Integration-by-parts reductions via algebraic geometry

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Overview

1. Motivation

2. IBP identities on unitarity cuts

3. Syzygy equations and their solution

4. Main example:
Integration-by-parts reductions

IBP identities arise from the vanishing integration of total derivatives,

\[ \int \prod_{i=1}^{L} \frac{d^D \ell_i}{\pi^{D/2}} \sum_{j=1}^{L} \frac{ \partial}{\partial \ell_j^\mu} \frac{v_j^{\mu} P}{D_1^{a_1} \cdots D_k^{a_k}} = 0. \]

where \( P \) and \( v_j^{\mu} \) are polynomials in \( \ell_i, \mu_j, \) and \( a_i \in \mathbb{N}. \)

Role in perturbative QFT calculations:

- **Reduction.** Reduce number of contributing loop integrals by factor of \( O(10^2) - O(10^6) \) to basis.

- **Computing master integrals.** Enable setting up differential equations for basis integrals \( I_j: \)

\[ \frac{\partial}{\partial x_m} I(x, \epsilon) = A_m(x, \epsilon) I(x, \epsilon) \]

where \( x_m \) denotes a kinematical invariant.
IBP reductions on unitarity cuts

Standard approach: enumerate all linear relations and apply Gauss-Jordan elimination to large linear systems


Idea here: use unitarity cuts to block-diagonalize system

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\]

We use the Baikov representation \( k = \frac{L(L+1)}{2} + LE \),

\[
I(N; a) \equiv \int \prod_{j=1}^{L} \frac{d^{D} \ell_j}{i\pi^{D/2}} \frac{N}{D_1^{a_1} \cdots D_k^{a_k}} = \int \frac{dz_1 \cdots dz_k}{z_1^{a_1} \cdots z_k^{a_k}} \text{Gram}(z) \frac{D-L-E-1}{2} N
\]


in which cuts are straightforward to apply,

\[
\int \frac{dz_i}{z_i^{a_i}} \overset{\text{cut}}{\rightarrow} \int_{\Gamma_{\epsilon}(0)} \frac{dz_i}{Z_i^{a_i}} \quad i \in S_{\text{cut}}
\]
Example: Zurich-flag cut

Let us construct IBP identities on the Zurich-flag cut

Define \( S_{\text{cut}} = \{1, 2, 4, 5, 7\} \) and \( G = \text{Gram}(\vec{p}, \ell) \).

On \( S_{\text{cut}} \), the double-box integral takes the form

\[
I_{\text{DB}}^{\text{cut}}[P] = \prod_{i \in S_{\text{cut}}} \oint_{\Gamma_{\epsilon}(0)} \frac{d\tilde{z}_i}{\tilde{z}_i} \int \prod_{j \notin S_{\text{cut}}} d\tilde{z}_j \frac{G(\tilde{z})^{\frac{D-6}{2}}}{\tilde{z}_3 \tilde{z}_6} P(\tilde{z})
\]
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\]

Relabeling \( z\{1,2,3,4\} = \tilde{z}\{3,6,8,9\} \), this becomes

\[
I_{\text{DB}}^{\text{cut}} [P] = \int \frac{dz_1 \, dz_2 \, dz_3 \, dz_4}{z_1 z_2} \frac{G(z)}{z_3 z_6} \frac{G(z)}{z_3 z_6} \quad P(z)
\]
Need to find IBP identities which involve

\[ l_{\text{cut}}^{\text{DB}}[P] = \int \frac{dz_1 \, dz_2 \, dz_3 \, dz_4}{z_1z_2} G(z) \frac{D-6}{2} P(z) \]

**Total derivatives → IBP identities.** Generic total derivative on cut:

\[
0 = \int \left[ \sum_{i=1}^{4} \frac{\partial}{\partial z_i} \left( \frac{a_i(z) G(z) \frac{D-6}{2}}{z_1z_2} \right) \right] dz_1 \cdots dz_4
\]

\[
= \int \left[ \sum_{i=1}^{4} \left( \frac{\partial a_i}{\partial z_i} + \frac{D-6}{2G} \frac{\partial G}{\partial z_i} \right) - \sum_{j=1,2} \frac{a_j}{z_j} \right] \frac{G(z) \frac{D-6}{2}}{z_1z_2} dz_1 \cdots dz_4
\]

The red term corresponds to an integral in \((D - 2)\) dimensions, and the purple term in general produces doubled propagators.
To avoid dimension shifts and doubled propagators in

$$0 = \int \left[ \sum_{i=1}^{4} \left( \frac{\partial a_i}{\partial z_i} + \frac{D - 6}{2G} a_i \frac{\partial G}{\partial z_i} \right) - \sum_{j=1,2}^{2} \frac{a_j}{z_j} \right] G(z) \frac{D-6}{2} \frac{1}{z_1 z_2} dz_1 \cdots dz_4$$

we demand that each term is polynomial,

$$\sum_{i=1}^{4} a_i \frac{\partial G}{\partial z_i} + bG = 0$$

$$a_j + b_j z_j = 0$$

with $a_i, b_i, b$ polynomials in $z$. Such eqs. are known as syzygy equations.

[Gluza, Kajda, Kosower, PRD83(2011)045012], [Schabinger, JHEP01(2012)077], [Ita, PRD94(2016)116015]

Obtain IBPs by plugging $(a_i, b)$ into the top equation.
Note: $(qa_i, qb)$ is also a solution, for polynomial $q$. 
Strategy to solve syzygy equations

Solve syzygy equations with $c$ cuts

\begin{align*}
    a_j + b_j z_j &= 0, \quad j = 1, \ldots, k-c \\
    \sum_{j=1}^{m-c} a_j \frac{\partial G}{\partial z_k} + b G &= 0
\end{align*}

as follows.

1) The generators of (1) are trivial:
\[ \mathcal{M}_1 = \langle z_1 e_1, \ldots, z_k e_k, e_{k+1}, \ldots, e_m \rangle \]

2) Generators $\mathcal{M}_2 = \langle (a_1, \ldots, a_m, b), \ldots \rangle$ of (2) for the off-shell case $c = 0$ can be explicitly found:
\[ (a_\alpha, b) = \left( \sum_{k=1}^{E+L} (1+\delta_{ik}) x_{jk} \frac{\partial z_\alpha}{\partial x_{ik}}, 2\delta_{ij} \right) \]
where $x_{ij} = v_i \cdot v_j$ with $v_{i,j} \in \{p_1, \ldots, p_E, \ell_1, \ldots, \ell_L\}$.

[B"ohm, Georgoudis, KJL, Schulze, Zhang, PRD 98(2018)025023]

3) Take module intersection $\mathcal{M}_1|_{\text{cut}} \cap \mathcal{M}_2|_{\text{cut}}$
Example 1: syzygies of planar double box

Set \( P_{12} = p_1 + p_2 \) and

\[
\begin{align*}
    z_1 &= \ell_1^2, \\
    z_2 &= (\ell_1 - p_1)^2, \\
    z_3 &= (\ell_1 - P_{12})^2 \\
    z_4 &= (\ell_2 + P_{12})^2, \\
    z_5 &= (\ell_2 - p_4)^2, \\
    z_6 &= \ell_2^2 \\
    z_7 &= (\ell_1 + \ell_2)^2, \\
    z_8 &= (\ell_1 + p_4)^2, \\
    z_9 &= (\ell_2 + p_1)^2
\end{align*}
\]

Only need to find explicit relation \( z = Ax + B \). Here \( A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ -2 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \)

Set \( t_{i,j} = (a_\alpha, b) \). The syzygy generators are linear in the \( z_k \)

\[
\begin{align*}
    t_{4,1} &= (z_1 - z_2, z_1 - z_2, -s + z_1 - z_2, 0, 0, 0, z_1 - z_2 - z_6 + z_9, t + z_1 - z_2, 0, 0) \\
    t_{4,2} &= (s + z_2 - z_3, z_2 - z_3, z_2 - z_3, 0, 0, 0, z_2 - z_3 + z_4 - z_9, -t + z_2 - z_3, 0, 0) \\
    t_{4,3} &= (-s + z_3 - z_8, t + z_3 - z_8, z_3 - z_8, 0, 0, 0, z_3 - z_4 + z_5 - z_9, z_3 - z_8, 0, 0) \\
    t_{4,4} &= (2z_1, z_1 + z_2, -s + z_1 + z_3, 0, 0, 0, z_1 - z_6 + z_7, z_1 + z_8, 0, -2) \\
    t_{4,5} &= (-z_1 - z_6 + z_7, -z_1 + z_7 - z_9, s - z_1 - z_4 + z_7, 0, 0, 0, -z_1 + z_6 + z_7, -z_1 - z_5 + z_7, 0, 0) \\
    t_{5,1} &= (0, 0, 0, s - z_6 + z_9, -t - z_6 + z_9, z_9 - z_6, z_1 - z_2 - z_6 + z_9, 0, z_9 - z_6, 0) \\
    t_{5,2} &= (0, 0, 0, z_4 - z_9, t + z_4 - z_9, -s + z_4 - z_9, z_2 - z_3 + z_4 - z_9, 0, z_4 - z_9, 0) \\
    t_{5,3} &= (0, 0, 0, z_5 - z_4, z_5 - z_4, s - z_4 + z_5, z_3 - z_4 + z_5 - z_8, 0, -t - z_4 + z_5, 0) \\
    t_{5,4} &= (0, 0, 0, s - z_3 - z_6 + z_7, -z_6 + z_7 - z_8, -z_1 - z_6 + z_7, z_1 - z_6 + z_7, 0, -z_2 - z_6 + z_7, 0) \\
    t_{5,5} &= (0, 0, 0, -s + z_4 + z_6, z_5 + z_6, 2z_6, -z_1 + z_6 + z_7, 0, z_6 + z_9, -2)
\end{align*}
\]
Example 2: syzygies of non-planar double pentagon

Set $P_{i,j} \equiv p_i + p_j$ and

$$z_1 = \ell_1^2,$$
$$z_2 = (\ell_1 - p_1)^2,$$
$$z_3 = (\ell_1 - P_{1,2})^2,$$
$$z_4 = (\ell_2 - P_{3,4})^2,$$
$$z_5 = (\ell_2 - p_4)^2,$$
$$z_6 = \ell_2^2,$$
$$z_7 = (\ell_1 + \ell_2)^2,$$
$$z_8 = (\ell_1 + \ell_2 + p_5)^2,$$
$$z_9 = (\ell_1 + p_3)^2,$$
$$z_{10} = (\ell_1 + p_4)^2,$$
$$z_{11} = (\ell_2 + p_1)^2$$

Here $z = Ax + B$ with

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\
-2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \\
-2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

and the syzygy generators are again compact:
Computing module intersections

Given $\mathcal{M}_1 = \langle v_1, \ldots, v_p \rangle$ and $\mathcal{M}_2 = \langle w_1, \ldots, w_q \rangle$ with $v_i, w_j$ $m$-tuples of polynomials. Let $Q$ denote the $m \times (p+q)$ matrix

$$Q = \begin{pmatrix}
\vdots & \cdots & \cdots & \cdots & \cdots \\
 v_1 & \cdots & v_p & w_1 & \cdots & w_q \\
\vdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}$$

Then compute wrt. POT and variable order $[z_1, \ldots, z_m] > [s_{ij}]$

$$\langle h_1, \ldots, h_t \rangle \equiv \text{Gröbner basis of column space of } Q \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Selecting $h_i = (0, \ldots, 0, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^{p} x_j v_j + \sum_{k=1}^{q} y_k w_k$$
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$$Q = \begin{pmatrix} v_1 & \cdots & v_p & w_1 & \cdots & w_q \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Then compute wrt. POT and variable order $[z_1, \ldots, z_m] \succ [s_{ij}]$

$$\langle h_1, \ldots, h_t \rangle \equiv \text{Gröbner basis of column space of } \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Selecting $h_i = (0, \ldots, 0, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^p x_j v_j + \sum_{k=1}^q y_k w_k \implies \sum_{j=1}^p x_j v_j = -\sum_{k=1}^q y_k w_k \in \mathcal{M}_1 \cap \mathcal{M}_2$$

Hence $\sum_{j=1}^p x_j v_j$ generate $\mathcal{M}_1 \cap \mathcal{M}_2$, taking $(x_1, \ldots, x_p)$ from each $h_i$. 
Spanning set of cuts for IBPs

To find the complete IBP reduction, we must consider the cuts associated with “uncollapsible” masters:

A bit more explicitly, the cuts we need to consider are

A bit more explicitly, the cuts we need to consider are
Main example: non-planar hexagon box

**Task:** IBP reduce non-planar hexagon box with numerator insertions of degree four in the $z_i$

There are 10 cuts to consider:

- [Chicherin, Henn, Mitev JHEP 05(2018)164]
- [Badger, Brønnum-Hansen, Hartanto, Peraro, PRL 120(2018)092001]
- [Abreu, Cordero, Ita, Page, Zeng, PRD 97(2018)116014]
- [Chawdhry, Lim, Mitov, 1805.09182]
- [S. Abreu, B. Page, M. Zeng, 1807.11522]
- [D. Chicherin, T. Gehrmann, J. Henn, N.A. Lo Presti, V. Mitev, P. Wasser, 1809.06240]
Construct and solve IBP identities on a spanning set of cuts.

Cut $\{1, 5, 7\}$
Non-planar hexagon box: spanning set of cuts

Construct and solve IBP identities on a spanning set of cuts.

Cut \{2, 5, 7\}
Construct and solve IBP identities on a spanning set of cuts.

\[ \mathcal{I}_{1,2,3}, \mathcal{I}_{4,5,6}, \mathcal{I}_{7,8}, \mathcal{I}_{9,10}, \mathcal{I}_{11,12,13}, \mathcal{I}_{14,15}, \mathcal{I}_{16,17}, \mathcal{I}_{18,19}, \mathcal{I}_{20,21}, \mathcal{I}_{22,23}, \mathcal{I}_{24}, \mathcal{I}_{25,26}, \mathcal{I}_{27,28}, \mathcal{I}_{29}, \mathcal{I}_{30,31}, \mathcal{I}_{32}, \mathcal{I}_{33}, \mathcal{I}_{34}, \mathcal{I}_{35}, \mathcal{I}_{36}, \mathcal{I}_{37}, \mathcal{I}_{38}, \mathcal{I}_{39}, \mathcal{I}_{40}, \mathcal{I}_{41,42}, \mathcal{I}_{43}, \mathcal{I}_{44}, \mathcal{I}_{45}, \mathcal{I}_{46}, \mathcal{I}_{47}, \mathcal{I}_{48}, \mathcal{I}_{49}, \mathcal{I}_{50}, \mathcal{I}_{51}, \mathcal{I}_{52,53}, \mathcal{I}_{54,55}, \mathcal{I}_{56}, \mathcal{I}_{57}, \mathcal{I}_{58}, \mathcal{I}_{59}, \mathcal{I}_{60}, \mathcal{I}_{61}, \mathcal{I}_{62}, \mathcal{I}_{63}, \mathcal{I}_{64}, \mathcal{I}_{65}, \mathcal{I}_{66}, \mathcal{I}_{67}, \mathcal{I}_{68}, \mathcal{I}_{69}, \mathcal{I}_{70}, \mathcal{I}_{71}, \mathcal{I}_{72}, \mathcal{I}_{73}, \mathcal{I}_{74}, \mathcal{I}_{75} \]

Cut \( \{2,5,8\} \)
Construct and solve IBP identities on a spanning set of cuts.

Cut \( \{2, 6, 7\} \)
Construct and solve IBP identities on a spanning set of cuts.

Cut \{3, 5, 8\}
Construct and solve IBP identities on a spanning set of cuts.

Cut \(\{3, 6, 7\}\)
Construct and solve IBP identities on a spanning set of cuts.

Cut \{3, 6, 8\}
Construct and solve IBP identities on a spanning set of cuts.
Non-planar hexagon box: spanning set of cuts

Construct and solve IBP identities on a spanning set of cuts.

Cut \{1, 4, 5, 8\}
Construct and solve IBP identities on a spanning set of cuts.

Cut \{1, 4, 6, 7\}
Syzygies for the non-planar hexagon box

Syzygies for ensuring $D$-dimensionality:

$$M_1 = \langle (z_1 - z_2, z_1 - z_2, -s_{12} + s_1 - s_2, -s_{12} - s_{13} + s_1 - s_2, s_{14} - s_1 - s_2 - s_8 + s_{10}, z_1 - s_2 - s_8 + s_{10}, 0, 0, -s_{12} - s_{13} - s_{14} + s_1 - s_2, 0, 0) \mid M_{11} \cap M_{12} \cap \text{cut} \rangle$$

$$M_2 = \langle (z_1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \mid \text{cut} \cap \langle (0, z_2, 0, 0, 0, 0, 0, 0, 0, 0, 0) \rangle \rangle$$

Syzygies for ensuring no doubled propagators:

$$M_1 = \langle (s_{12} + s_8 - s_3, -s_3 - s_2, z_1 - s_2 - s_8 + s_{10}, z_1 - z_2 - s_8 + s_{10}, 0, 0, -s_{12} - s_{13} - s_{14} + s_1 - s_2, 0, 0) \rangle$$

Compute intersection of $M_1|_{\text{cut}} \cap M_2|_{\text{cut}}$ on each of the 10 cuts.
Complexity of IBP systems

- Resources to compute $M_1|_{\text{cut}} \cap M_2|_{\text{cut}}$: 25-800 s and 1-14 GB RAM (on 24 cores, 3.40 GHz)

- Size of generating systems after trimming: 1.5-10 MB

Plug resulting generators into ansatz for total derivative:

$$0 = \int \left[ \sum_{i=1}^{m-c} \left( \frac{\partial a_{ri}}{\partial z_{ri}} + \frac{D-L-E-1}{2G(z)} a_{ri} \frac{\partial G}{\partial z_{ri}} \right) - \sum_{i=1}^{k-c} \frac{a_{ri}}{z_{ri}} \right] G(z) \left( \frac{D-L-E-1}{2} \right) d z_{r_1} \cdots d z_{r_{m-c}}$$

- Resulting linear systems to solve: 700-1200 equations, size 1 MB, density 1.5%
Gauss-Jordan elimination of IBP systems

To find the IBP reductions, Gauss-Jordan eliminate IBP systems.

Some remarks:

- To preserve sparsity, use a *total pivoting* strategy (i.e., allow column swaps)

- For cut \(\{1, 4, 6, 7\}\), the RREF can be performed fully analytically, requiring 31 minutes on one core and 1.5 GB RAM.

- For \(\{3, 6, 7\}\), assigned numerical values to two \(s_{ij}\).
  Ran 440 points on cluster (2.5 h and 1.8 GB RAM per job).
  Used interpolation code to get analytical results (23 min and 15 GB RAM on one core).

[von Manteuffel and Schabinger, PLB 744(2015)101]
[Peraro, JHEP12(2016)030]
Merging on-shell IBP reductions

By solving the IBP identities on the following cuts

we reconstruct the complete IBP reductions by merging the partial results.

An example of an IBP relation produced by our method ($\chi \equiv t/s$):

\[
\begin{align*}
&\quad (\bullet\ldots\bullet)^2 = \frac{(D-4)s^2\chi}{8(D-3)}
\quad \frac{(3D-2\chi-12)s}{4(D-3)}
\quad \frac{(4-D)(9\chi+7)}{4(D-3)}
\quad \frac{2(D\chi+1)-8\chi-7}{2(D-4)s}
\quad \frac{9(3D-10)(3D-8)}{4(D-4)^2s^2\chi}
\quad \frac{(3D-10)(3D-8)(2\chi+1)}{2(D-4)^2(D-3)s^2}
\end{align*}
\]
Results for IBP reductions

- Fully analytic IBP reductions of the 32 hexagon boxes

\[
\begin{align*}
&\{(1,1,1,1,1,1,1,0,0,-4), (1,1,1,1,1,1,1,0,-3,-1), (1,1,1,1,1,1,1,-1,-1,-2), \\
&(1,1,1,1,1,1,1,-2,0,-2), (1,1,1,1,1,1,1,-3,0,-1), (1,1,1,1,1,1,1,0,0,-3), \\
&(1,1,1,1,1,1,1,0,-3,0), (1,1,1,1,1,1,1,-1,-2,0), (1,1,1,1,1,1,1,-3,0,0), \\
&(1,1,1,1,1,1,1,0,-2,0), (1,1,1,1,1,1,1,0,0,-1)\} \\
&\{(1,1,1,1,1,1,1,0,-1,-3), (1,1,1,1,1,1,1,0,-4,0), (1,1,1,1,1,1,1,-1,-2,-1), \\
&(1,1,1,1,1,1,1,-2,-1,-1), (1,1,1,1,1,1,1,0,-1,-2), (1,1,1,1,1,1,1,-3,-1,0), \\
&(1,1,1,1,1,1,1,0,-1,-2), (1,1,1,1,1,1,1,-1,0,-2), (1,1,1,1,1,1,1,-2,0,-1), \\
&(1,1,1,1,1,1,1,0,-2), (1,1,1,1,1,1,1,0,-1,0,-1), (1,1,1,1,1,1,1,0,-1,0), \\
&(1,1,1,1,1,1,1,0,-1)\}
\end{align*}
\]

can be downloaded from (268 MB compressed / 790 MB uncompressed)
https://github.com/yzhphy/hexagonbox_reduction/releases/download/1.0.0/hexagon_box_degree_4_Final.zip

- Our results agree with fully numerical results from FIRE5 C++
(6 hours per point).

[A. Smirnov, CPC 189(2015)182]
Conclusions

- New formalism for IBP reductions. Main ideas: cuts, IBP identities from syzygies, total pivoting, rational reconstruction.

- Obtained the fully analytic IBP reductions of

  ![Diagram of a Feynman diagram with numerator insertions up to degree 4 in the $z_i$.]

  with numerator insertions up to degree 4 in the $z_i$.

- Powerful framework. IBP reductions for further $2 \rightarrow 3$ two-loop processes seem well within reach.