_0^2 + t_2 + t_1^2(1 + t_2) + t_1(2 + 2t_2) + t_0(2 + t_2 + t_1(2 + t_2)))]^{-1},$$

- Three Feynman parameters after iterated decomposition, three kinematic scales, m_1, m_2 and u.

Method - BT1

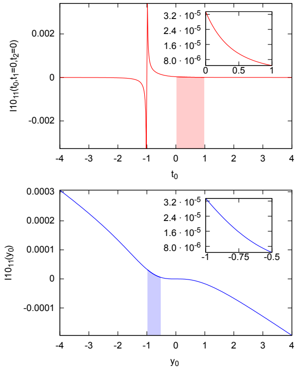
Below Threshold step 1 (BT1) - maximise distance to nearest point of non-analyticity

- The iterated sectors G_l^F - still have non-analytic points outside the integration region.
- To move these as far away as possible, import G_l^F into Mathematica (Wolfram), apply conformal mappings,

$$t_j = \frac{ay_j + b}{cy_j + d}.$$

- Thus far: optimum mapping found.

Illustration of the method - BT1



- Plot of a one dimensional integrand,

  $$\mathrm{I10}_1(t_0, t_1 = 0, t_2 = 0) = 1/((1 + t_0)(-m_2 t_0 + m_1^2(1 + 2t_0 + t_0^2))),$$

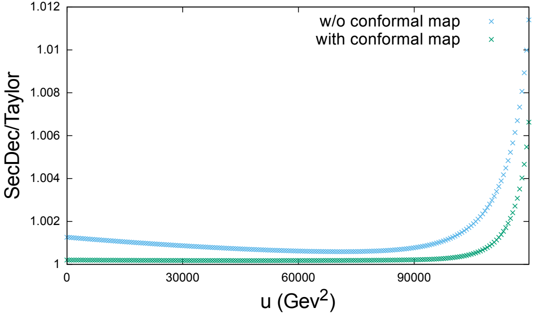
  before and after a conformal mapping $y_0 = \frac{-1-t_0}{t_0}$.

Method - BT2 and BT3

Below Threshold steps 2 and 3 (BT2-3) - Taylor expand and integrate in the Feynman parameters

- BT2 - Taylor expand the integrand in the re-mapped Feynman parameters y_j.
- BT3 - integrate over the y_j.
- Done in FORM (Kuipers, Ueda, Vermaseren, arXiv:1310.7007).

Illustration of the method - BT2 and BT3



- Ratio of SecDec and TayInt calculation of the ϵ^0 coefficient of I10.

Going over threshold (OT1-OT6)

- Above lowest threshold of an integral - discontinuities on the real axis. A Taylor expansion won't converge!
- TayInt returns to the result of U2, the iterated sector integrands $G_l^{\mathsf{F}}(t_j)$. The Feynman $+i\delta$ prescription is implemented in Mathematica.
- TayInt determines the contour configuration in the complex plane to avoid the discontinuities.
- Over threshold part of TayInt is fully automated in Mathematica.

Method - OT1

Over Threshold step 1 (OT1) - generate all possible contour configurations for each iterated sector integrand

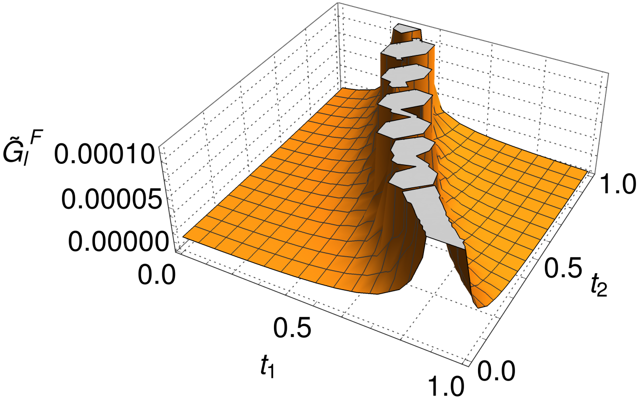
- The first Over Threshold step (OT1) - transform the Feynman parameters of the $J-1$ iterated sectors, $t_j \to \frac{1}{2} + \frac{1}{2}\exp\left(i\theta_j\right)$.
- Generate representative sample of the kinematic region.
- A nested list of values $\mathcal{K} = \{\{s_1, \ldots, s_\beta\}_1, \ldots, \{s_1, \ldots, s_\beta\}_\gamma\} = \{\mathcal{K}_1, \ldots, \mathcal{K}_\gamma\}$ for a β scale integral, sample size of γ points.

Method - OT2

Over Threshold step 2 (OT2) - select contour configuration optimised for a Taylor expansion

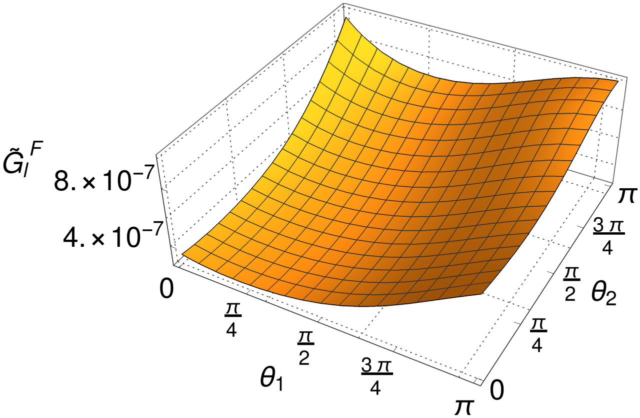
- OT2 - calculate the mean absolute value of the θ_j derivatives (MAD) of the $G_I^{\mathrm{F}}(\theta_j)$.
- Kinematic scales first set to the mean of the sample, MAD calculated at the edges.
- Kinematic scales then set to each sample value, MAD calculated over bulk.
- MAD calculated for all possible contour configurations, $\Theta_{o(1),\ldots o(J-1)}$ - $o(j) = \pm$ is the orientation of the jth contour in the θ_j.
- Contour configuration which minimises the MAD selected.

Illustration of the method - OT1 and OT2



- Slice of I10$_2$ without a complex mapping...

Illustration of the method - OT1 and OT2



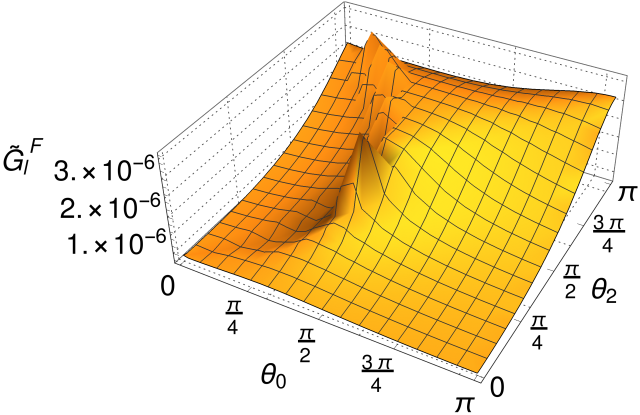
- with a complex mapping, contour orientation $\{o(1), ...o(J-1)\}$ determined via TayInt.

Method - OT3 and OT4

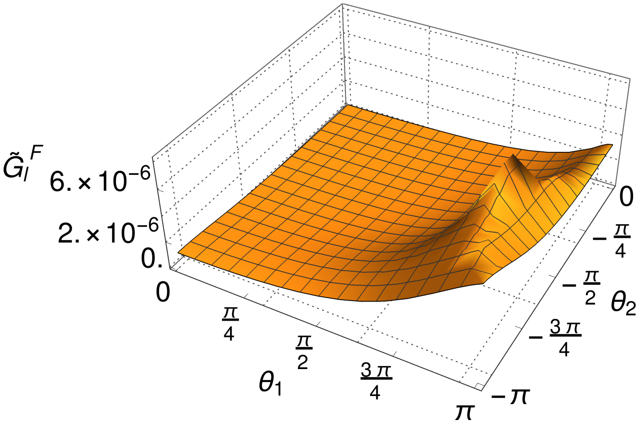
Over Threshold steps 3 and 4 (OT3-4) - generate all possible post-integration contour configurations for each iterated sector integrand - select optimum one.

- OT3 - perform all possible one-fold integrations in the θ_j exactly.
- In OT4 the resultant $J-2$ variable contour configuration with the lowest MAD is selected - same process as in OT2.
- If MAD is lower than without integration, this contour configuration is used.
- The θ_j integrated to yield the selected contour configuration is θ_j^*.

Illustration of the method - OT3 and OT4 I

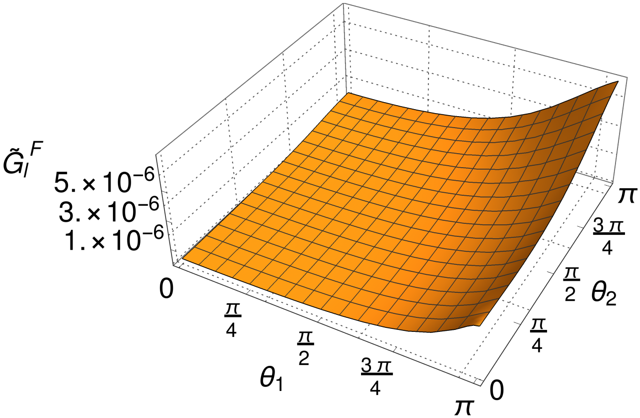


- $I10_2$: pre-integration contour chosen by TayInt, arbitrary post-integration contour.



- $I10_2$: pre-integration contour arbitrary, post-integration contour chosen by TayInt.

Illustration of the method - OT3 and OT4 III



- $I10_2$: full TayInt algorithm.

Method - OT5 and OT6

Over Threshold steps 5 and 6 (OT5-6) - maximise the convergence of the Taylor expansion

- OT5 - determine the optimal partitioning, $\mathcal{P}_j = \{(l,h)_1, ..., (l,h)_N\}_j$, of the integrals in θ_j,

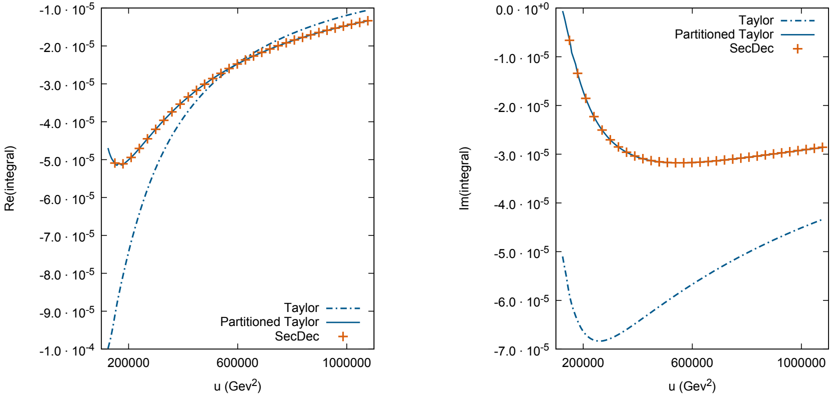
$$\int_0^{\pm\pi} d\theta_j = \sum_{k=1}^{N} \int_{l_{k,j}}^{h_{k,j}} d\theta_j \ ,$$

- $h_{N,\ j} = \pm\pi$ and $l_{1,\ j} = 0$.

- New integrands expanded and integrated within each partition in OT6 - allows target precision to be met.

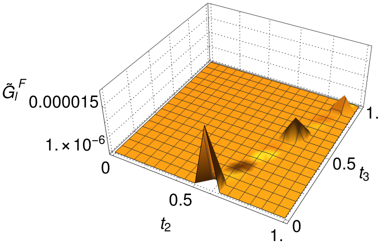
- Results are functions of the kinematic scales valid everywhere above threshold!

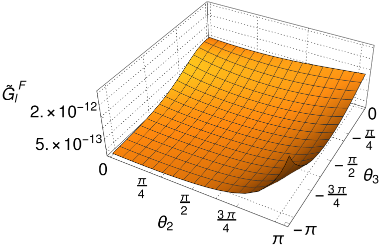
Illustration of the method - OT5 and OT6



- I10 calculated to ϵ^0 over the $4m_1^2$ threshold with and without partitioning.

Illustration of the method - multiple thresholds I





- Slice of $I246_1$ at ϵ^0 in the first over threshold region before and after applying TayInt.

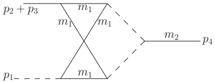
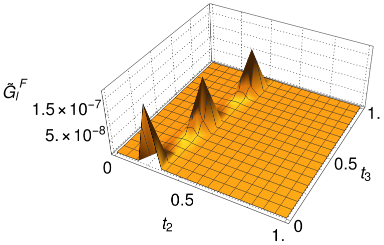
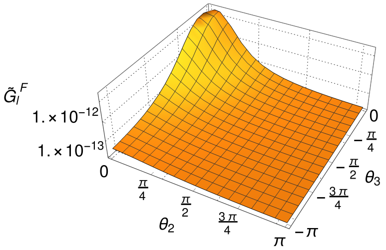
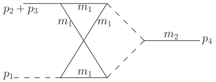


Illustration of the method - multiple thresholds II



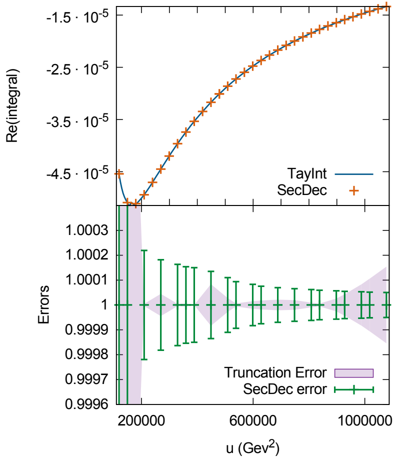


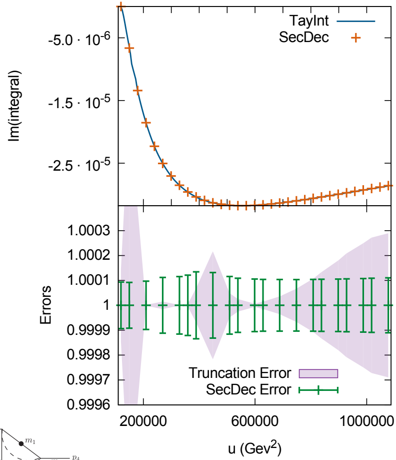
- Slice of I246$_1$ at ϵ^0 in the second over threshold region before and after applying TAYINT.

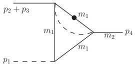


Example of results: I10, $\mathcal{O}\left(\epsilon^0\right)$

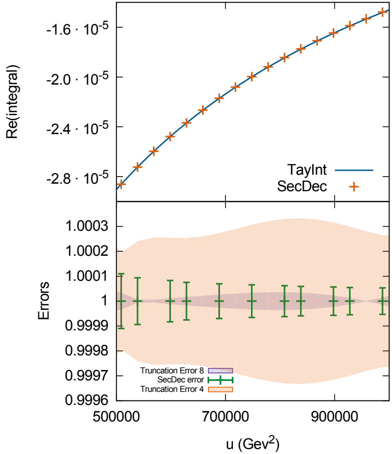
I10: $\mathcal{O}\left(\epsilon^0\right)$, $u > 4m_1^2$, $m_2^2 = 0.5m_1^2$, $m_1 = 173\text{GeV}$

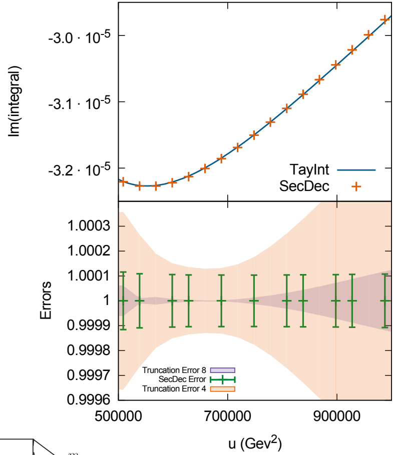


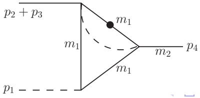




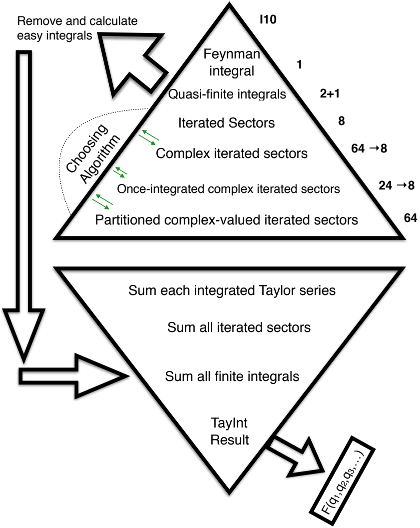
Example of results: I10, $\mathcal{O}\left(\epsilon^0\right)$



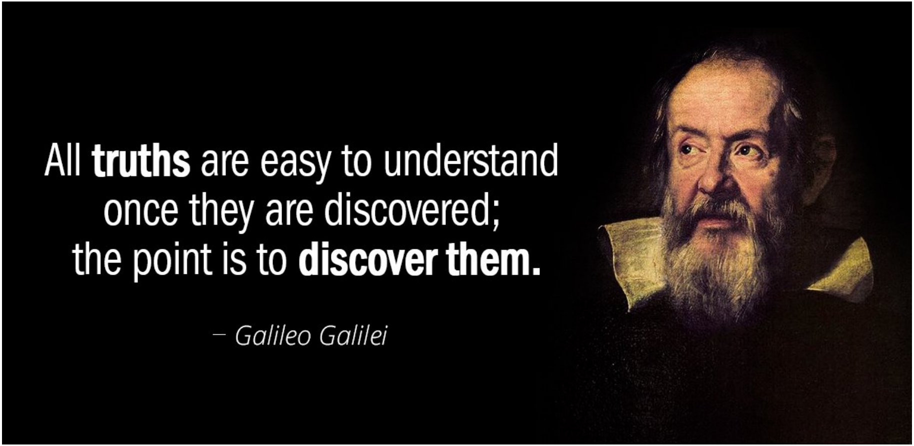




Summary of the method II



TayIntv2 - easier and truer



TAYINTv2 -Changes to the method I

New method of contour deformation, removed dependence on Reduze

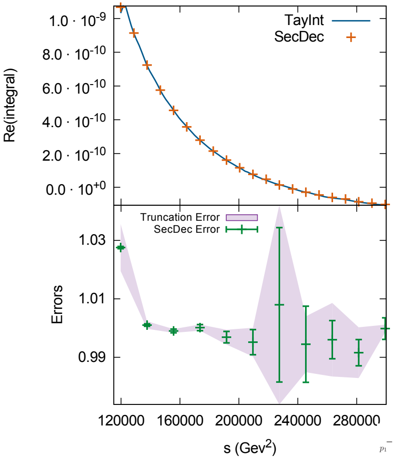
- U1 removed - divergent integrals can be computed.
- OT3-4 removed - exact integration no longer performed as precision can be more reliably controlled with partitions.
- OT2 changed - MAD computed at a sample of kinematic points - select the contour with the most invariant MAD over the sample.
- Check - is the chosen contour the optimal one at the majority of points in the kinematic sample?

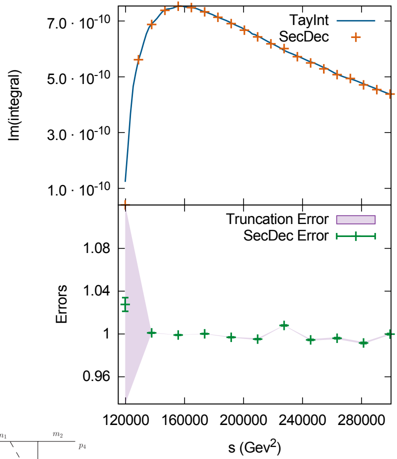
TAYINTv2 -Changes to the method II

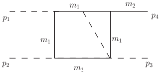
- If not - use partial complex contours - one Feynman parameter not mapped - and perform the same contour selection. This continues until a contour is optimum for the majority of points in the kinematic sample.
- The new method of contour selection is more reliable as it cannot be distorted by small regions of extreme change.
- It is based on our observation that the smoothest surface changes the least when you change the kinematics.
- This invariance is a more general indicator of a surfaces suitability for a Taylor expansion than the mean MAD over a kinematic sample.

Example of new results: I59, $\mathcal{O}\left(\epsilon^0\right)$

I59: $\mathcal{O}\left(\epsilon^0\right)$, $s > 4m_1^2$, $u = -2m_1^2$, $m_2^2 = 0.5m_1^2$, $m_1 = 173\text{GeV}$

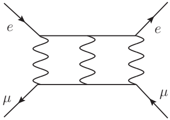


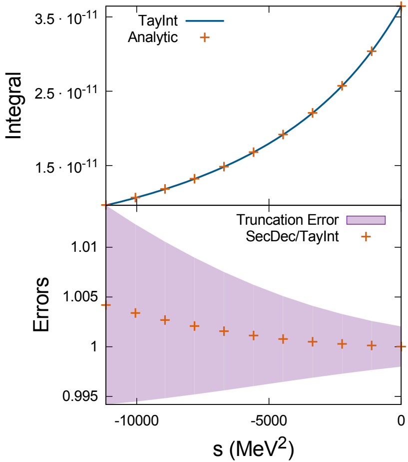




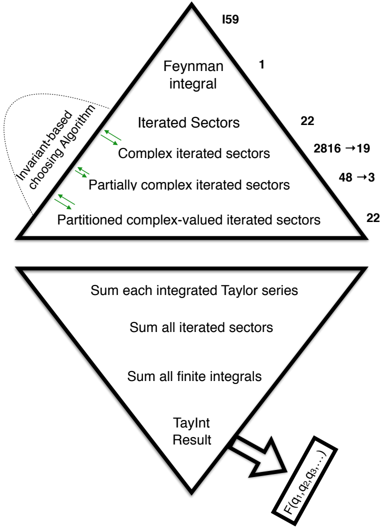
Example of new results: I503, $\mathcal{O}\left(\epsilon^{-3}\right)$

I503: $\mathcal{O}\left(\epsilon^{-3}\right)$, $0 < s > -m_\mu^2$, $u = -0.5m_\mu^2$, $u = -0.5m_\mu^2$, $m_\mu = 106\text{MeV}$





Summary of the new method II



Conclusions

Three aims: to produce an algebraic integral library with full phase space validity for any kind of integral

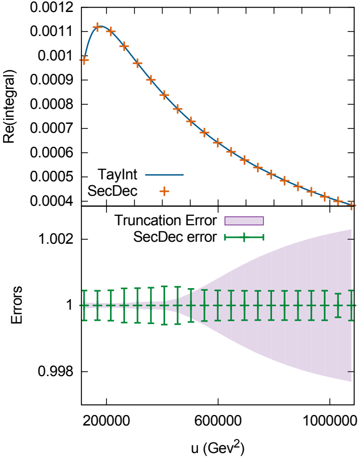
- TayInt is flexible. Results generated for different:
- numbers of propagators
- numbers of external scales
- ϵ orders
- kinematic regions, above and below thresholdS
- diagrams relevant for Higgs+jet production at two loop.

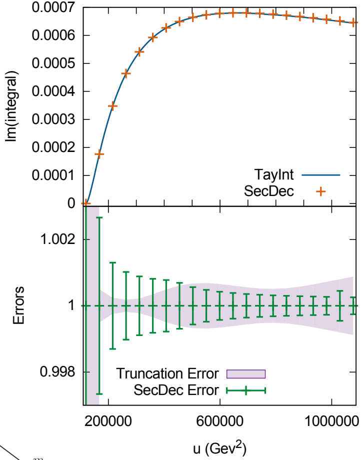
Outlook

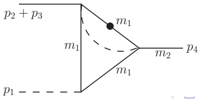
- Future goals:
- Do we need the Sector decomposition?
- Automate fully,
- Apply to phenomenological processes,
- Apply to five-point integrals.

Thank you very much for listening!

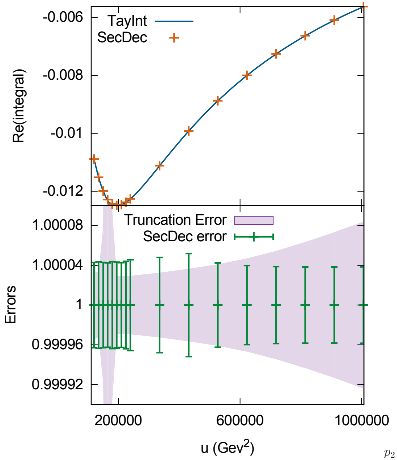
BACKUP-I10 ϵ^1 $u > 4m_1^2,\ m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$

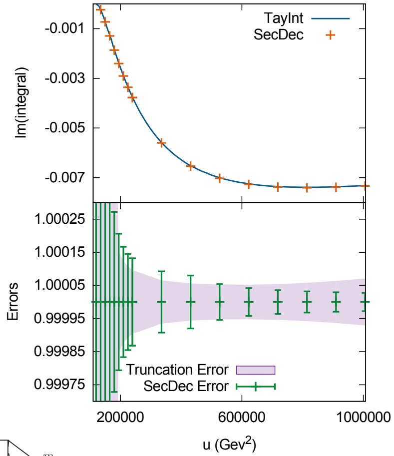


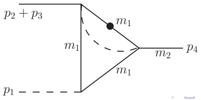




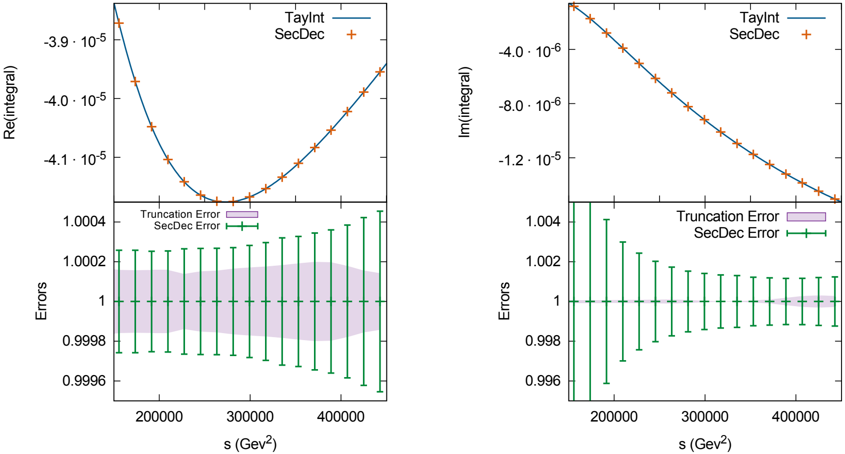
BACKUP-I10 ϵ^2 $u > 4m_1^2,\ m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$

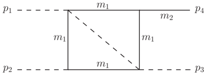




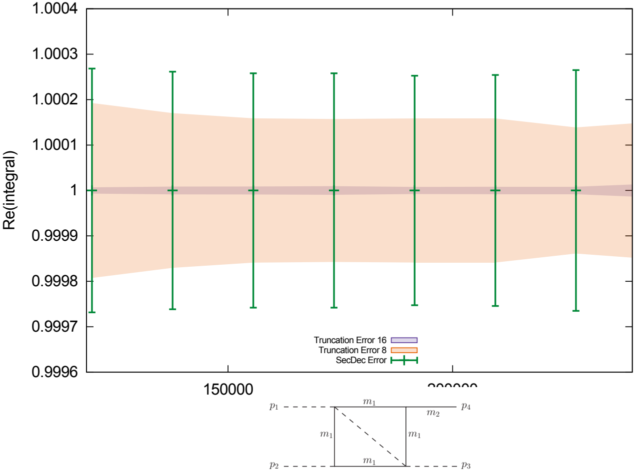


BACKUP-I39 ϵ^0, $s > 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$

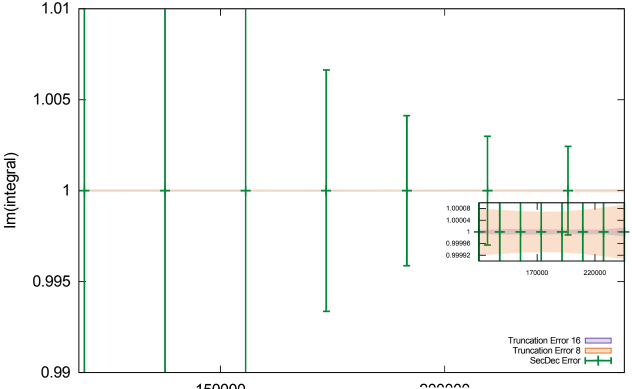


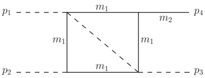


BACKUP-I39 ϵ^0, $s > 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$

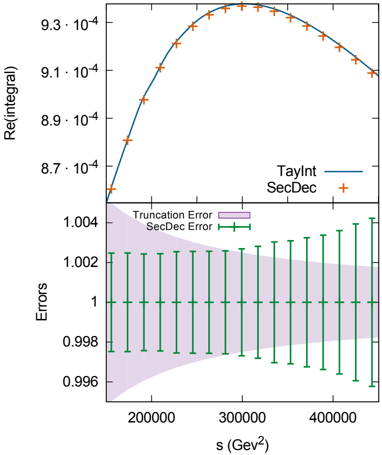


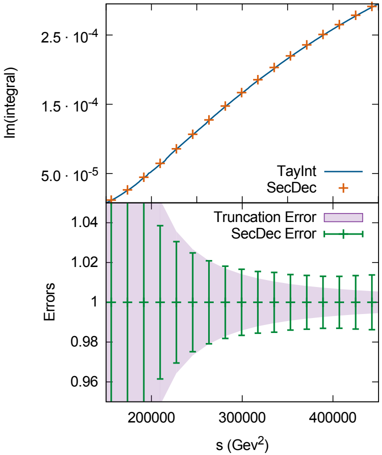
BACKUP-I39 ϵ^0, $s > 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$

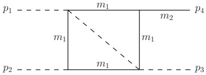




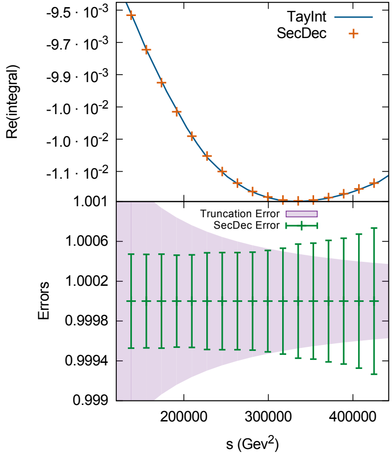
BACKUP-I39 ϵ^1, $s > 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173$GeV

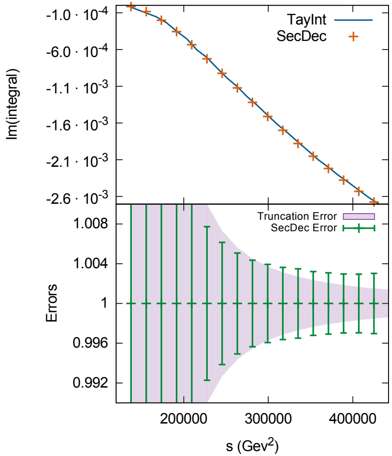


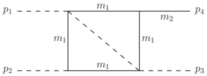




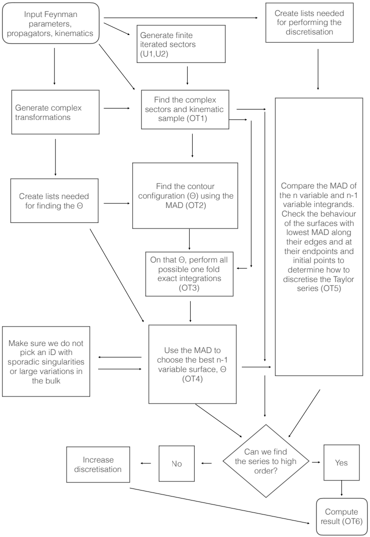
BACKUP-I39 ϵ^2, $s > 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$







BACKUP-Summary of the Method II



BACKUP - Convergence Study

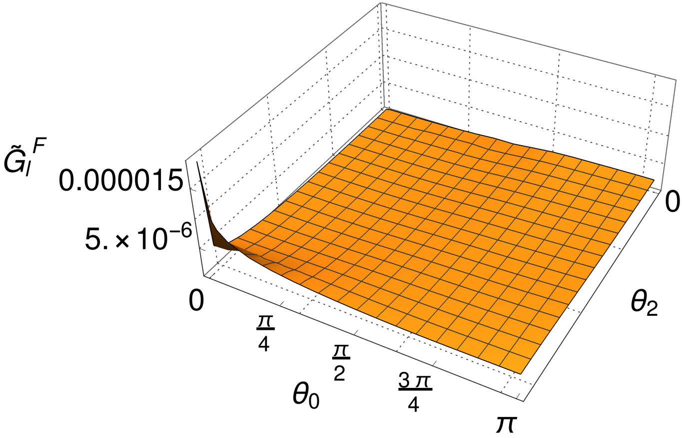
		Mean TayInt Error			
		Number of Partitions			
		4		8	
		Re	**Im**	**Re**	**Im**
Order	0	0.530165	0.623989	0.0812167	0.242449
	2	0.0221554	0.0242271	0.000642405	0.00237282
	4	0.00278254	0.00242541	0.000163342	0.000079292
	6	0.000284179	0.000281809	0.0000239721	0.000038864

BACKUP - Summary of Results

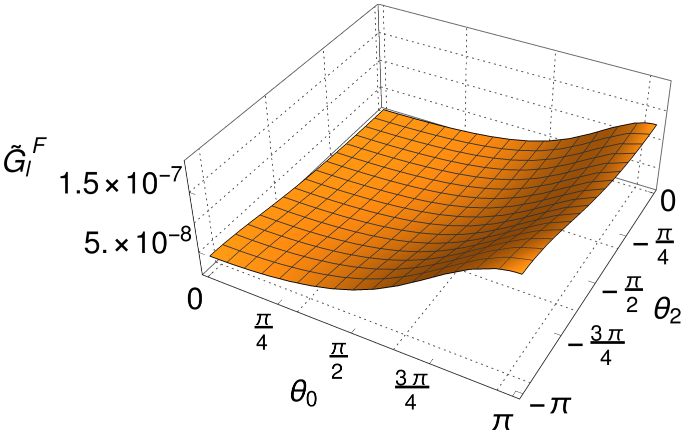
Graph	Re (Δ)	Im (Δ)
I10	0.000658179	0.000270775
I21	0.00126601	0.000277579
I39	0.0000763027	0.0000668706

- The mean difference Δ between TayInt, using a sixth order expansion, and SecDec, normalised to the SecDec result.

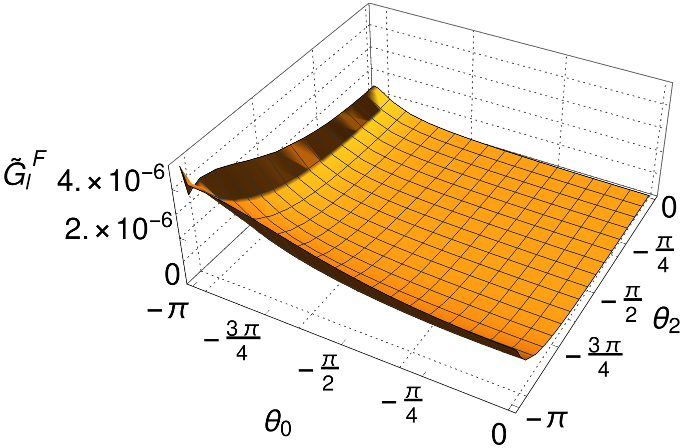
BACKUP - Convergence Sensitivity to Kinematics I



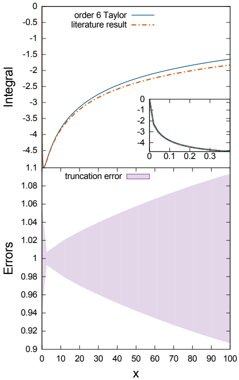
BACKUP - Convergence Sensitivity to Kinematics II



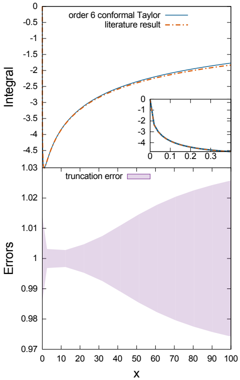
BACKUP - Convergence Sensitivity to Kinematics III



BACKUP - T41



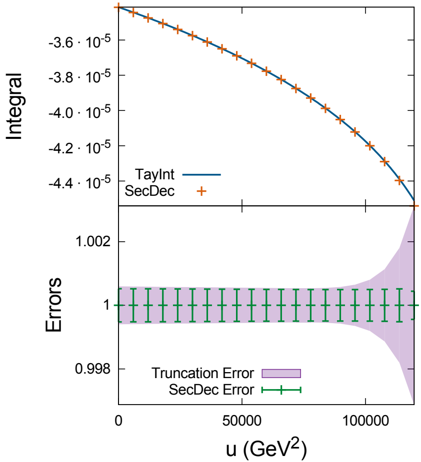
(k)



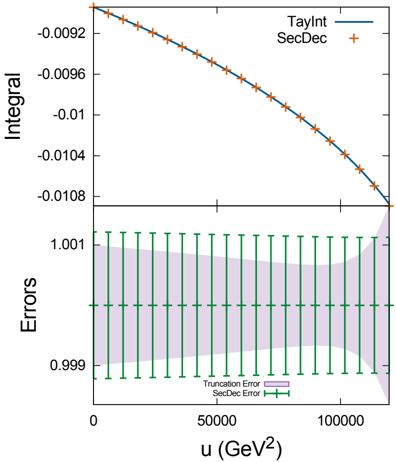
(l)

$$x = \frac{\sqrt{s + 4\,m^2} - \sqrt{s}}{\sqrt{s + 4\,m^2} + \sqrt{s}}$$

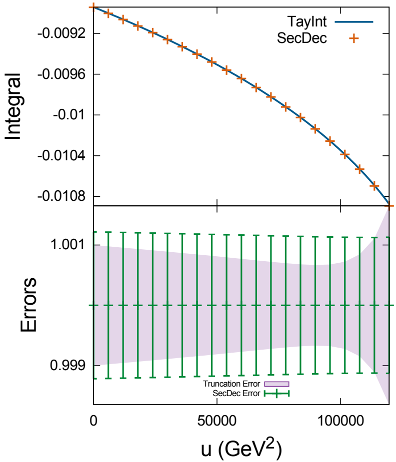
BACKUP - I10 ϵ^0, $u < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$



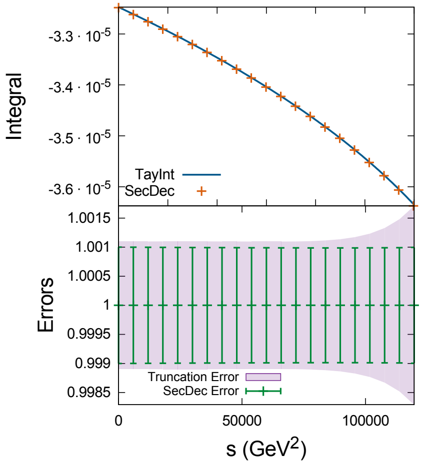
BACKUP - I10 ϵ^1, $u < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173$GeV



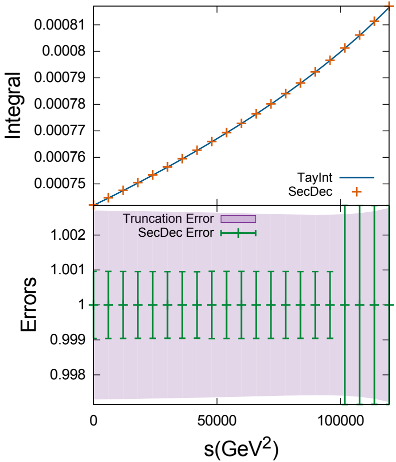
BACKUP - I10 ϵ^2, $u < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173$GeV



BACKUP - I39 ϵ^0, $s < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173$GeV



BACKUP - I39 ϵ^1, $s < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$



BACKUP - I39 ϵ^2, $s < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$