Amplitudes in the LHC Era

Adventures in Loop Integration

Jacob Bourjaily

Niels Bohr International Academy

based on work in collaboration with

Caron-Huot, Herrmann, Trnka; Dixon, Dulat, Panzer;

He, ***McLeod****, Spradlin,* ***von Hippel****,* ***Wilhelm****; Duhr, Dulat, Penante,…*

The Niels Bohr International Academy

VILLUM FONDEN

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Roadmap (past Elliptic Polylogs)

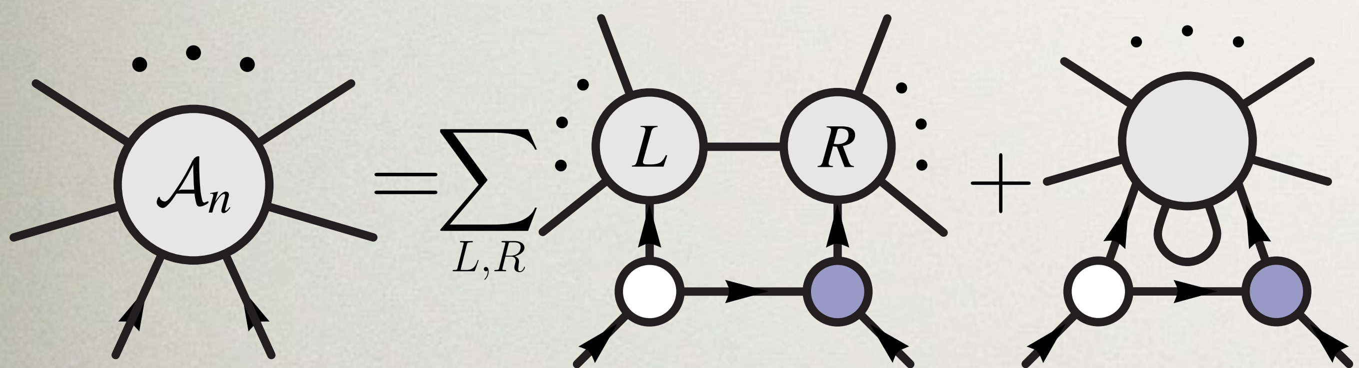
- **Loop** ***Integrands*** (***prescriptive*** *representations of*)
  [**JB**, Herrmann, Trnka (2016)]
  [**JB**, Herrmann, McLeod, Trnka (*in prep*)]
- **Loop** ***Integration*** (*better technology for*)
  - ***dual-conformal sufficiency*** [**JB**, Dixon, Dulat, Panzer (*to appear*)]
  - ***momentum twistor reducibility***
    [**JB**, McLeod, von Hippel, Wilhelm (2018)]
- **Loop** ***Integrals*** (*general structure of*)
  - beyond multiple polylogarithms
  - beyond elliptic polylogarithms
  - *a bestiary of irreducible loop-integral geometries*
    [**JB**, McLeod, Spradlin, von Hippel, Wilhelm (2017)]
    [**JB**, He, McLeod, von Hippel, Wilhelm (2018)]
    [**JB**, McLeod, von Hippel, Wilhelm (2018)]
    [**JB**, McLeod, von Hippel, Wilhelm (*in prep.*)]

Constructing Integrands for Loop Amplitudes (constructively)

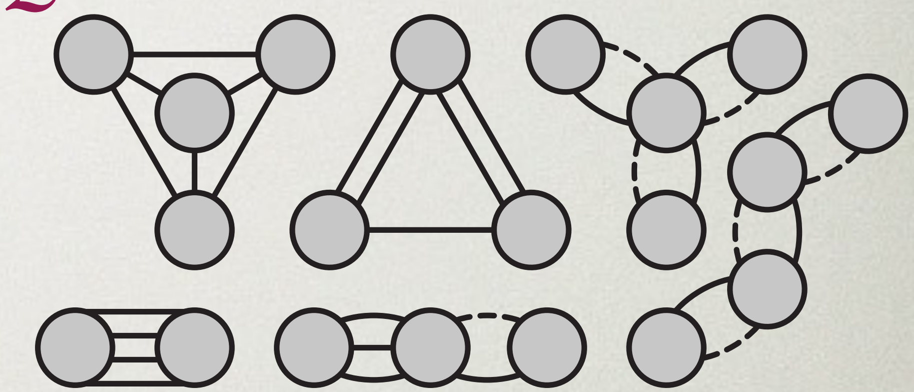
Novel Representations of Integrands

- **Powerful new tools** now exist for ***understanding*** and ***computing*** integrands in perturbation theory

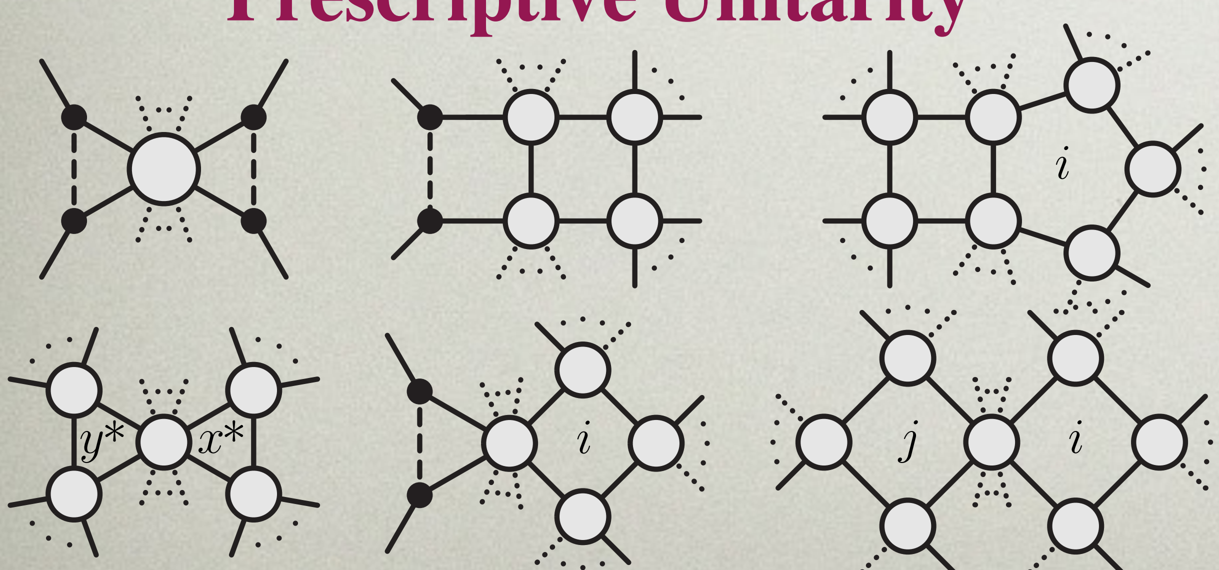
Recursion Relations



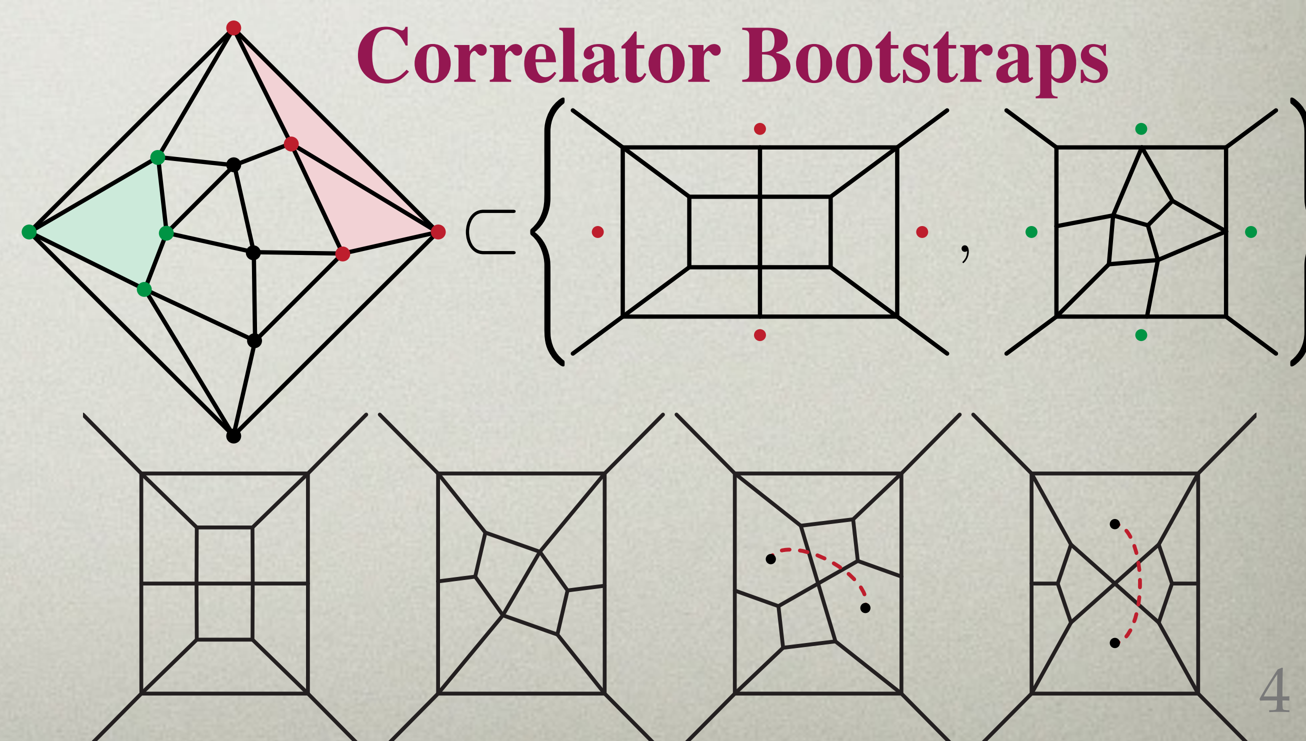
***Q*-cuts and Forward Limits**



Prescriptive Unitarity



Correlator Bootstraps



Generalized/Prescriptive Unitarity

[Bern, Dixon, Kosower; Dunbar; …]

- Integrands are rational functions—so may be expanded into an *arbitrary* (but complete) basis:

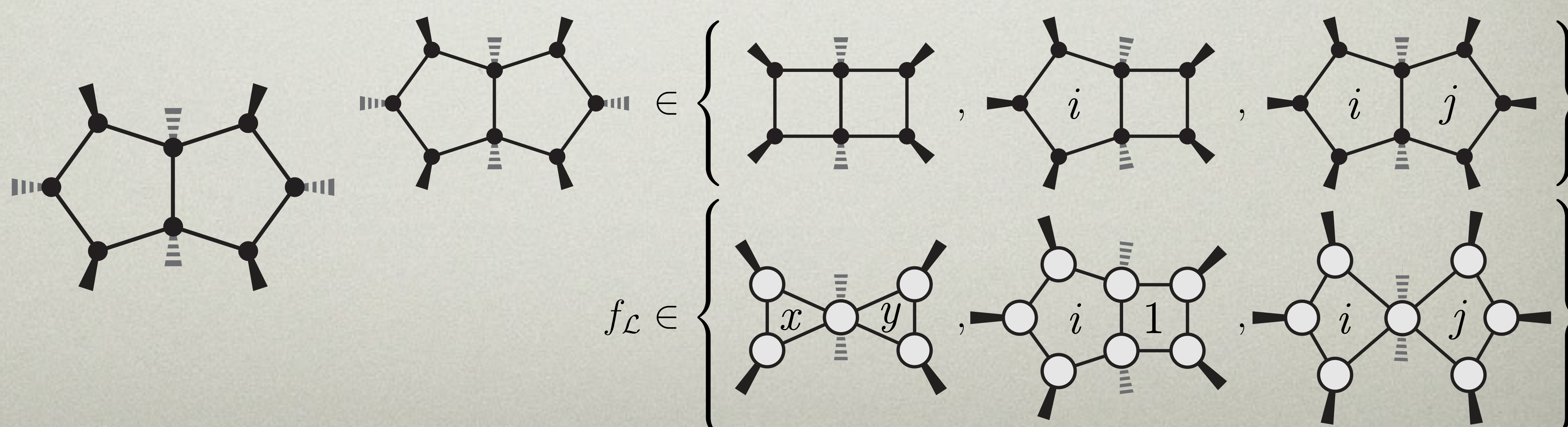
$$\mathcal{A}^L = \sum_i c_i\, \mathcal{I}_i$$

with coefficients c_i determined by cuts

[**JB**, Herrmann, Trnka]

- A representation is called ***prescriptive*** if all the coefficients are *individual* field-theory residues

$$\mathcal{A}_n^{L=2} = \sum_{\mathcal{L}} f_{\mathcal{L}}$$



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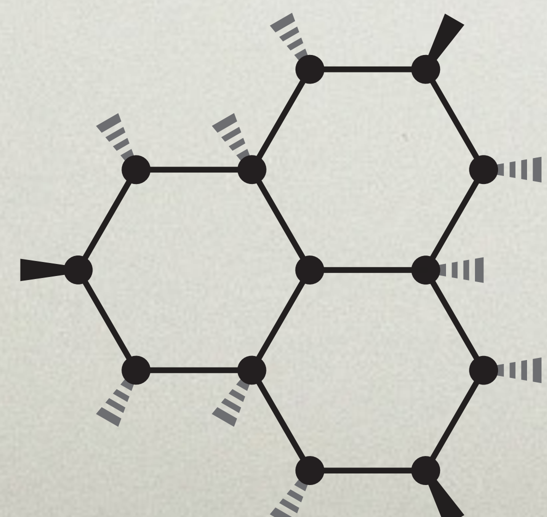
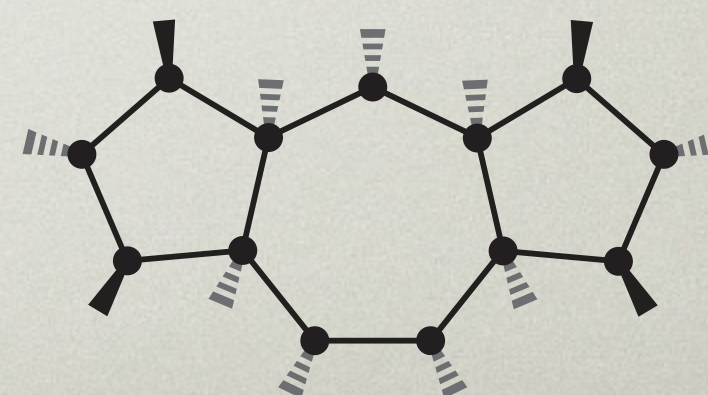
$$\mathcal{A}_n^{L=3} = \sum_{\mathcal{W}} f_{\mathcal{W}}$$  $$+ \sum_{\mathcal{L}} f_{\mathcal{L}}$$ 

Building Bases of Loop Integrands

- In order to define a (finite-dimensional) basis of loop integrands, two things must be specified:
  - A *fixed* spacetime dimension d (or $(d{-}2\epsilon)$)
  - A *bound* on the (loop-dependent) polynomial degrees—the "power-counting" of the theory

A Notational Triviality: WLOG

use inverse propagators for everything! $(\ell,Q)\equiv(\ell-Q)^2$

$$[\ell]\equiv\mathrm{span}\big\{(\ell-Q)^2\big\}=\mathrm{span}\big\{\ell^2,\ell\cdot k_i,1\big\}\qquad\mathrm{rnk}([\ell])=(d{+}2)$$

$$[\ell]^k\equiv\mathrm{span}\big\{\big[(\ell-Q)^2\big]^k\big\}\qquad\mathrm{rnk}([\ell]^k)=\binom{d{+}k}{d}+\binom{d{+}k{-}1}{d}$$

$$\textit{nota bene:}\ 1\in[\ell]\ \Rightarrow[\ell]^a\subseteq[\ell]^{a+b}\quad\forall\,a,b\geq0$$

Basics of Basis Reduction

[Passarino, Veltman; van Neerven, Vermaseren]

- Consider one-loop integrands in *d* dimensions
  A classic result (of P-V) is that all integrands with $(d{+}2)$ propagators or more are *reducible*:

$$1 \in [\ell] = \text{span}\{(\ell, a_1), \ldots, (\ell, a_{d+2})\}$$
$$\frac{1}{(\ell, a_1)\cdots(\ell, a_{d+2})} \subset \frac{[\ell]}{(\ell, a_1)\cdots(\ell, a_{d+2})}$$

- Moreover, the only independent integrands with $(d{+}1)$ propagators can be chosen to be *parity-odd*

$$1 \in [\ell] = \text{span}\{\underbrace{(\ell, a_1), \ldots, (\ell, a_{d+1})}_{\text{“contact terms”}}, i\epsilon(\ell, a_1, \ldots, a_{d+1})\}$$

Power-Counting & Constructibility

- An integrand has 'p-gon power-counting' if:

$$\lim_{\ell\to\infty}(\mathcal{I})=\frac{1}{(\ell^2)^p}\big(1+\mathcal{O}(1/\ell^2)\big)$$

(much less* ambiguous for integrand bases than amplitudes)

- Let $\mathcal{B}_p$ denote a complete basis of integrands with p-gon power-counting. Because $1\in[\ell]$, $\mathcal{B}_{p+1}\subset\mathcal{B}_p$

$$\widehat{\mathcal{B}}_p\equiv\mathcal{B}_p\backslash\mathcal{B}_{p+1} \qquad \mathcal{B}_p=\mathcal{B}_d\oplus\widehat{\mathcal{B}}_{d-1}\oplus\cdots\oplus\widehat{\mathcal{B}}_p$$

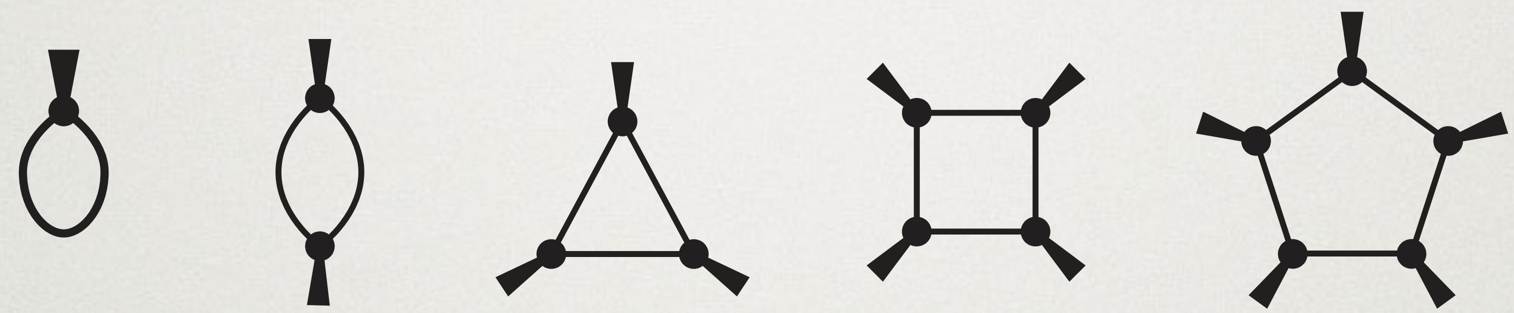
- An *amplitude* is 'p-gon **constructible**' if: $\mathcal{A}\subset\mathcal{B}_p$
  (*nota bene*: this may be loop-order (L) dependent!)

$$\mathcal{A}_p\equiv\mathcal{A}\cap\widehat{\mathcal{B}}_p \qquad \mathcal{A}=\mathcal{A}_d\oplus\mathcal{A}_{d-1}\oplus\cdots$$

One Loop Unitarity Redux (4d)

- Re-considering one loop unitarity in 4 dimensions

[Ossola, Papadopoulos, Pittau; Forde, Kosower]

1 

$\mathcal{B}_4$					1	$[\ell]^1$
$\mathcal{B}_3$				1	$[\ell]^1$	$[\ell]^2$
$\mathcal{B}_2$			1	$[\ell]^1$	$[\ell]^2$	$[\ell]^3$
$\mathcal{B}_1$		1	$[\ell]^1$	$[\ell]^2$	$[\ell]^3$	$[\ell]^4$
$\mathcal{B}_0$	1	$[\ell]^1$	$[\ell]^2$	$[\ell]^3$	$[\ell]^4$	$[\ell]^5$

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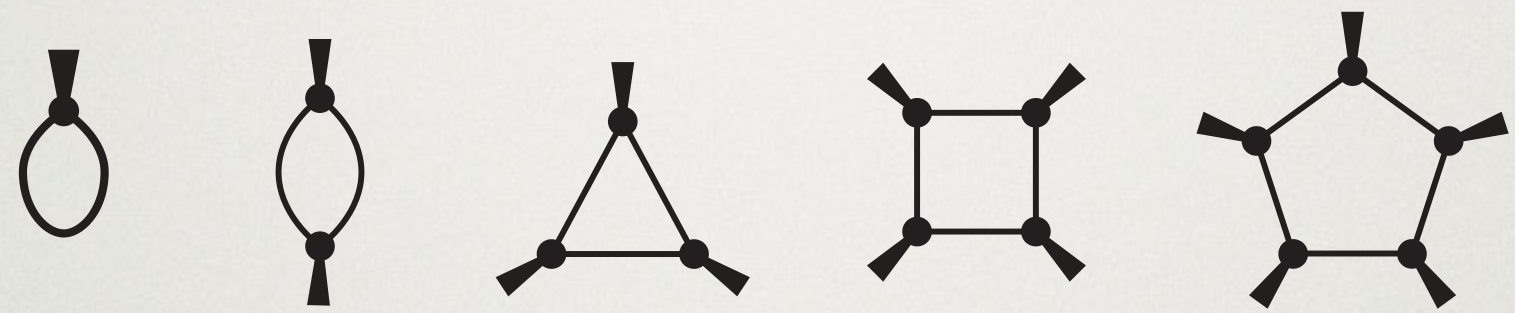
1 

$\mathcal{B}_4$					1	6
$\mathcal{B}_3$				1	6	20
$\mathcal{B}_2$			1	6	20	50
$\mathcal{B}_1$		1	6	20	50	105
$\mathcal{B}_0$	1	6	20	50	105	196

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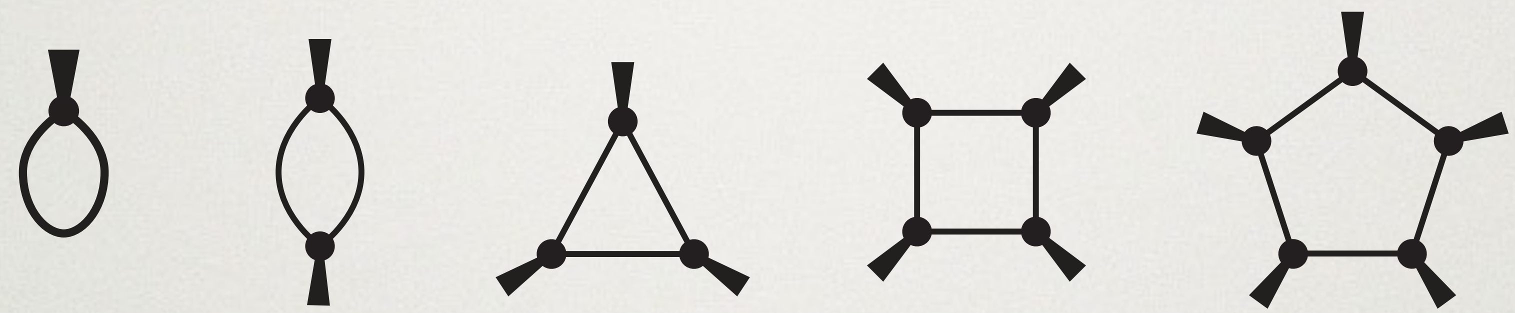
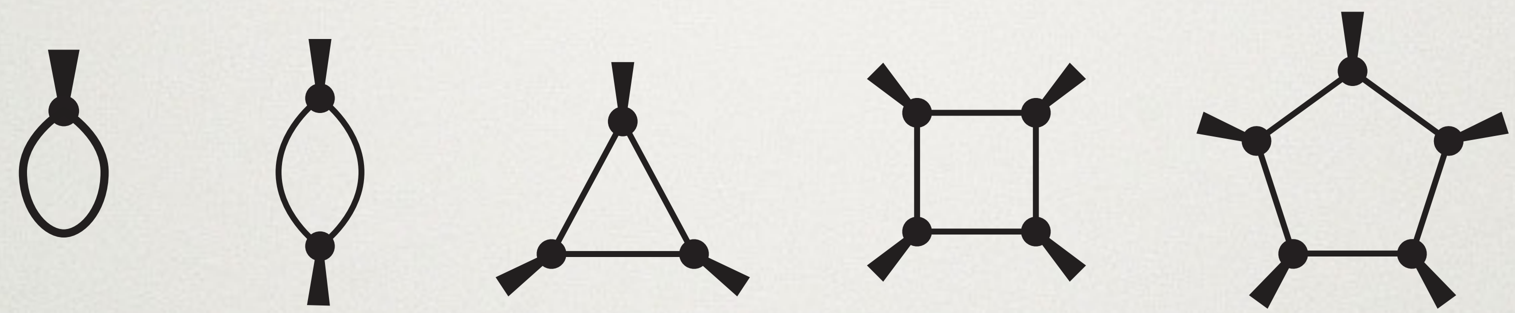
1 

$\mathcal{B}_4$					**1**+0	**1**+5
$\mathcal{B}_3$				**1**+0	**2**+4	**0**+20
$\mathcal{B}_2$			**1**+0	**3**+3	**2**+18	**0**+50
$\mathcal{B}_1$		**1**+0	**4**+2	**5**+15	**2**+48	**0**+105
$\mathcal{B}_0$	**1**+0	**5**+1	**9**+11	**7**+43	**2**+103	**0**+196

One Loop Unitarity Redux (4d)

- Re-considering one loop unitarity in 4 dimensions

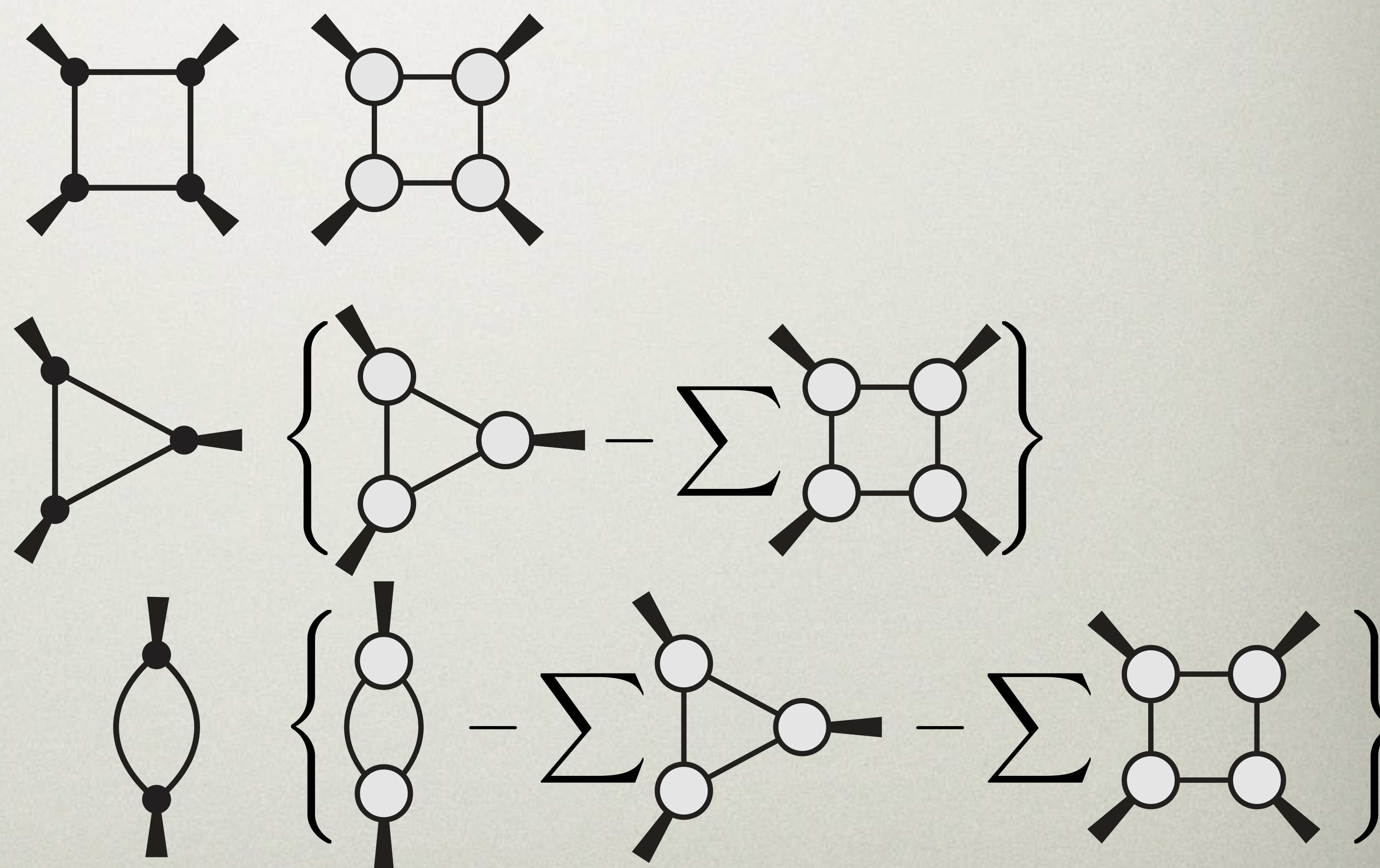
[Ossola, Papadopoulos, Pittau; Forde, Kosower]



	1	tadpole	bubble	triangle	box	pentagon	
$\widehat{\mathcal{B}}_4$					1+0	0+5	weight = 2
$\widehat{\mathcal{B}}_3$				1+0	1+4	0+14	
$\widehat{\mathcal{B}}_2$			1+0	2+3	0+14	0+30	weight = 1
$\widehat{\mathcal{B}}_1$		1+0	3+2	2+12	0+30	0+55	weight = 0
$\widehat{\mathcal{B}}_0$	1+0	4+1	5+9	2+28	0+55	0+91	

Triangularity & Diagonalization

- Consider a theory that is bubble constructible (such as $\mathcal{N} \geq 1$ SYM)



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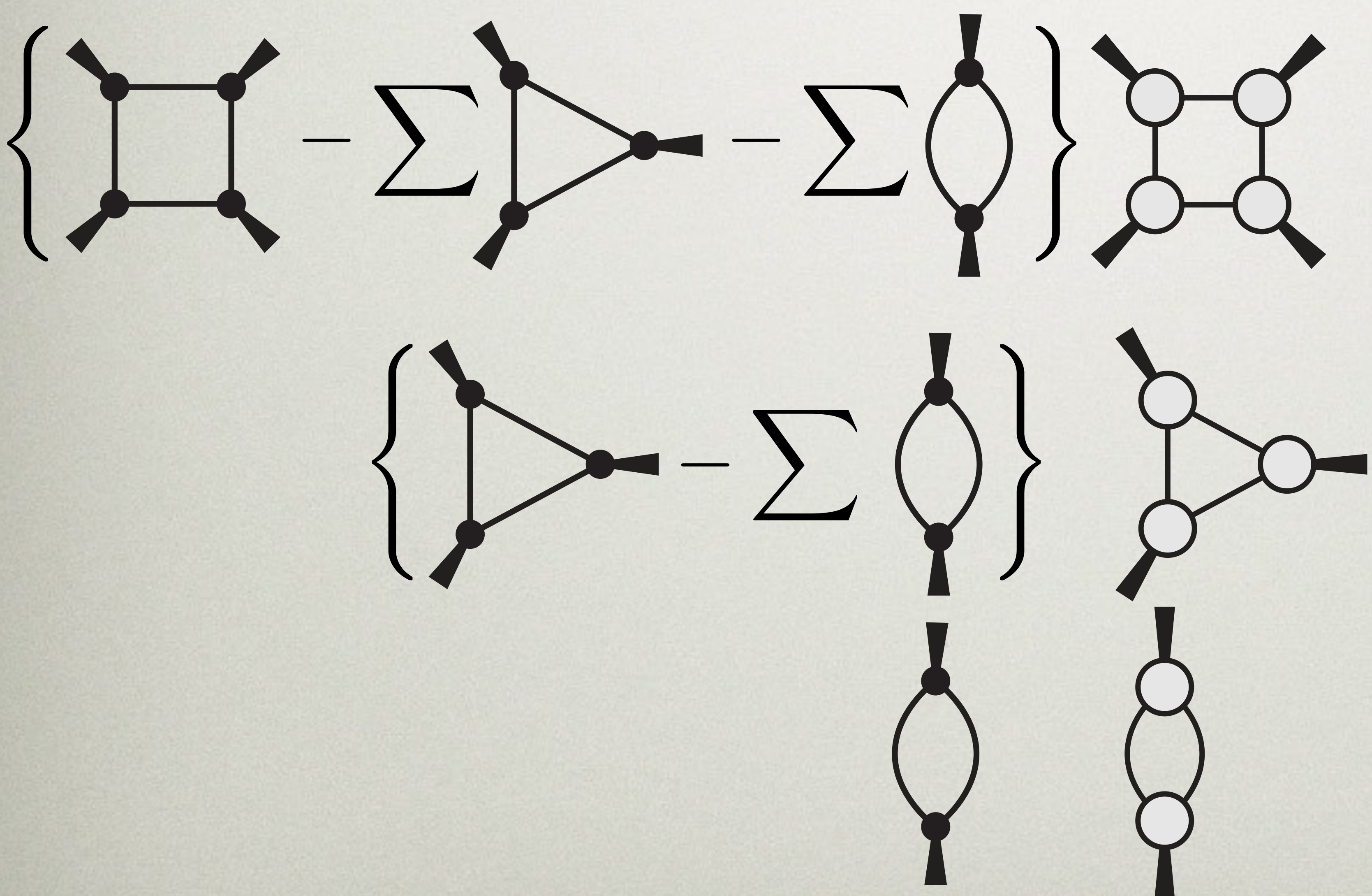
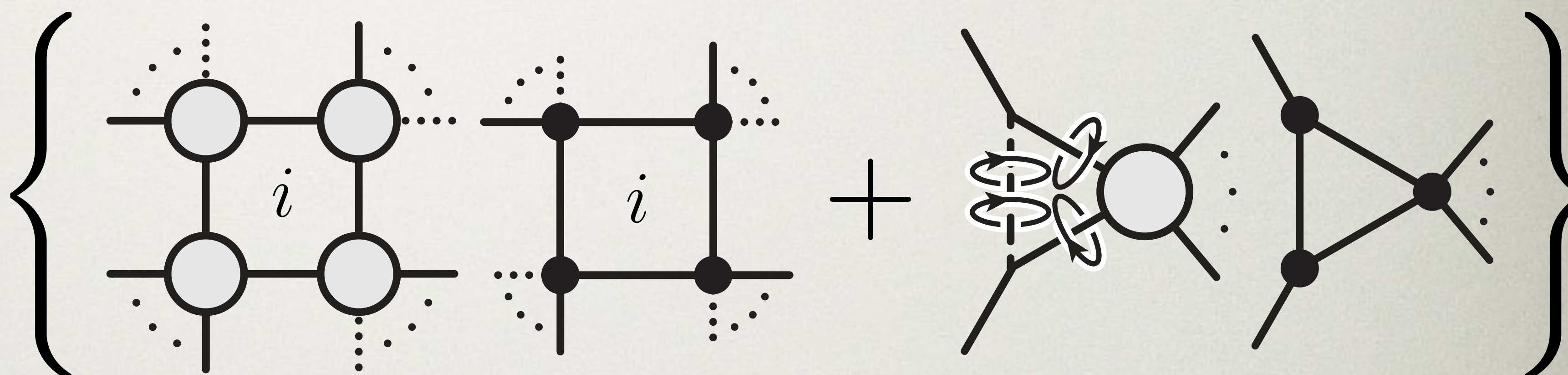
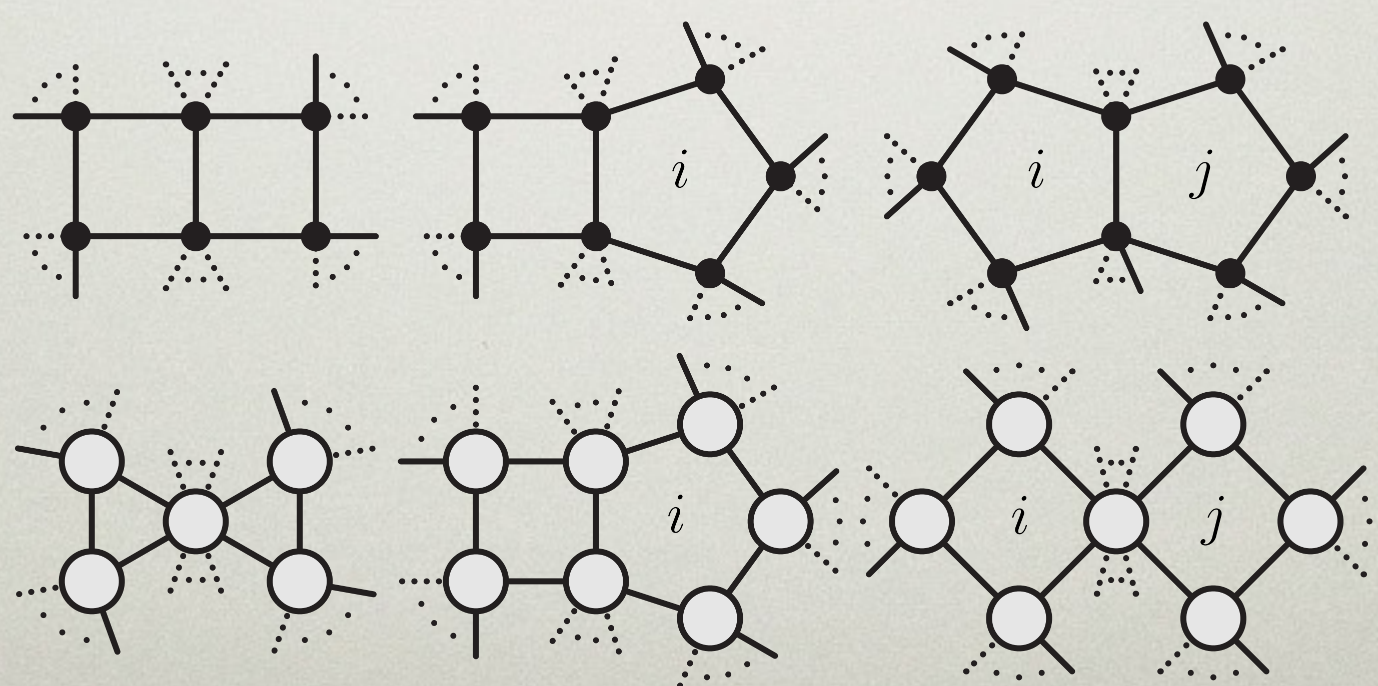


Using Prescriptive *Unitarity*

- Rather than starting from an arbitrary basis of loop integrands, tailor each to *manifestly* match one cut
- one loop:
  [**JB**, Caron-Huot, Trnka (2013)]



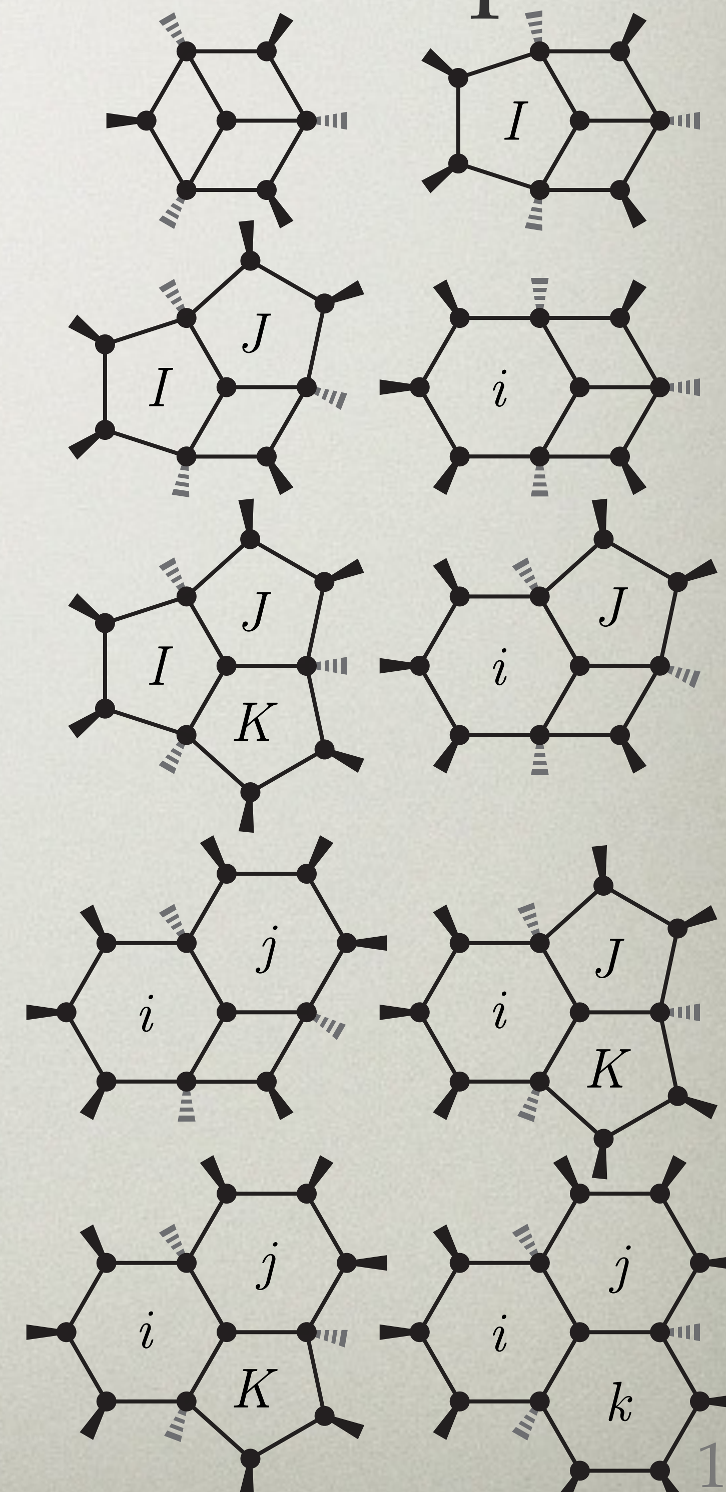
- two loop:
  [**JB**, Trnka (2015)]

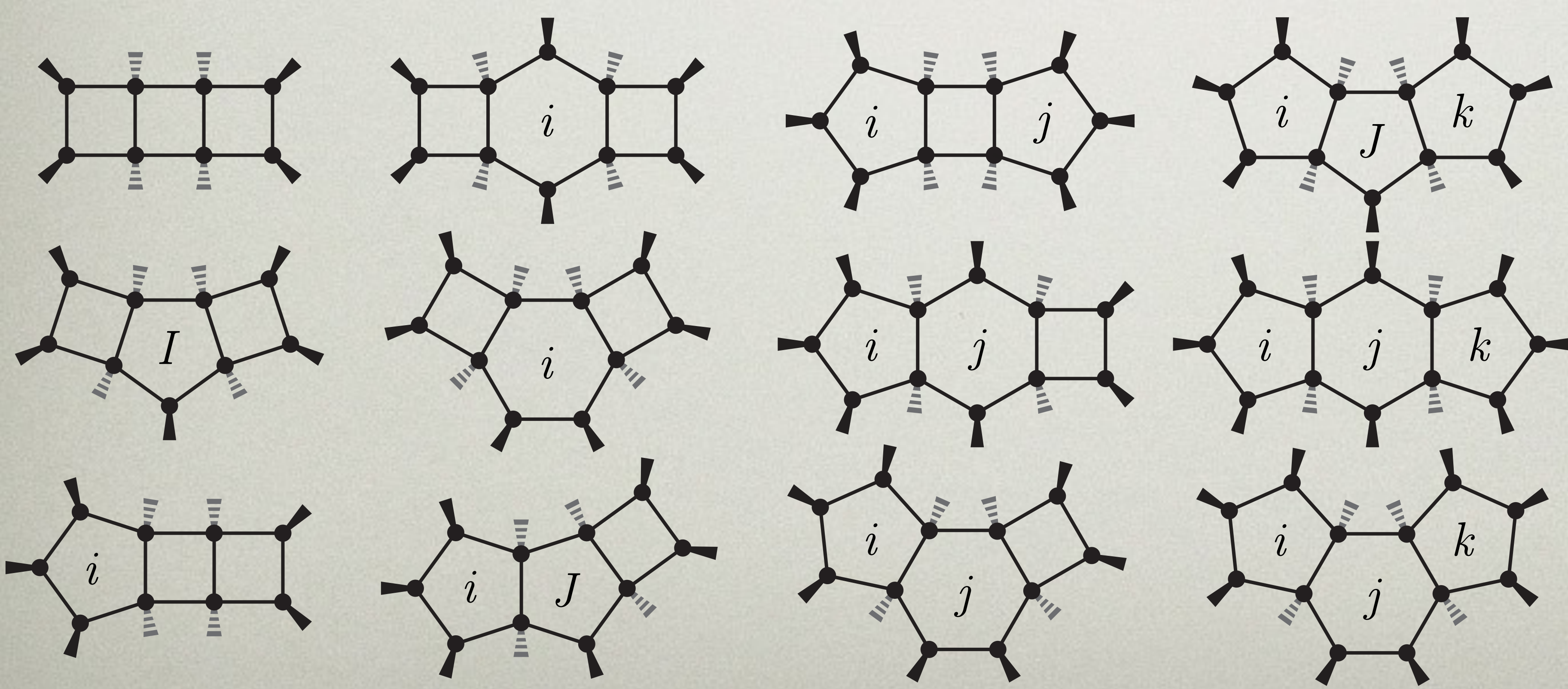


Extending the Prescriptive Reach

- This procedure continues to work at three loops:
  [**JB**, Herrmann, Trnka (2017)]

- Generalizing this to non-planar theories is quite easy, *provided less-than-best power counting is considered*
  [**JB**, Herrmann, McLeod, Trnka (*in prep*)]



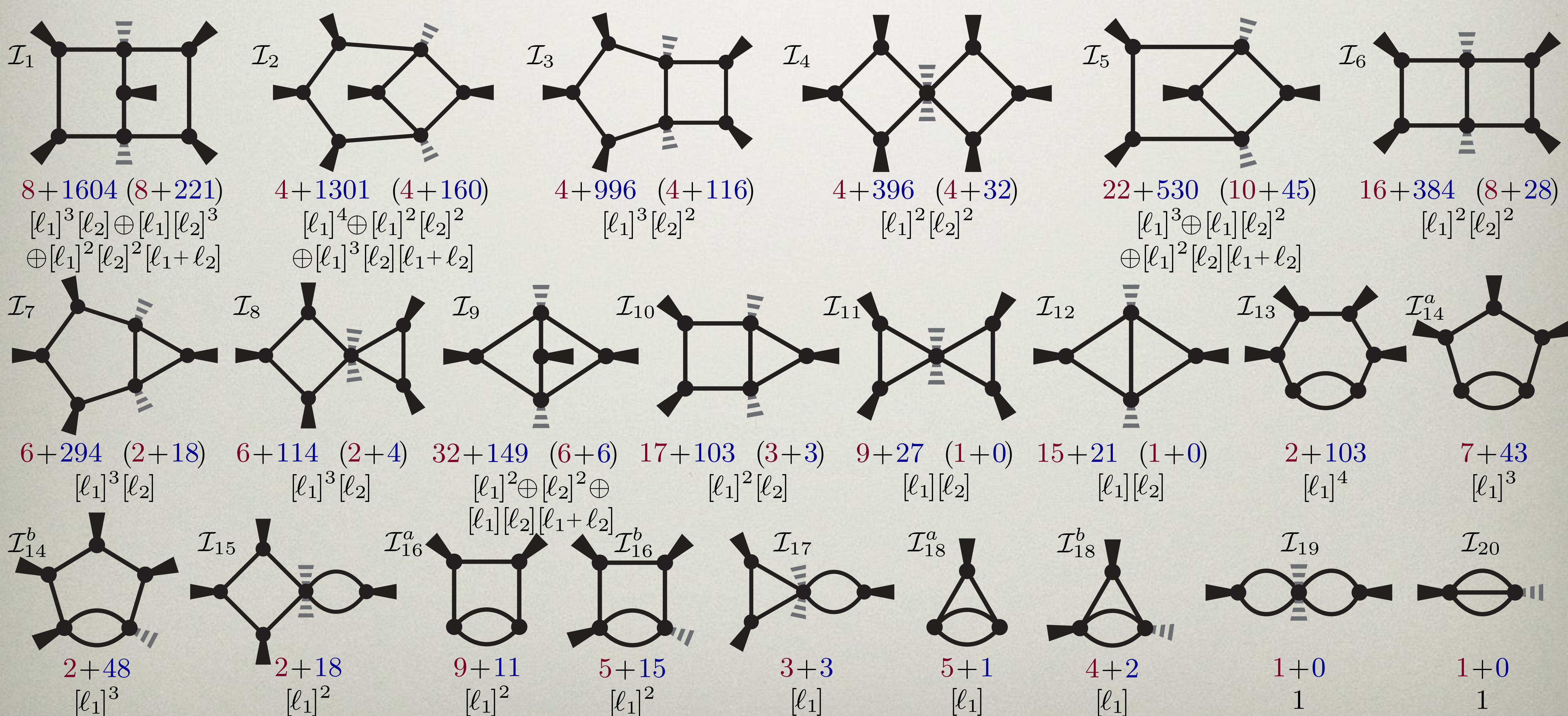


Beyond Planar Prescriptivity

- Abandoning box power-counting, however, immediately allows for *prescriptive* bases

[Feng, Huang]

[**JB**, Herrmann, McLeod, Trnka (*in prep.*)]



Recent Advances in Loop Integration Technology

When it's been Integrated

Obviously, "loop ***integrands*** should be ***integrated***"

but what this *really means* depends on who's talking (& why)

This is so even when the integral is "just" a number

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$$-0.656957931158387569164345 6$$

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$$\left(\frac{3}{2}\zeta_3 - \pi^2\log(2) + \zeta_2 + \frac{197}{72}\right)$$

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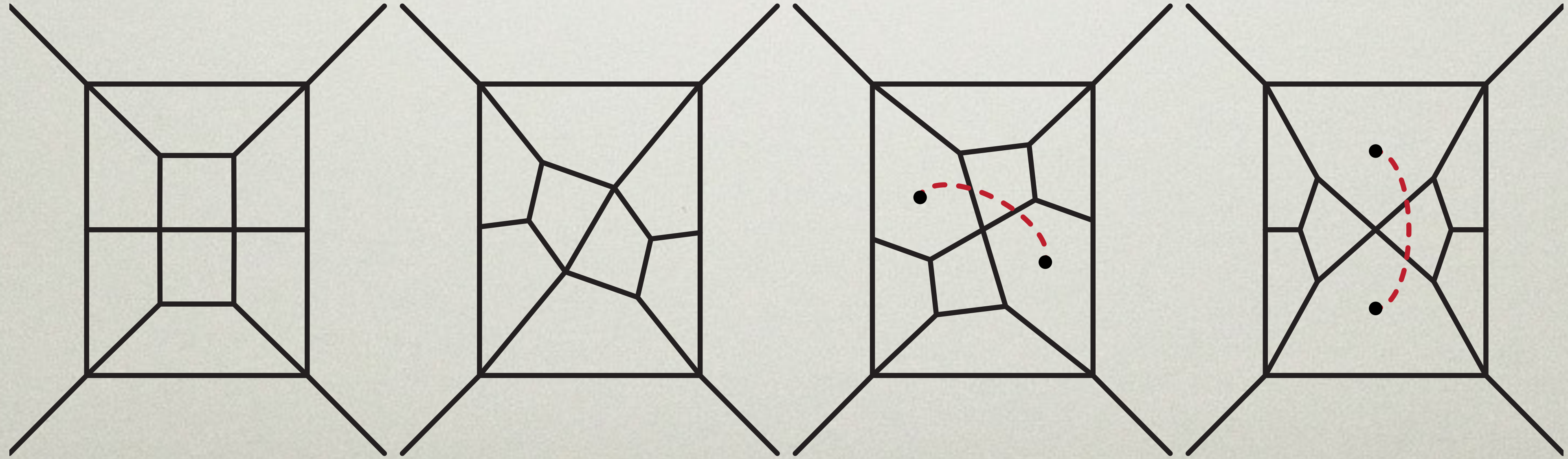
Are ***these "numbers"*** MZVs? **YES!** [O. Schnetz (*private corr.*)]



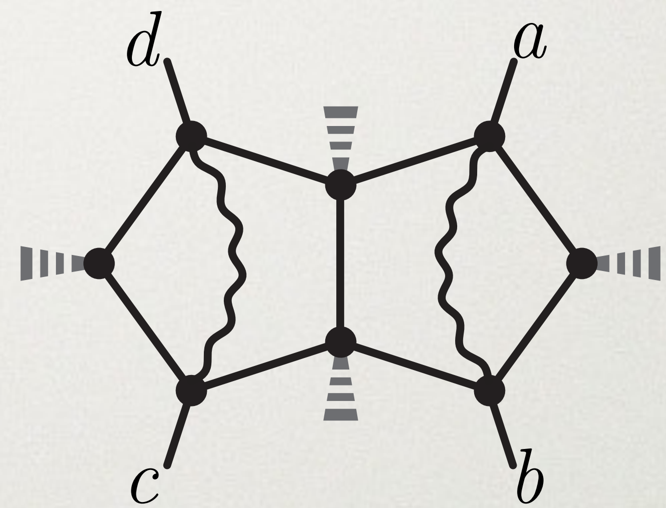
[**JB**, Heslop, Tran (2015)]

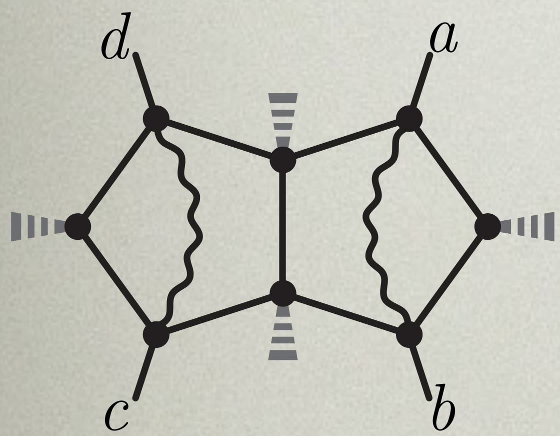
implications for BES…

When's it been Integrated?

When the result is a *function,* this is more subtle—depending on various (often valid) criteria

$$\mathcal{A}_n^{L=2,\mathrm{MHV}} = \sum_{a<b<c<d<a}$$





$$\equiv \frac{(\ell_1,N_1)(\ell_2,N_2)}{(\ell_1,a)(\ell_1,a{+}1)(\ell_1,b)(\ell_1,b{+}1)(\ell_1,\ell_2)(\ell_2,c)(\ell_2,c{+}1)(\ell_2,d)(\ell_2,d{+}1)}$$

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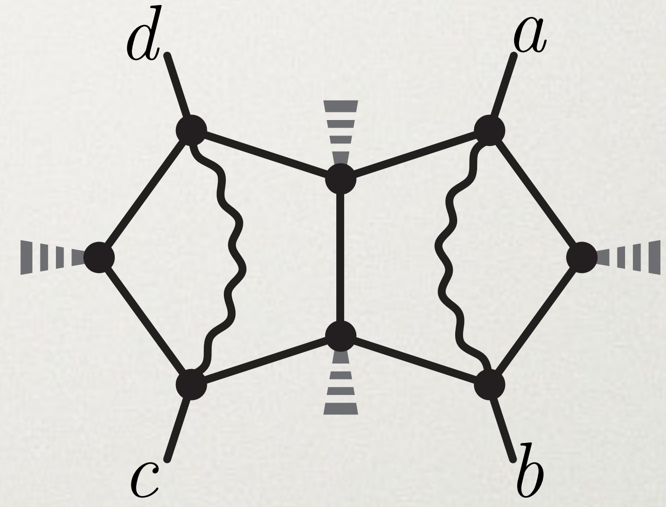

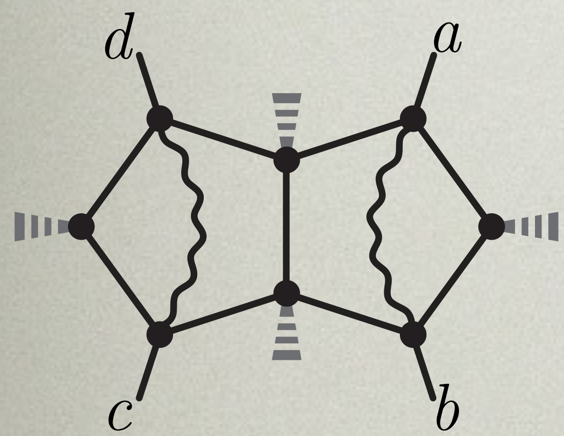

$$\equiv \int \frac{d^4\ell_1 d^4\ell_2 \qquad (\ell_1,N_1)(\ell_2,N_2)}{(\ell_1,a)(\ell_1,a{+}1)(\ell_1,b)(\ell_1,b{+}1)(\ell_1,\ell_2)(\ell_2,c)(\ell_2,c{+}1)(\ell_2,d)(\ell_2,d{+}1)}$$

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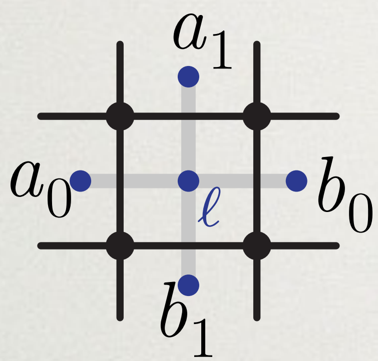

$$\Rightarrow \int d^4\ell \frac{(a_0,b_0)(a_1,b_1)}{(\ell,a_0)(\ell,a_1)(\ell,b_1)(\ell,b_0)} = \int_0^\infty d^2\vec{\alpha}\,\frac{1}{f_1 f_2}$$

$$= \int_{-i\infty}^{i\infty} \bar{d}^2\vec{z}\;\; \Gamma(-z_1)^2\Gamma(-z_2)^2\Gamma(1+z_1+z_2)^2\, u^{z_1} v^{z_2}$$

[Symanzik (1972)]

$$\propto \mathrm{Li}_2(\widetilde{u}) + \mathrm{Li}_2(\widetilde{v}) + \frac{1}{2}\log(u)\log(v) - \log(\widetilde{u})\log(\widetilde{v}) - \zeta_2$$

[Hodges (1977)]

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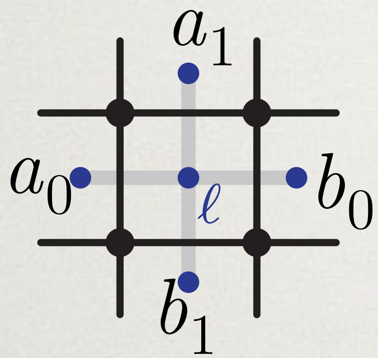

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[Hodges (1977)]

- built of functions *known to*
  - undergrads (Euler / Abel / …)
  - Goncharov / Brown / Bloch…
  - Mathematica / GiNaC…

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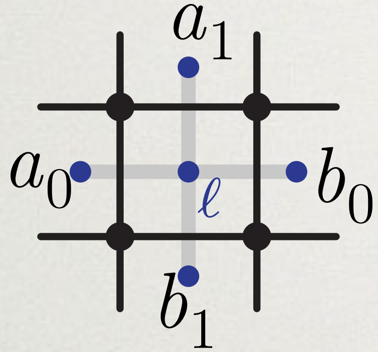

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[Hodges (1977)]

- numerically fast (and reliable)
- manifest "(transcendental) weight"
- minimal cancellation among terms
- manifest physical symmetries (non-redundantly)

✦ built of functions *known to*

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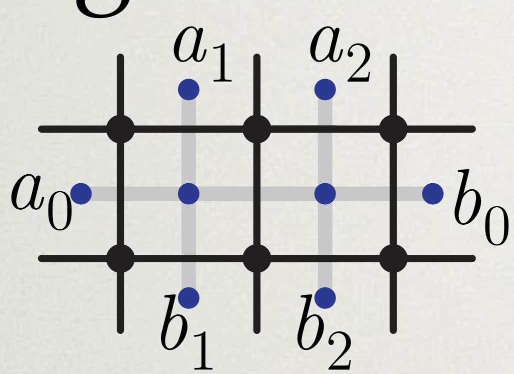

$$\Rightarrow \int d^8\vec{\ell}\,\frac{(a_0,b_0)(a_1,b_1)(a_2,b_2)}{(\ell_1,a_0)(\ell_1,a_1)(\ell_1,b_1)(\ell_1,\ell_2)(\ell_2,a_2)(\ell_2,b_2)(\ell_2,b_0)}$$

$$= \int_0^\infty d^6\vec{\alpha}\ \frac{\mathscr{U}}{\mathscr{F}^3} \qquad = \int_{-i\infty}^{i\infty} \bar{d}^7\vec{z}\left[\Gamma(-z_1)^2\cdots\right]\left[u_1^{z_1}\cdots u_7^{z_7}\right]$$

$$= \int_0^\infty d^4\vec{\alpha}\,\frac{1}{f_1 f_2\, g_2} = \int \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}} H_3(s)$$

- ***Certifiability***
  - against some reference (symbology, fibration *bases*,…)
  - by checking physical limits/branch cuts/…

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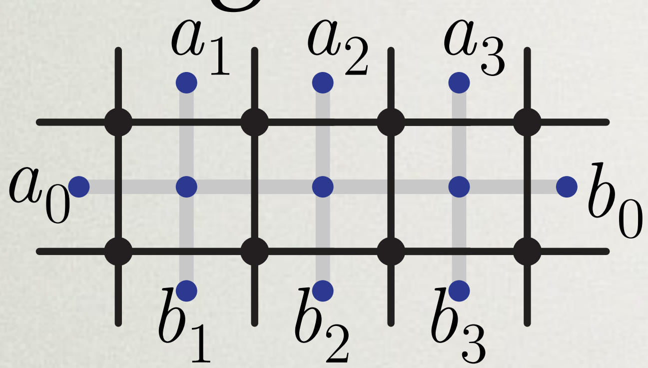

$$\Rightarrow \int d^{12}\vec{\ell}\, \frac{(a_0,b_0)(a_1,b_1)(a_2,b_2)(a_3,b_3)}{(\ell_1,a_0)(\ell_1,a_1)(\ell_1,b_1)(\ell_1,\ell_2)\cdots(\ell_3,b_0)}$$

$$= \int_0^\infty d^8\vec{\alpha}\ \frac{\mathscr{U}^2}{\mathscr{F}^4} = \int_{-i\infty}^{i\infty} đ^{16}\vec{z}\left[\Gamma(-z_1)^2\cdots\right]\left[u_1^{z_1}\cdots u_{16}^{z_{16}}\right]$$

$$= \int_0^\infty d^6\vec{\alpha}\, \frac{1}{f_1 f_2 f_3\, g_3} = \int \frac{ds\, dz}{\sqrt{4s^3 - g_2(z)s - g_3(z)}} H_4(s,z)$$

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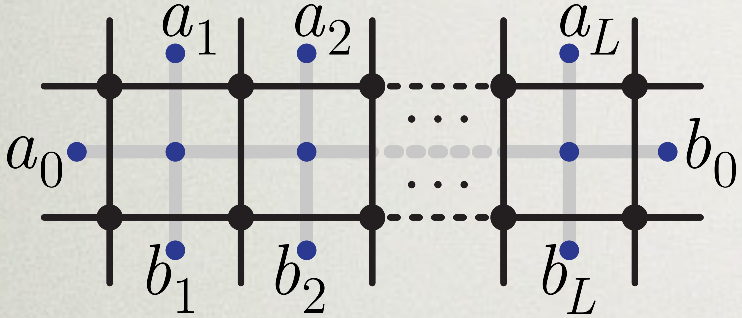


$$\Rightarrow \int d^{4L}\vec{\ell}\,\frac{(a_0,b_0)(a_1,b_1)\cdots(a_L,b_L)}{(\ell_1,a_0)(\ell_1,a_1)(\ell_1,b_1)(\ell_1,\ell_2)\cdots(\ell_L,b_0)}$$

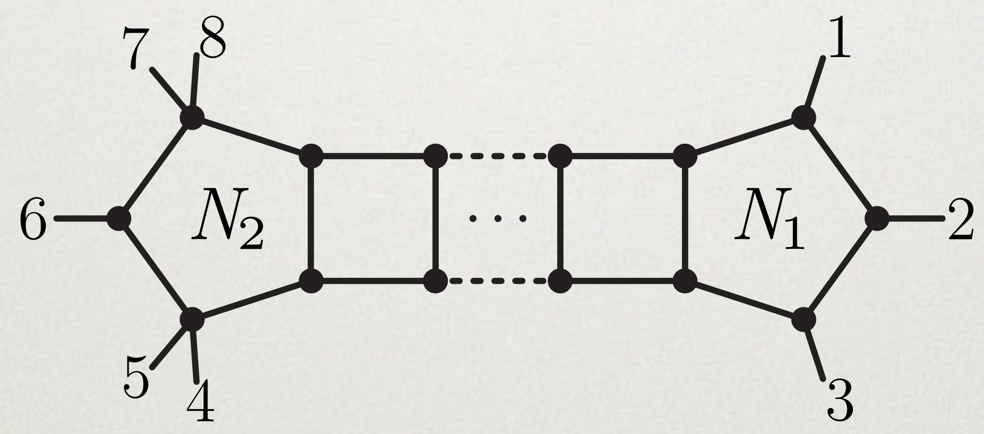
$$=\int_0^\infty d^{3L}\vec{\alpha}\,\frac{\mathscr{U}^{L-1}}{\mathscr{F}^{L+1}}=\int_{-i\infty}^{i\infty} d\!\!{}^{-}{}^{(2L^2-L+1)}\vec{z}\,\Big[\Gamma(-z_1)^2\cdots\Big]\prod_{i=1}^{2L^2-L+1}u_i^{z_i}$$

$$=\int_0^\infty d^{2L}\vec{\alpha}\,\frac{1}{(f_1\cdots f_L)g_L}=\int\frac{ds\,d^{L-2}\vec{z}}{\sqrt{4s^3-g_2(\vec{z})s-g_3(\vec{z})}}H_{L+1}(s,\vec{z})$$

- ***Certifiability***
  - against some reference (symbology, fibration *bases*,…)
  - by checking physical limits/branch cuts/…

When's it been Integrated?

When the result is a *function*, this is more subtle—depending on various (often valid) criteria



- ***Certifiability***
  - against some reference (symbology, fibration *bases*,…)
  - by checking physical limits/branch cuts/…

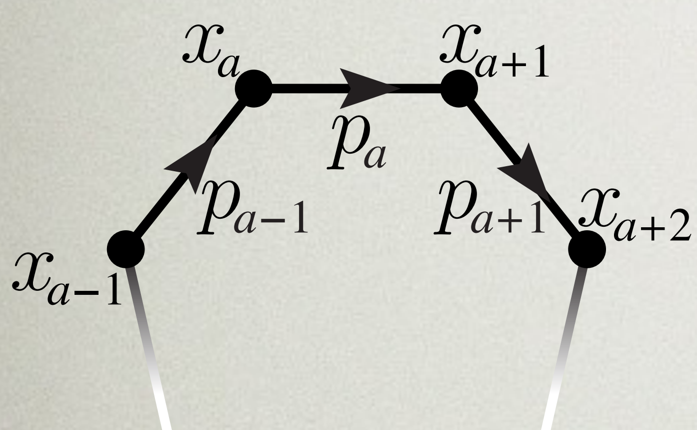
Rationalizing Loop Integration

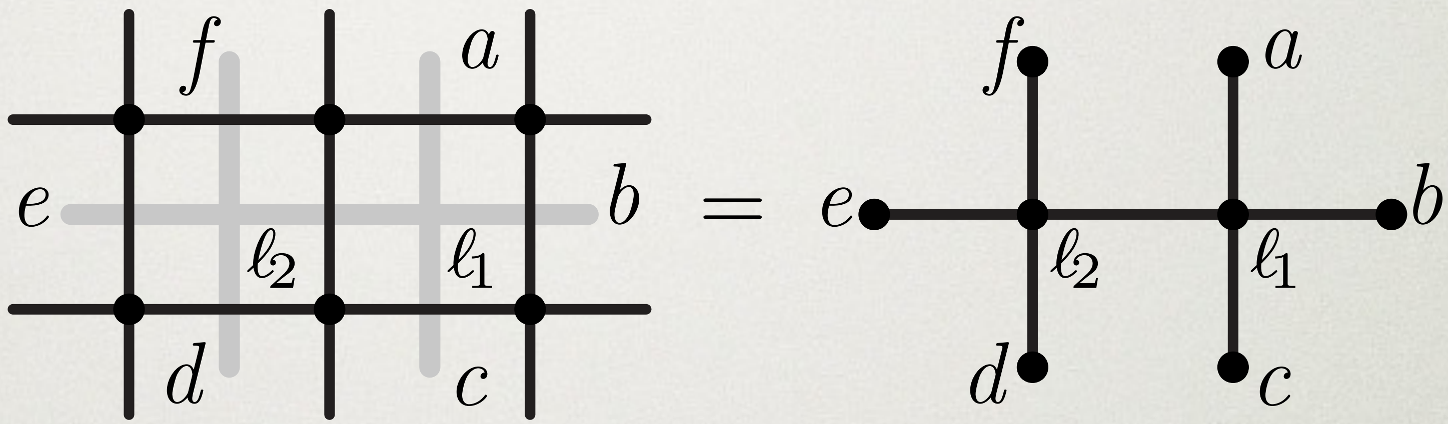
A surprisingly large class of planar **UV finite** multi-loop integrals can be *directly* integrated provided the right kind of naïveté (and mild cleverness):

- Feynman parameterize ***in 4d,*** one loop at a time
- Maintain manifest dual conformal invariance:
  - **regulate** IR divergences with \`DCI masses'
    [**JB**, Caron-Huot, Trnka (2013)]
  - **rescale** Feynman parameters to trivialize DCI
    [**JB**, Dixon, Dulat, Panzer (*to appear*)]
- Parameterize kinematic variables using: momentum twistors [**JB**, McLeod, von Hippel, Wilhelm (2018)]
  chosen non-redundantly
- *Partial fraction to death* (e.g. use **HyperInt**) [Panzer (2014)]

Planarity & Dual-Conformality

- We may parameterize momenta of planar loop (Feynman) integrals by their dual-graphs





$$p_a \equiv (x_{a+1} - x_a)$$

$$\equiv \int \frac{d^4\ell_1 d^4\ell_2 \qquad (a,c)(b,e)(d,f)}{(\ell_1,a)(\ell_1,b)(\ell_1,c)(\ell_1,\ell_2)(\ell_2,d)(\ell_2,e)(\ell_2,f)}$$

$$(a,b) = (b,a) \equiv (x_b - x_a)^2 = (p_a + \ldots + p_{b-1})^2 \equiv s_{a \cdots b-1} \quad \text{and} \quad (\ell, a) \equiv (x_\ell - x_a)^2$$

- **Dual-Conformal Invariance:** conformality in x's

$$(ab;cd) \equiv \frac{(a,b)(c,d)}{(a,c)(b,d)}$$

[Drummond, Henn, Smirnov, Sokatchev; Drummond, Korchemsky, Henn; …]

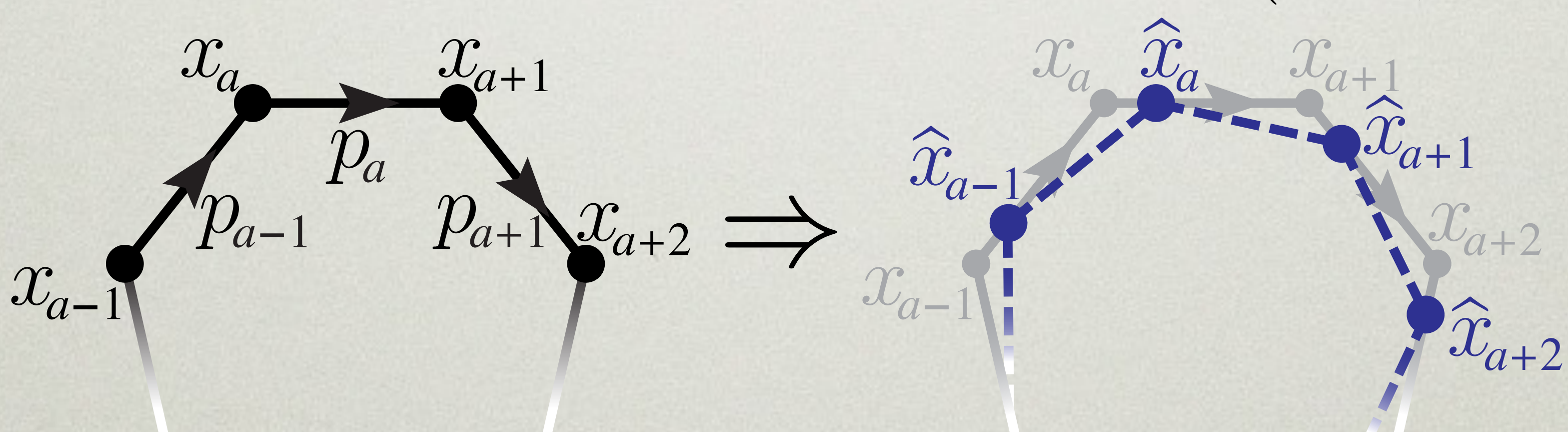
The Dual-Conformal Regulator

- The basic idea of the dual-conformal regulator is to give legs masses, but controlled by a parameter 'δ' that is ***dimensionless*** & ***has no conformal weight***

[**JB**, Caron-Huot, Trnka (2013)]

$$p_a^2 \mapsto p_a^2 + \delta \frac{(p_{a-1}+p_a)^2(p_a+p_{a+1})^2}{(p_{a-1}+p_a+p_{a+1})^2} \quad x_a \mapsto x_{\widehat{a}} \equiv x_a + \delta(x_{a+1}-x_a)\frac{(a-2,a)}{(a-2,a+1)}$$

$$(a,a+1) \mapsto (\widehat{a},\widehat{a+1}) = (a,a+1) + \delta\frac{(a-1,a+1)(a,a+2)}{(a-1,a+2)}$$



$$I = \int_{\mathbb{R}^{3,1}} \prod_{i=1}^{L} d^4\ell_i \, \mathcal{I} \;\mapsto\; I^\delta \equiv \int_{\mathbb{R}^{3,1}} \prod_{i=1}^{L} \left[d^4\ell_i \left(\prod_a \frac{(\ell_i,a)}{(\ell_i,\widehat{a})} \right) \right] \mathcal{I}$$

Perservering Dual-Conformality

- Using the dual-conformal regularization scheme,

$$p_a^2 \mapsto p_a^2 + \delta \frac{(p_{a-1}+p_a)^2(p_a+p_{a+1})^2}{(p_{a-1}+p_a+p_{a+1})^2}$$

all(?) UV-finite planar loop integrals take the form:

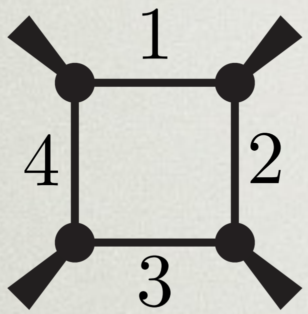
$$I \mapsto \sum_{k=0}^{2L} I_k \log^k(\delta)$$

[**JB**, Dixon, Dulat, Panzer (*to appear*)]

- Coefficients of each divergence can be obtained as *strictly finite* (Feynman-) parametric integrals—which can always be rendered *manifestly* DCI

Restoring Conformality

- Feynman parameterization is naïvely at odds with maintaining (dual) conformal invariance



$$\Rightarrow \int d^4\ell \frac{1}{(\ell,1)(\ell,2)(\ell,3)(\ell,4)} = \int_0^\infty [d^3\vec{\alpha}]\,\frac{1}{\mathscr{F}^2}$$

$$\mathscr{F} \equiv \alpha_1\alpha_2(1,2)+\alpha_2\alpha_3(2,3)+\alpha_1\alpha_3(1,3) +\alpha_1\alpha_4(1,4)+\alpha_2\alpha_4(2,4)+\alpha_3\alpha_4(3,4)$$

At least when integrating one loop (at a time), conformality is always(?) restorable by rescaling Feynman parameters:

$$\alpha_1 \mapsto \alpha_1(2,3) \quad \alpha_2 \mapsto \alpha_2(1,3) \quad \alpha_3 \mapsto \alpha_3(1,2) \quad \alpha_4 \mapsto \alpha_4 \frac{(1,2)(2,3)}{(2,4)}$$

$$\mathscr{F} \mapsto (1,2)(2,3)(1,3)\underbrace{\Big(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3+\alpha_4(\alpha_1 v+\alpha_2+\alpha_3 u)\Big)}_{(f_1+\alpha_4 f_2)}$$

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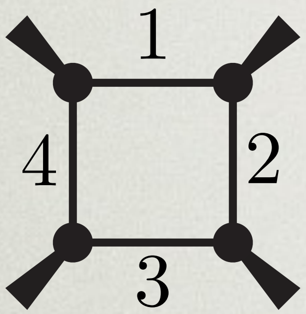

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$$\int_0^\infty [d^3\vec{\alpha}]\,\frac{1}{\mathscr{F}^2} \propto \int_0^\infty [d^2\vec{\alpha}] \int_0^\infty d\alpha_4 \frac{1}{(f_1+\alpha_4 f_2)^2} = \int_0^\infty [d^2\vec{\alpha}]\,\frac{1}{f_1 f_2}$$

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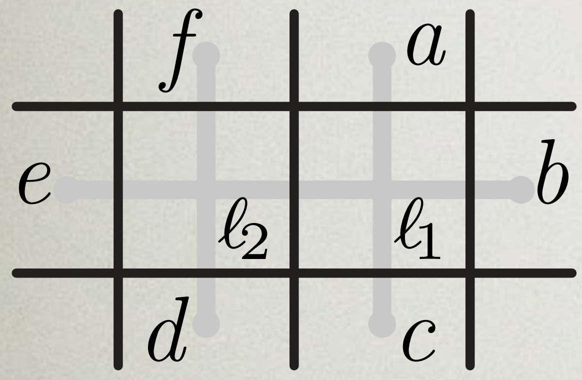
Symanzik Polynomial Obstructions

- It is easy to see that the trick just used breaks down at higher loops if one uses the Symanzik (graph) polynomial formalism. For example, consider:



$$I_{\text{db}}^{\text{ell}} \propto \int_0^\infty [d^6\vec{\alpha}]\, \frac{\mathscr{U}}{\mathscr{F}^3} \qquad \int_0^\infty [d\vec{\alpha}]\, \frac{\mathscr{U}^{n-2(L+1)}}{\mathscr{F}^{n-2L}}$$

$$\begin{aligned}\mathscr{F} = \quad & (a,c)\alpha_a\alpha_c\big(\beta_d+\beta_e+\beta_f+\gamma\big) \\ & +(d,f)\beta_d\beta_f\big(\alpha_a+\alpha_b+\alpha_c+\gamma\big)+\ldots\end{aligned}$$

- Nevertheless, it appears that this obstruction is **always** avoidable simply by parameterizing one loop at a time (true through at least three loops)

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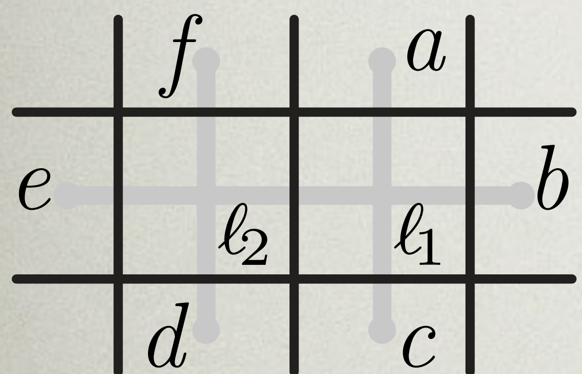

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$$f_1 \equiv \alpha(1+\beta_1)+\beta_1,\;\; f_2 \equiv 1+u_1\alpha+v_1\beta_1+u_2\beta_2+v_2\beta_3,$$
$$f_3 \equiv (1+u_3\alpha)\beta_2+(1+u_4\beta_1)\beta_3+\beta_2\beta_3+u_3u_4u_5 f_1,$$

$$u_1 \equiv (ab;ce),\; u_2 \equiv (bd;ef),\; u_3 \equiv (ab;cf),\; v_1 \equiv (ea;bc),\; v_2 \equiv (fb;de),\; u_4 \equiv (bc;da),\; u_5 \equiv (ac;df)$$

[**JB**, McLeod, Spradlin, von Hippel, Wilhelm (2017)]

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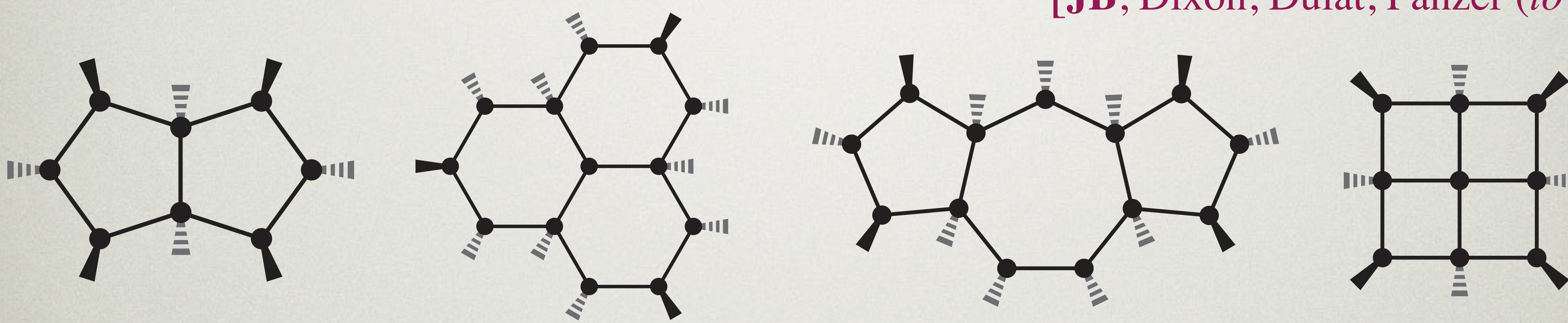
Dual-Conformal Sufficiency

- We may now (**regulate** &) **represent** all of the following integrals in the space of *finite*, *manifestly conformal* *(Feynman-)parametric integrals*

[**JB**, Dixon, Dulat, Panzer (*to appear*)]



$$I \mapsto \sum_{k=0}^{2L} I_k \log^k(\delta) \qquad I_k \in \text{span}\left\{ \int_0^\infty d\vec{\alpha}\ \frac{\mathfrak{N}(\vec{\alpha})}{\mathfrak{F}(\vec{\alpha},\vec{u})} \right\}$$

where *u's* are parity-even cross-ratios: $(ab;cd) \equiv \dfrac{(a,b)(c,d)}{(a,c)(b,d)}$

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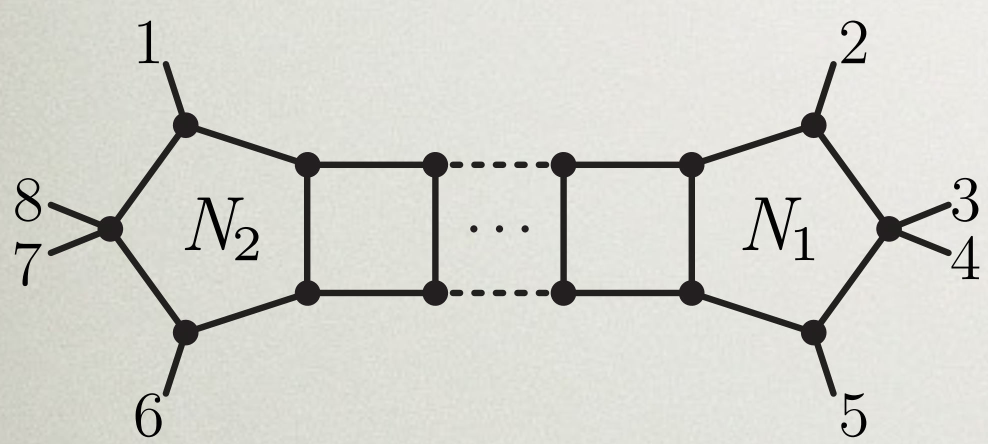

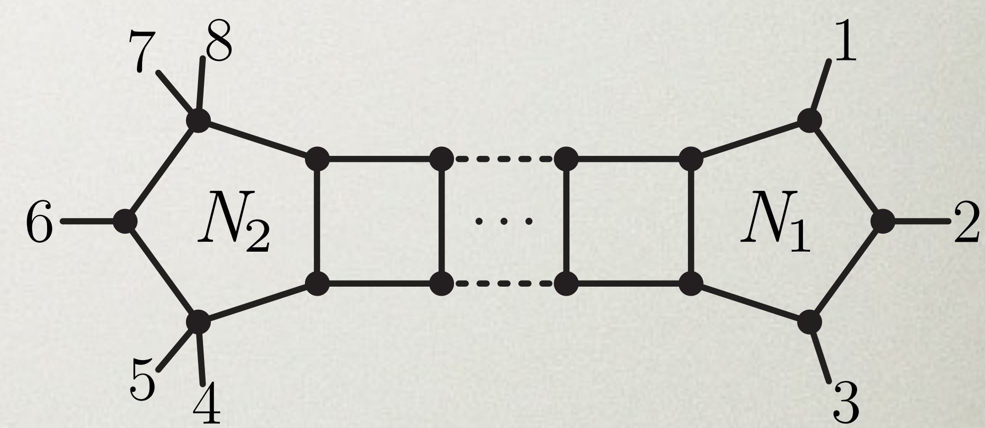

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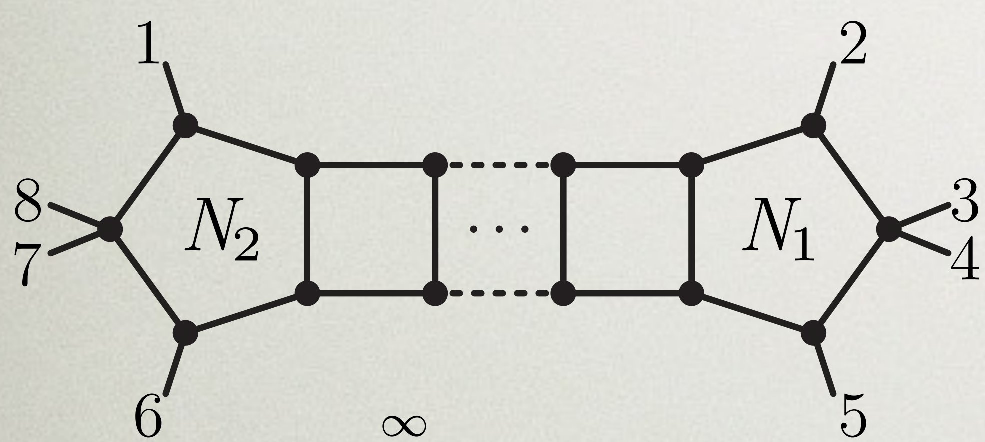

$$I_{8,\mathrm{B}}^{(L)} \equiv \int_0^\infty d^{2L}\vec{\alpha}\,[d\beta]\,\frac{u_1}{(f_1\cdots f_{L-1})\,g_1\,g_2\,g_3}\left(\frac{\alpha_2^L(\beta_1 n_1^1+\beta_2 n_2^1)}{g_1}+\frac{\beta_1 n_1^2+\beta_2 n_2^2}{g_2}-1\right)$$

$$f_k \equiv (\alpha_1^1+\ldots+\alpha_1^k)\beta_2 u_2 + (\alpha_2^1+\ldots+\alpha_2^k)\beta_1 u_3 + \beta_1\beta_2 u_2 u_3 u_4 + \sum_{i,j=1}^{k}\alpha_1^i\alpha_2^j\,;$$

$$g_1 \equiv f_{L-1}+(\alpha_2^1+\ldots+\alpha_2^{L-1})(\alpha_1^L+\alpha_2^L)+\alpha_1^L\beta_2\,u_2+\alpha_2^L(\beta_1+\beta_2);$$

$$g_2 \equiv (\alpha_1^1+\ldots+\alpha_1^L)+\alpha_2^L u_5+\beta_1+\beta_2 u_1; \qquad g_3 \equiv (\alpha_1^1+\ldots+\alpha_1^L)+\alpha_2^L+\beta_1 u_3,$$

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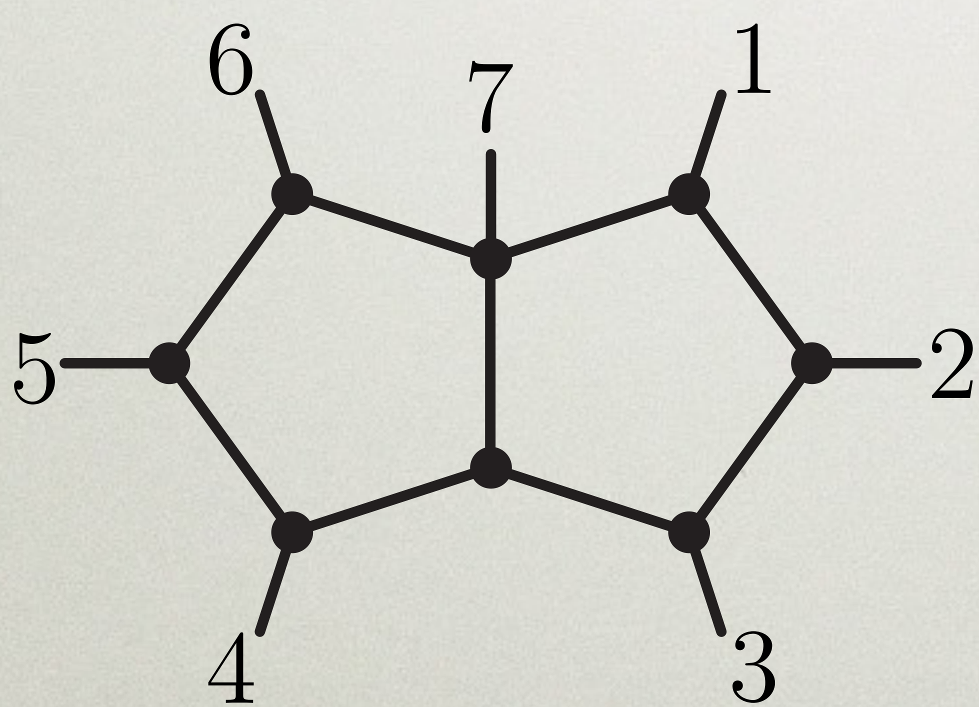
Conformal Complications

- Although a good start, we haven't yet eliminated ***all*** conformal redundancies—just the rescalings

—which is to say that parity-even cross-ratios are:

  - too great in number
  - the "wrong" variables…



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actually independent cross ratios: $3n-15$

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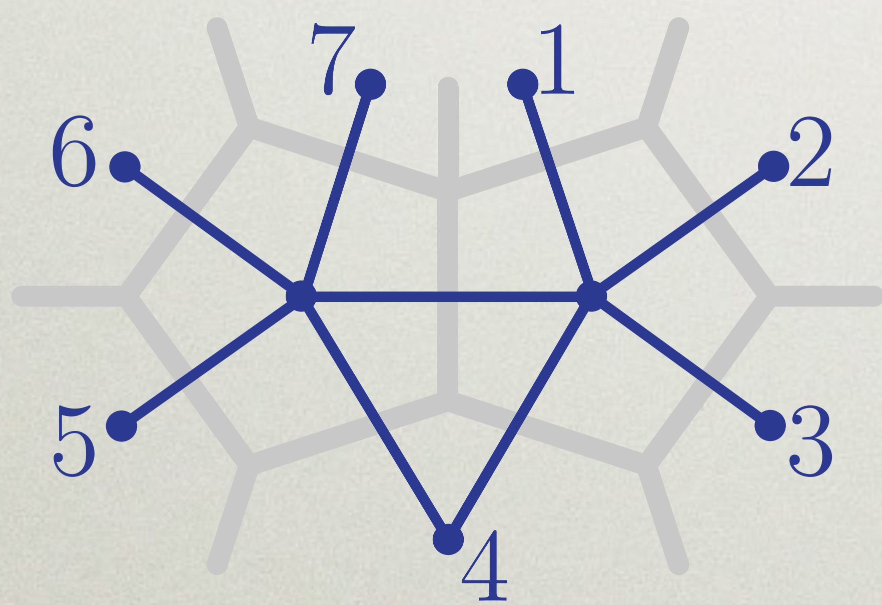

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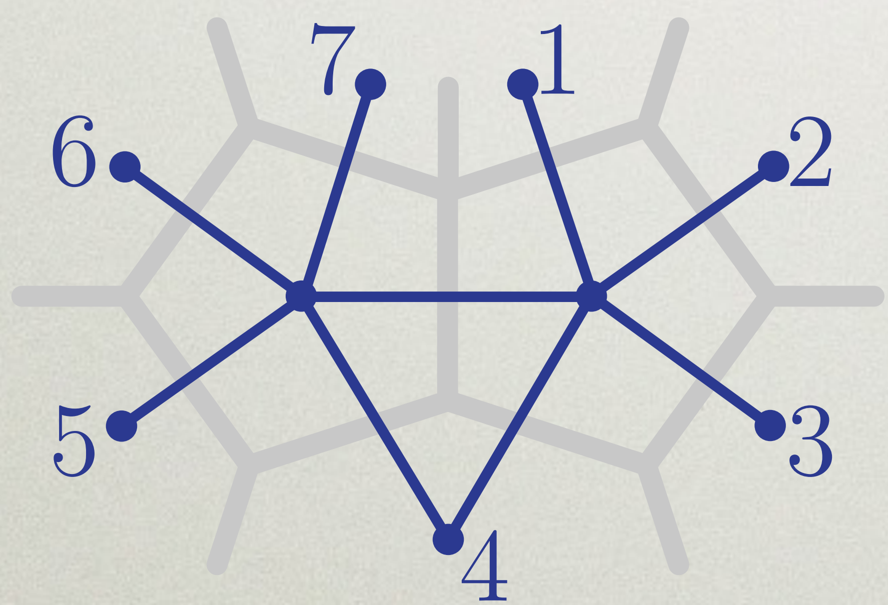

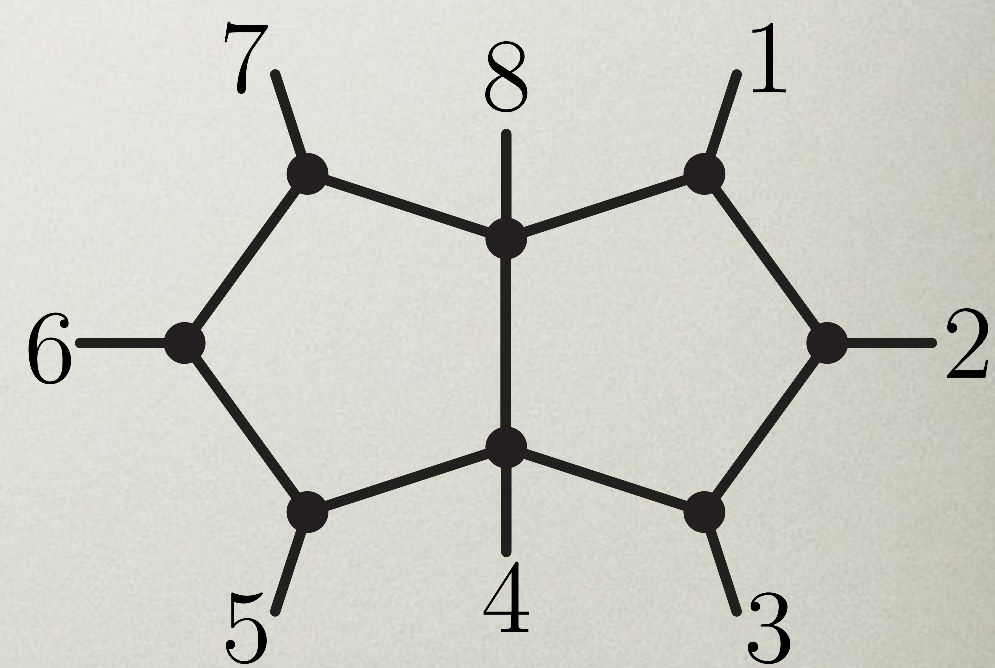

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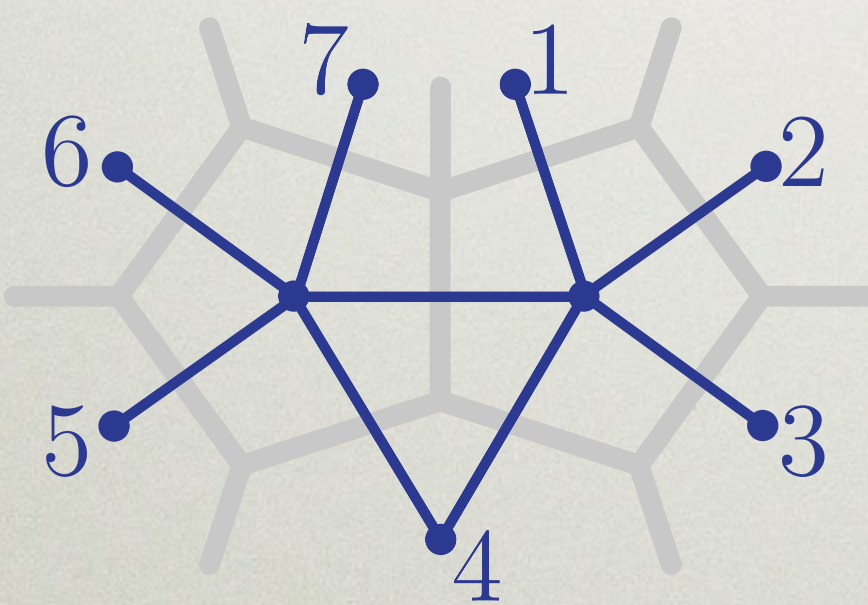

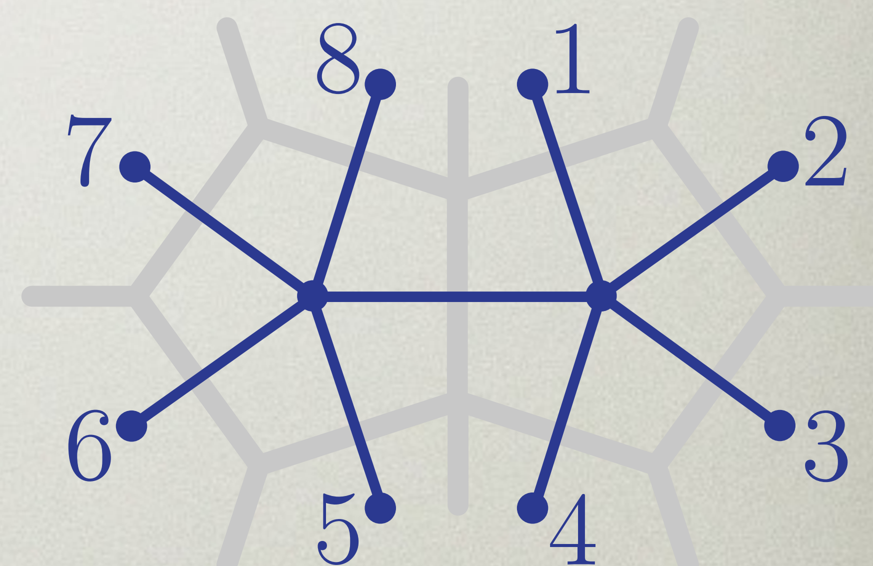

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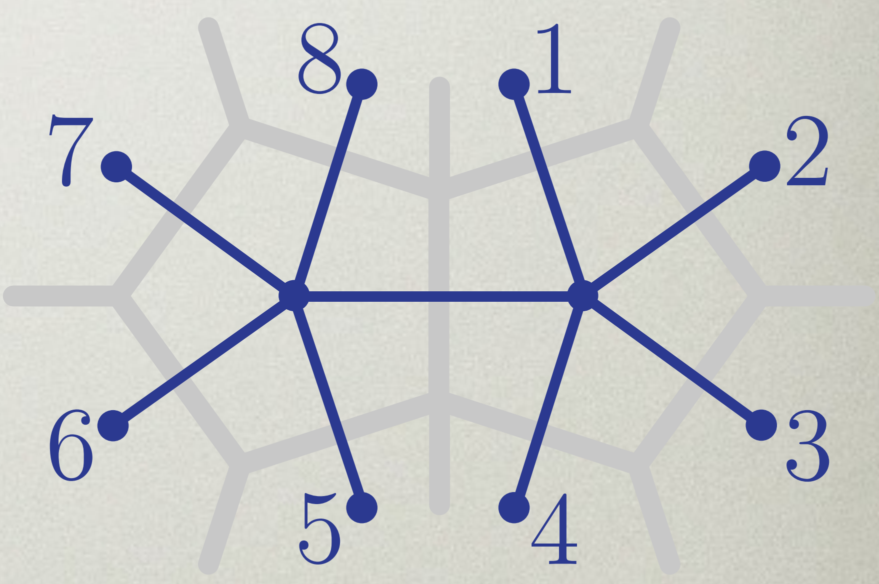
- over-count the degrees of freedom
- insensitive to the rank of the Gramian
- do not rationalize Gramian dets
- satisfy (complex) algebraic relations

$$\sqrt{(1-u-v-w)^2-4uvw}$$



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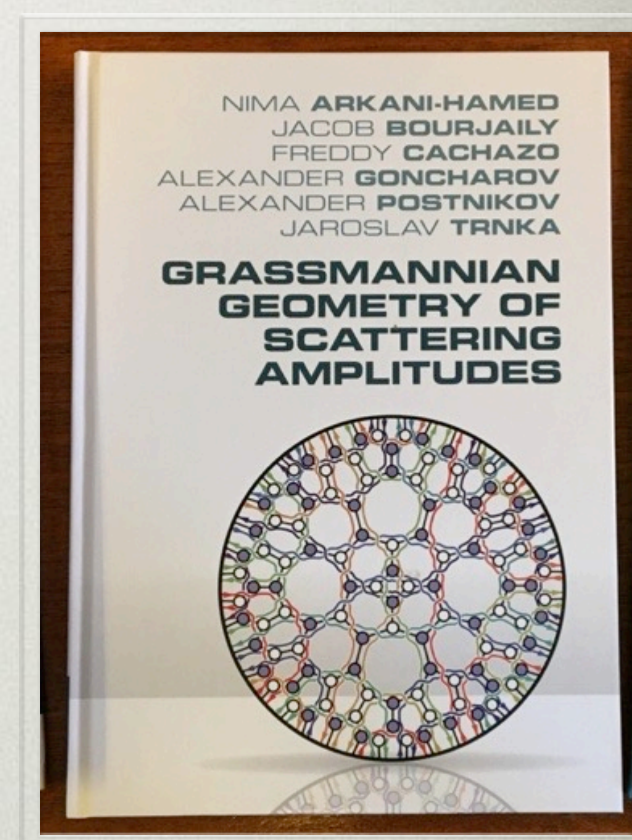
Momentum-Twistor Magic

[Hodges (2009)]

- Unsurprisingly (to most of us), ***momentum twistors*** are (closer to) the ***right*** kind of conformal variables

[Golden, Paulos, Spradlin,Volovich; Harrington; McLeod, …]

  - manifest the rank of the Gramian
    - no *constrained* extra degrees of freedom
    - rationalize all 6x6 Gram determinants
  - positive domain ⊂ Euclidean domain
  - positive domain is a ***cluster variety***
    - cluster coordinates given by *plabic graphs*



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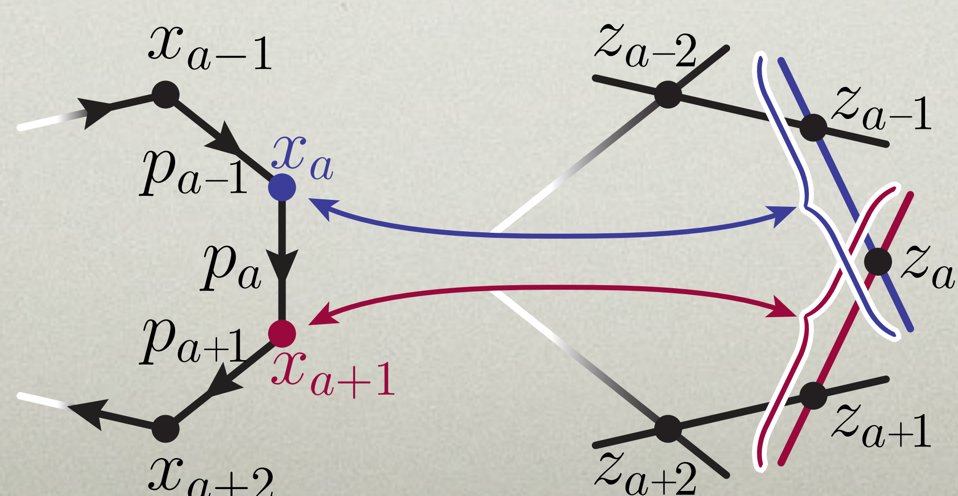

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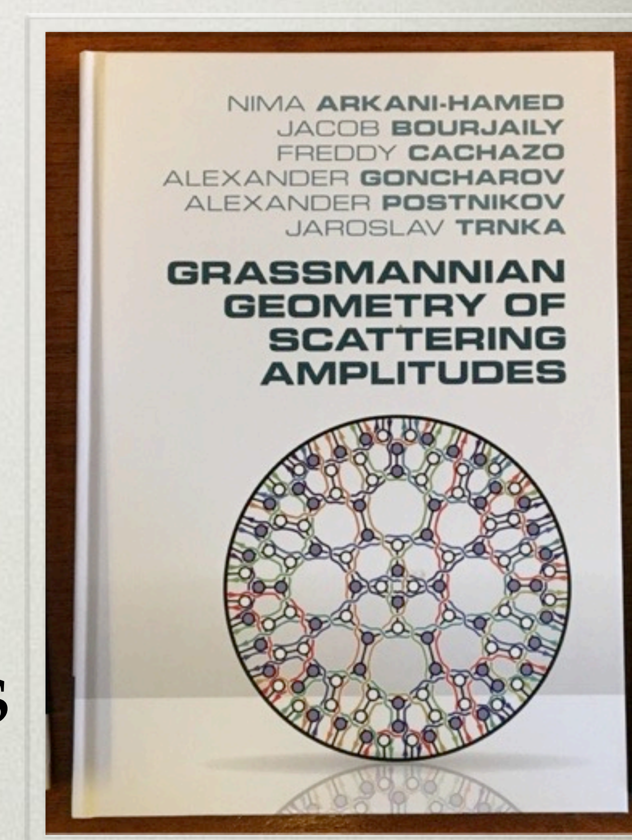
    - no *constrained* extra degrees of freedom
    - rationalize all 6x6 Gram determinants
  - positive domain ⊂ Euclidean domain
  - positive domain is a ***cluster variety***
    - cluster coordinates given by *plabic graphs*
  - easy to expose/probe kinematic boundaries
  - easy to eliminate redundant parameters



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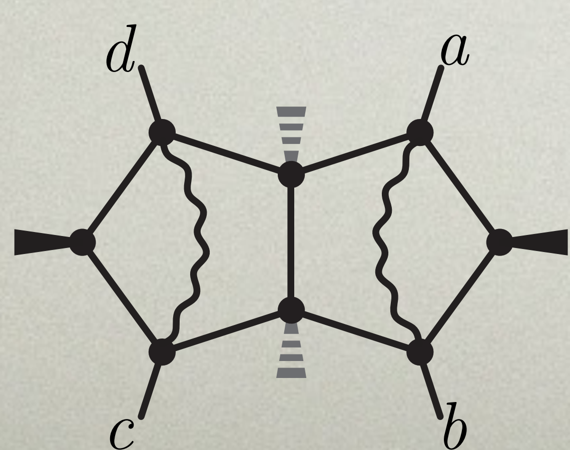

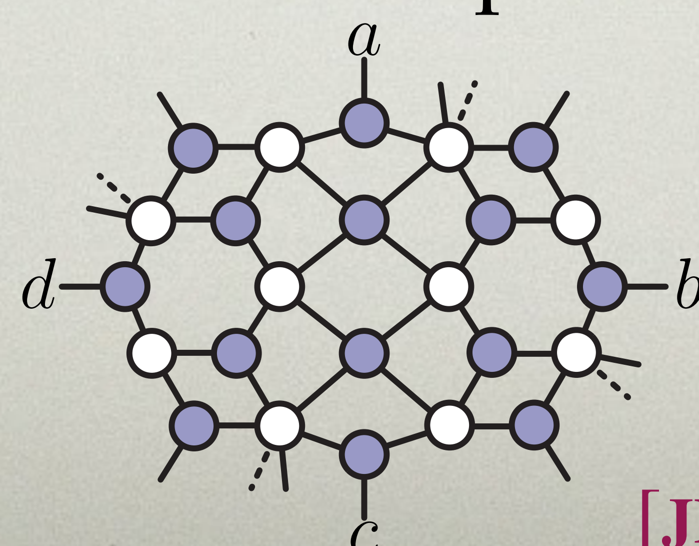

[**JB**, McLeod, von Hippel, Wilhelm (2018)]

Loop Integral Zoology

general complexity beyond polylogs

(& beyond elliptic polylogs)

The Two-Loop 'Master' Integrals

- How hard are the general "masters" at two loops?

[**JB**, Duhr, Dulat, McLeod, Penante, von Hippel, Wilhelm, (*in progress*)]

The Two-Loop 'Master' Integrals

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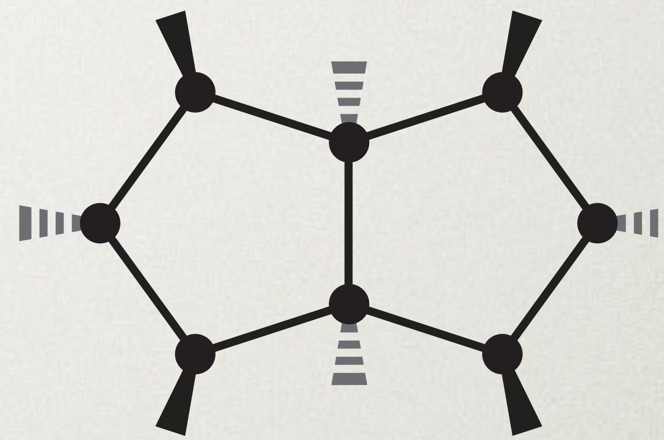
$$\mathcal{A}_n^{L=2} = \sum_{\mathcal{L}} f_{\mathcal{L}}$$



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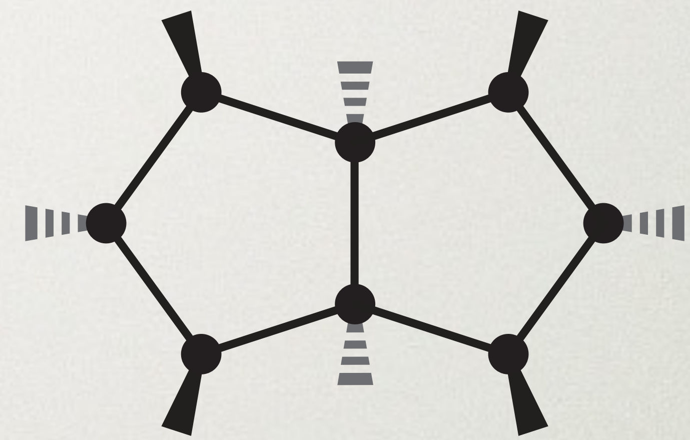

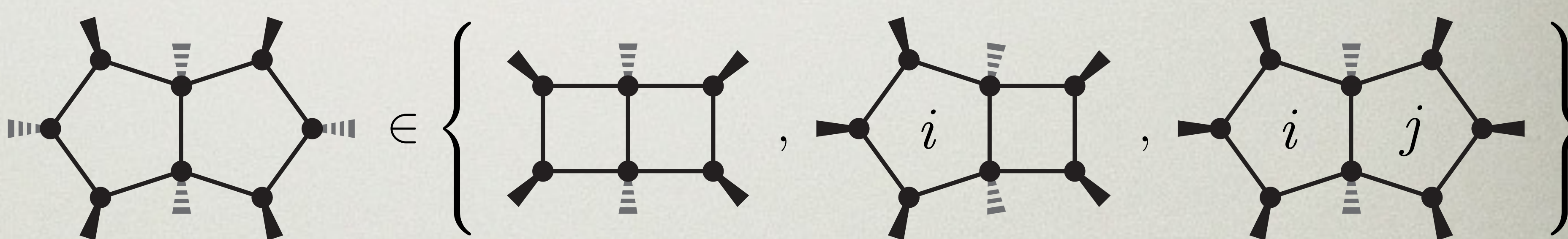

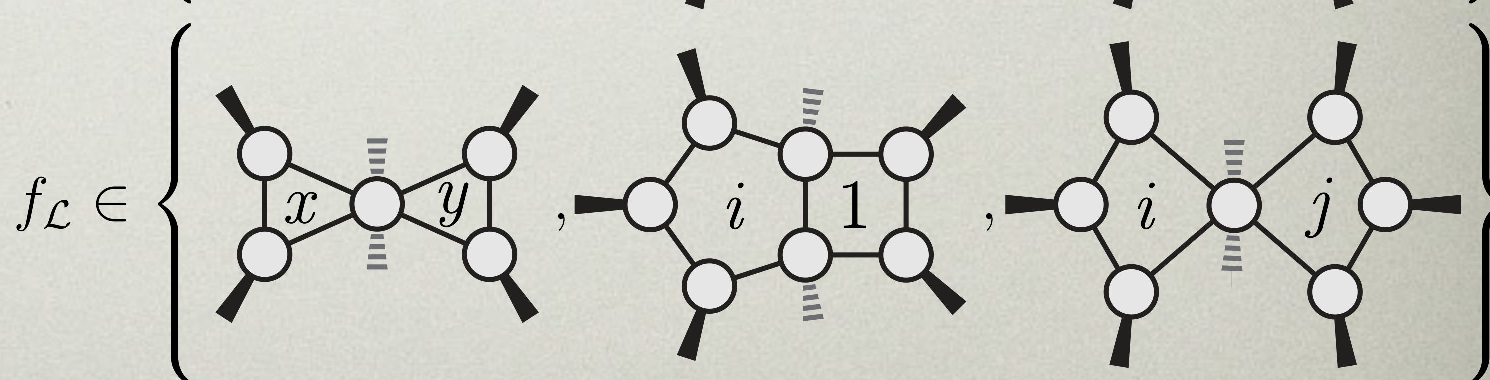

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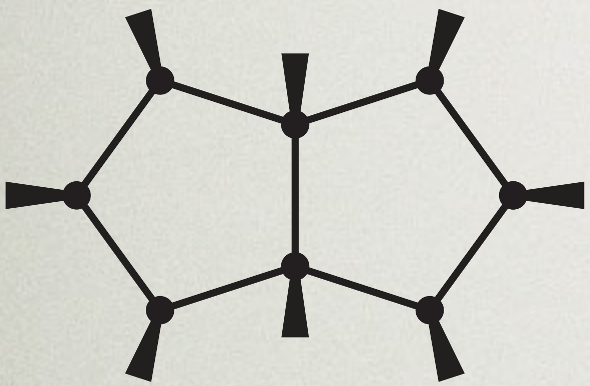

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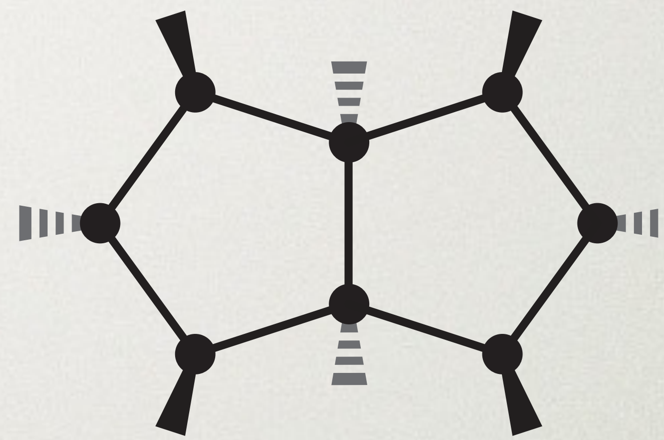
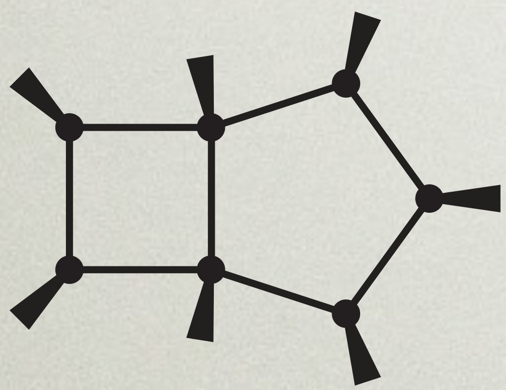
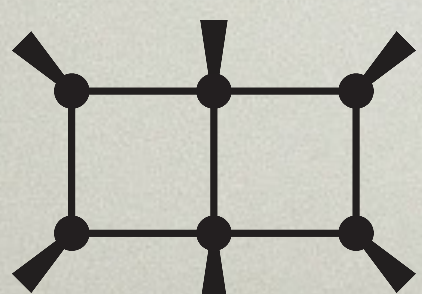




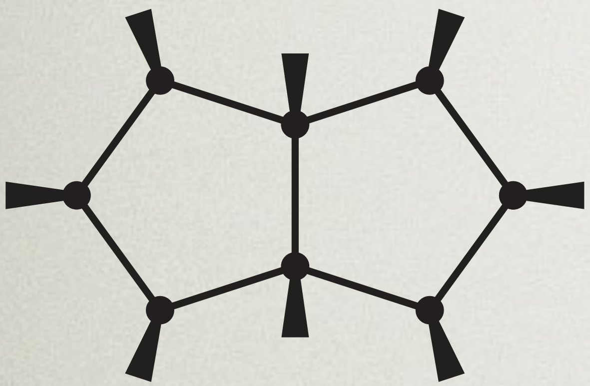


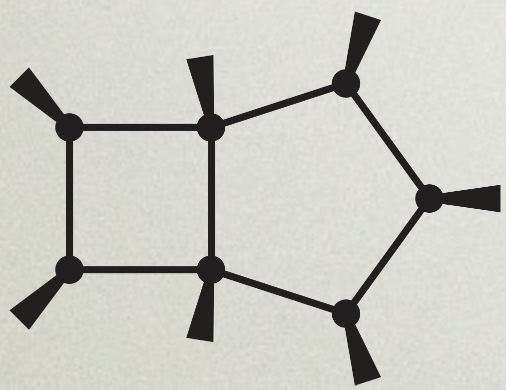
[**JB**, Duhr, Dulat, McLeod, Penante, von Hippel, Wilhelm, (*in progress*)]

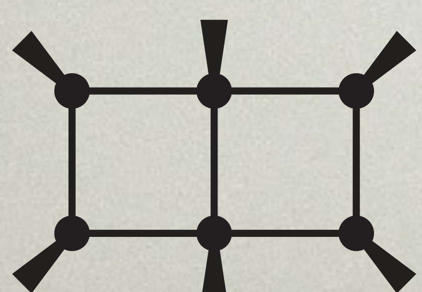
The Two-Loop 'Master' Integrals

✦ How hard are the general "masters" at two loops?

# d.o.f.	# cross ratios	# Kinematic Square Roots 4x4(+6x6)+cuts/coeffs	# elliptic curves







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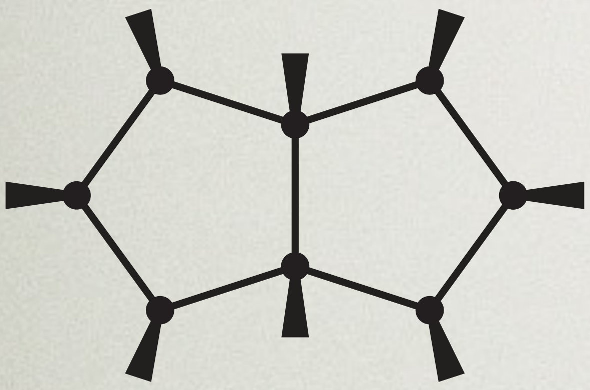

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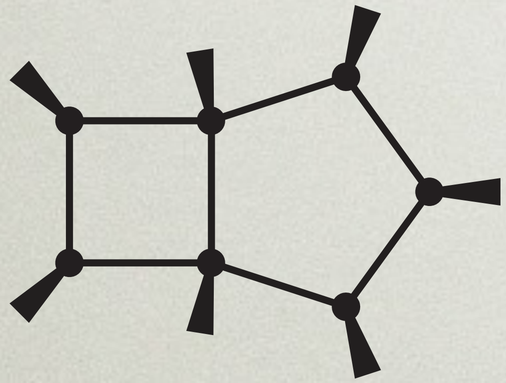

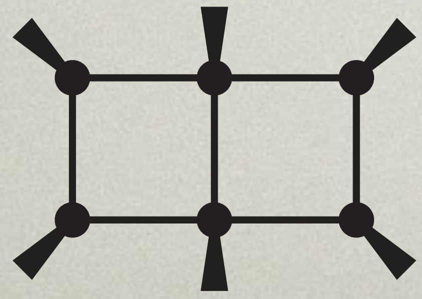

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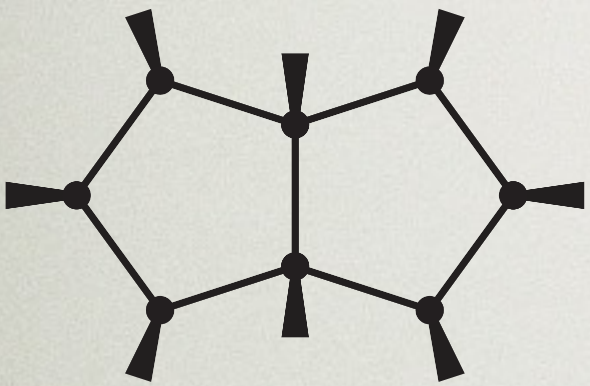
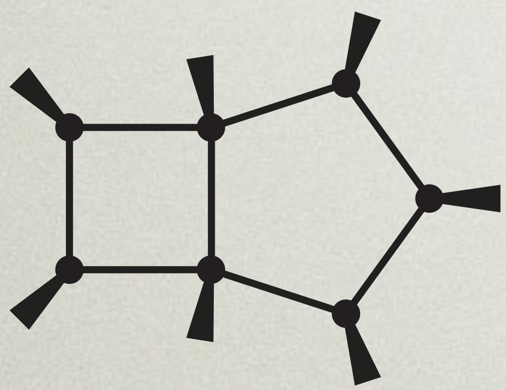
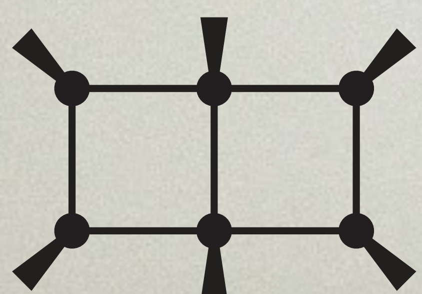
	17	20	70(+56)+10	16
	13	14	35(+7)+1	4



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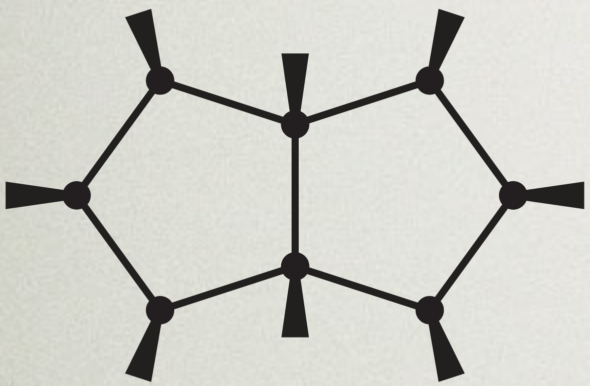
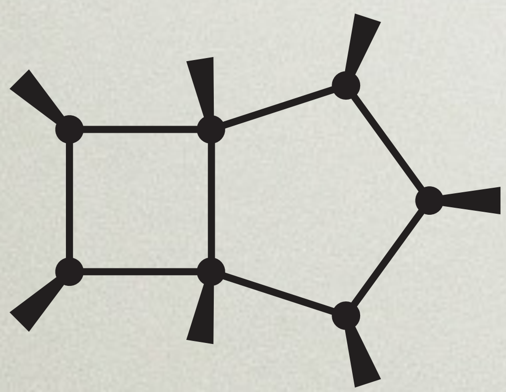
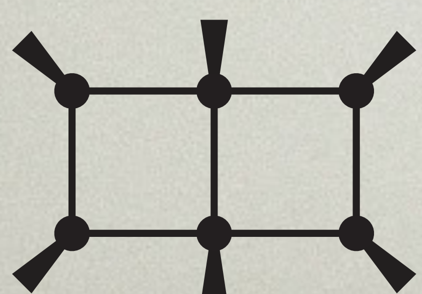
	17	20	70(+56)+10	16
	13	14	35(+7)+1	4
	9	9	15(+1)+0	1

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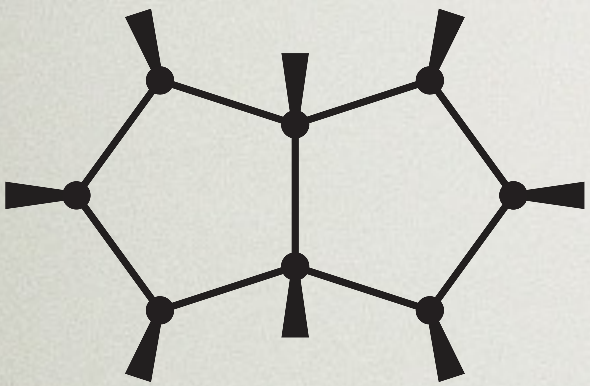
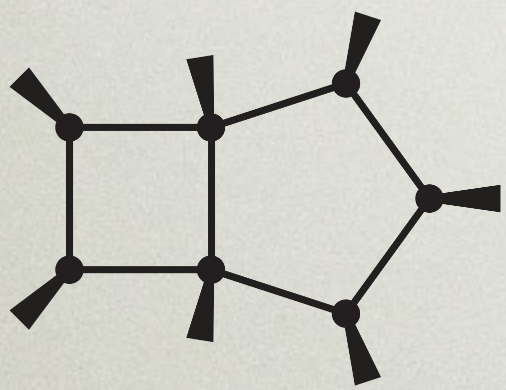
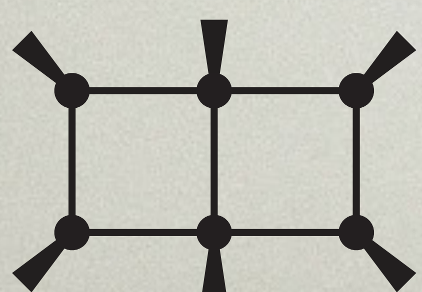
	17	20	70(+56)+10+16	16
	13	14	35(+7)+1+4	4
	9	9	15(+1)+0+1	1

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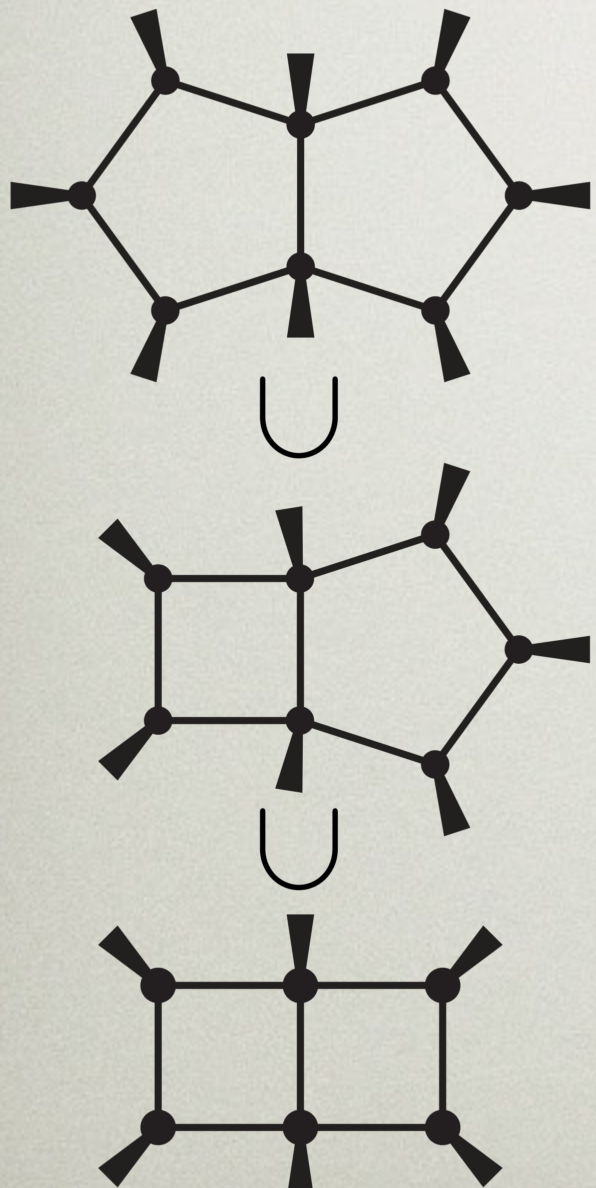

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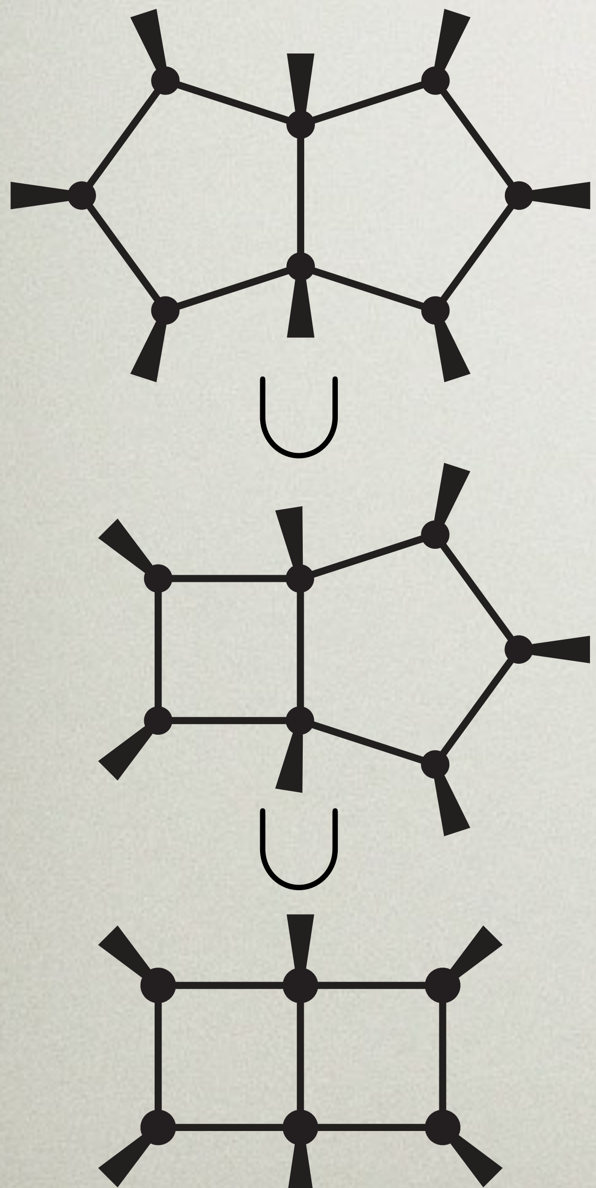

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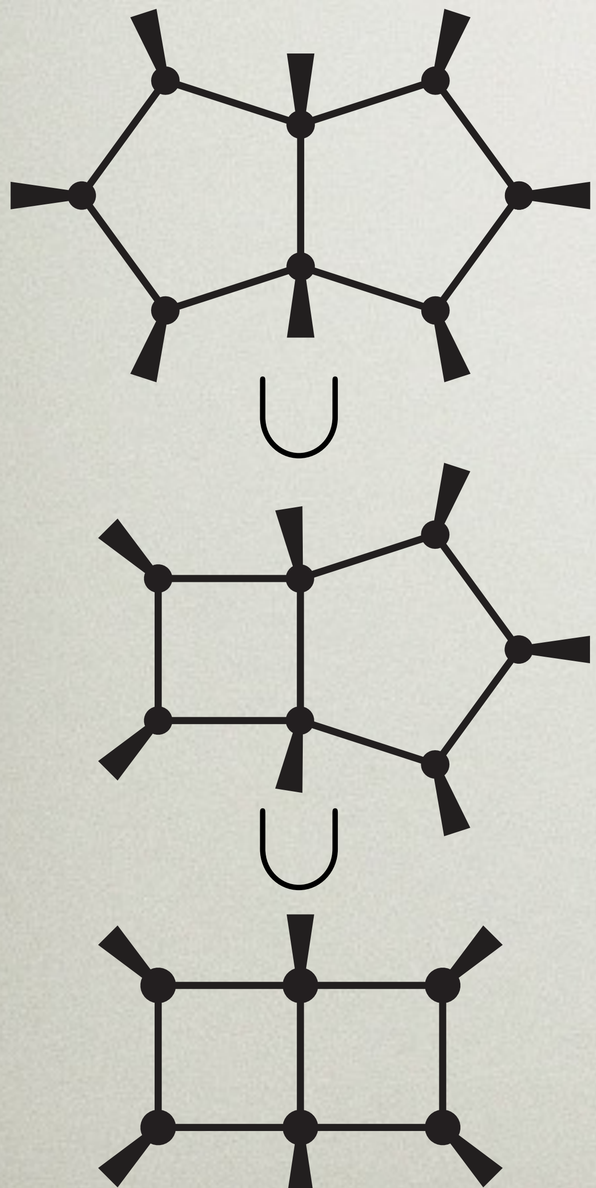

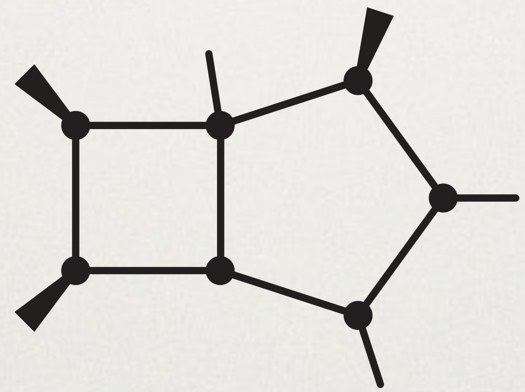

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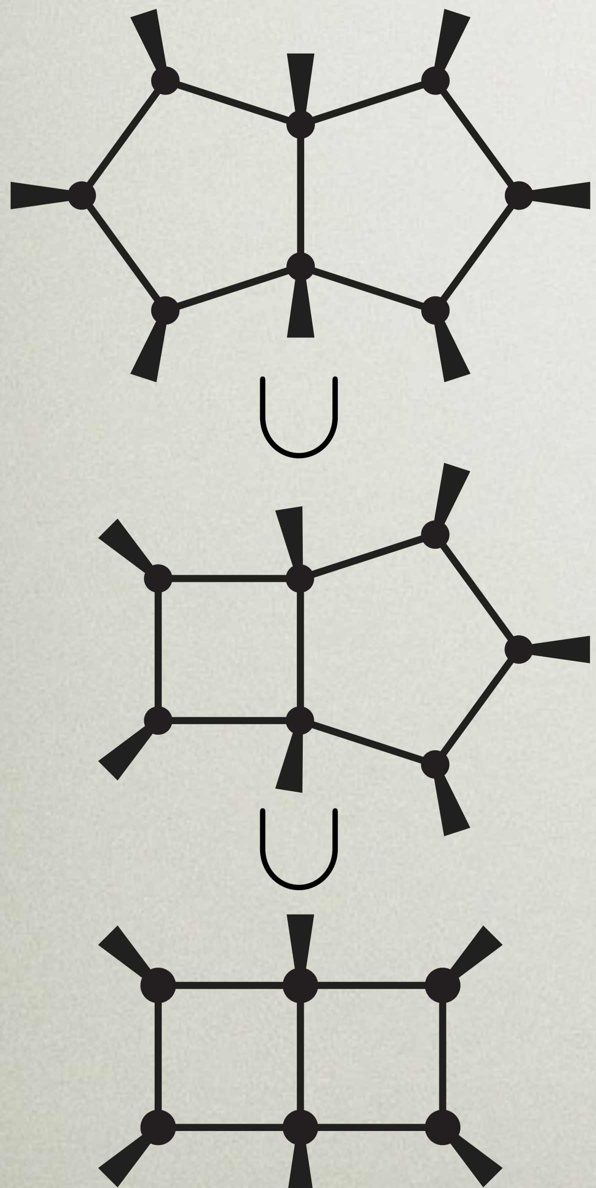

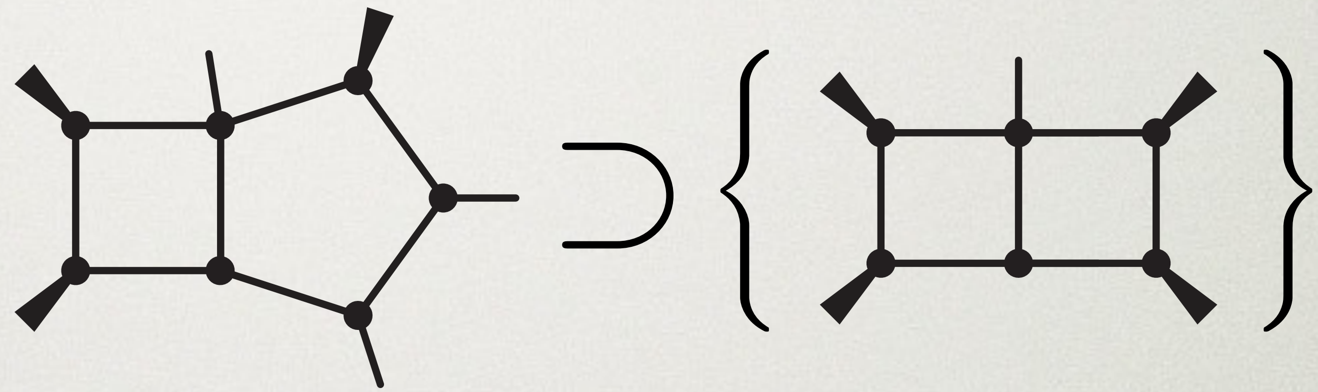

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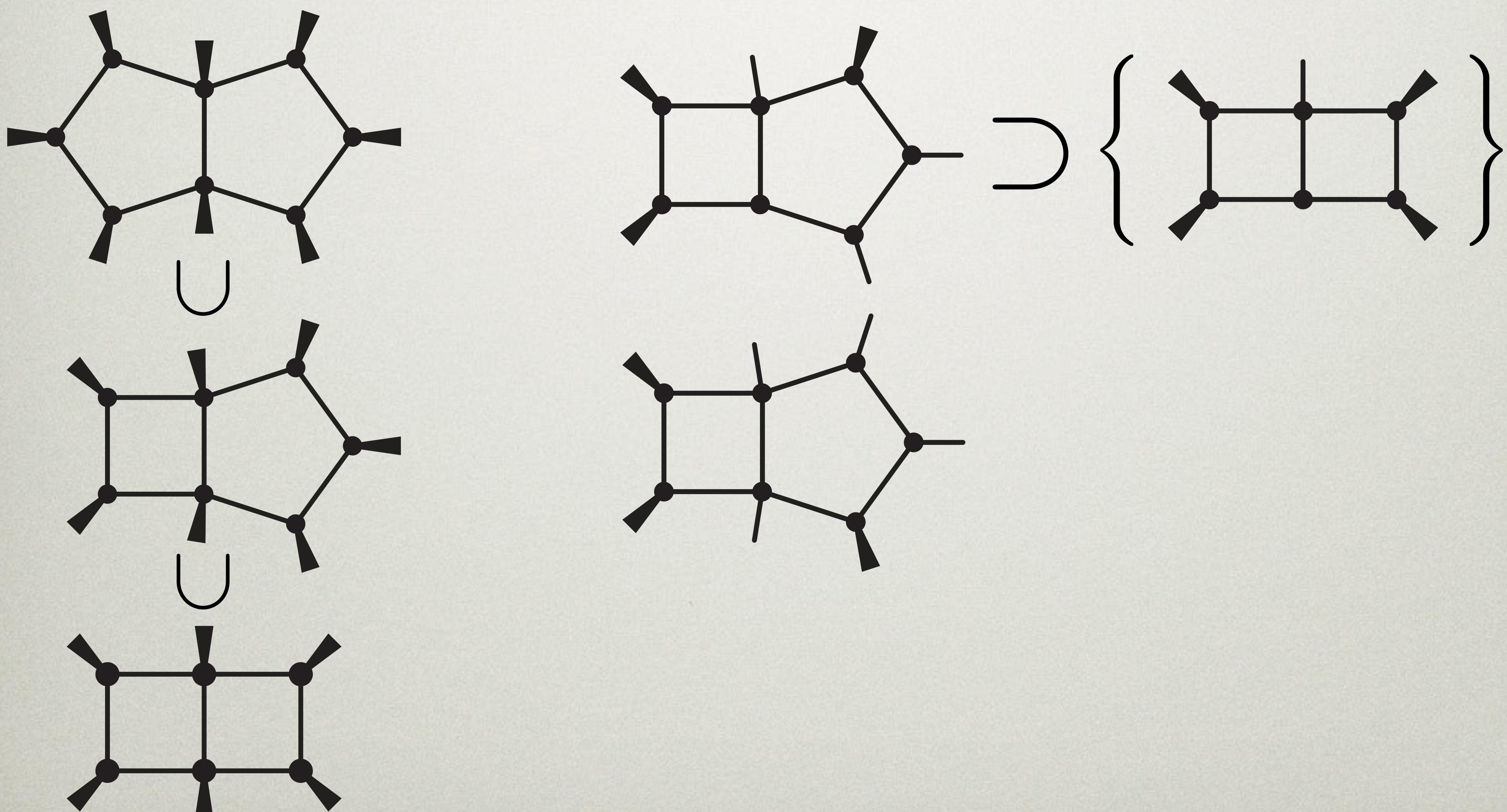

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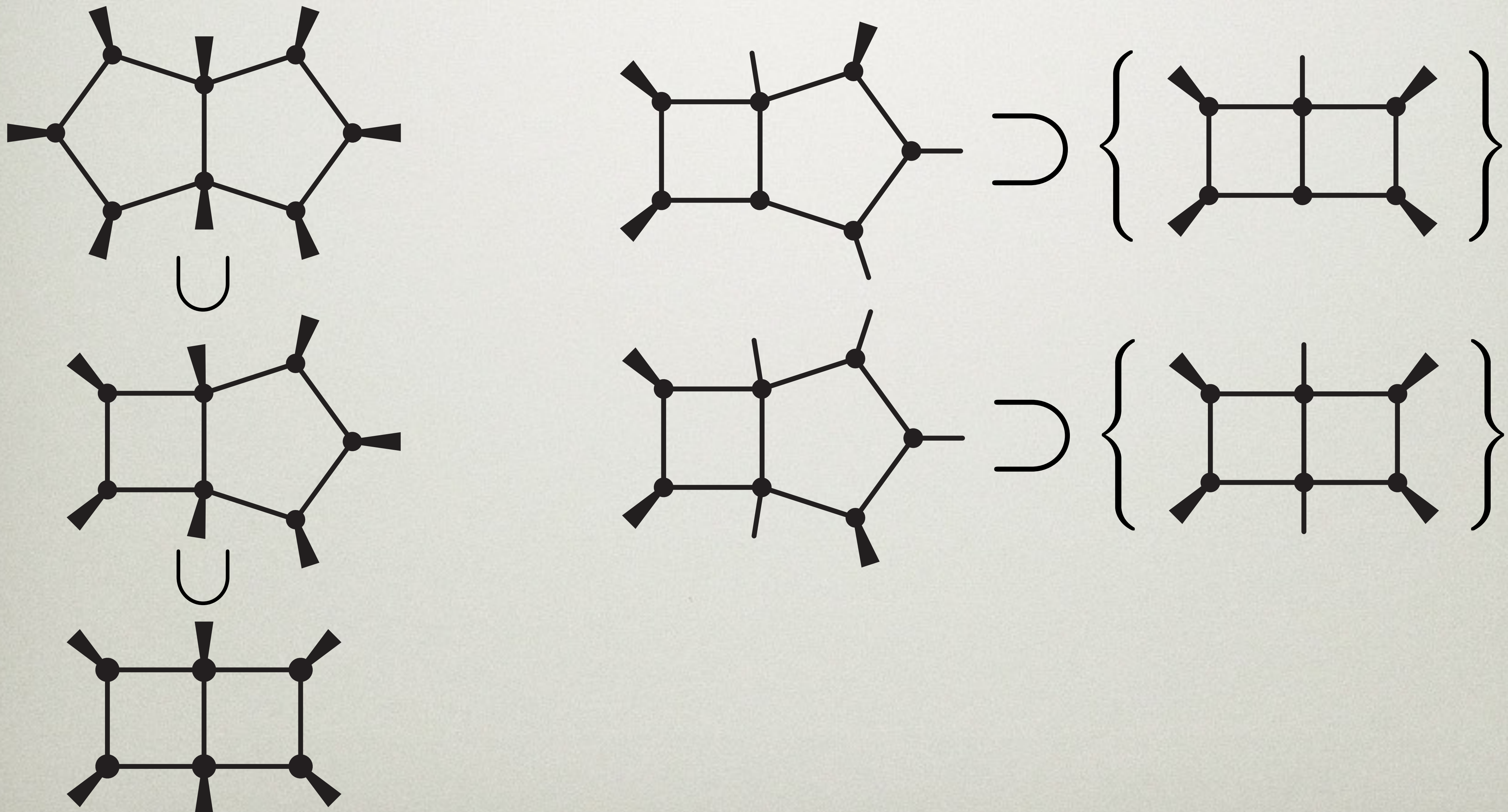

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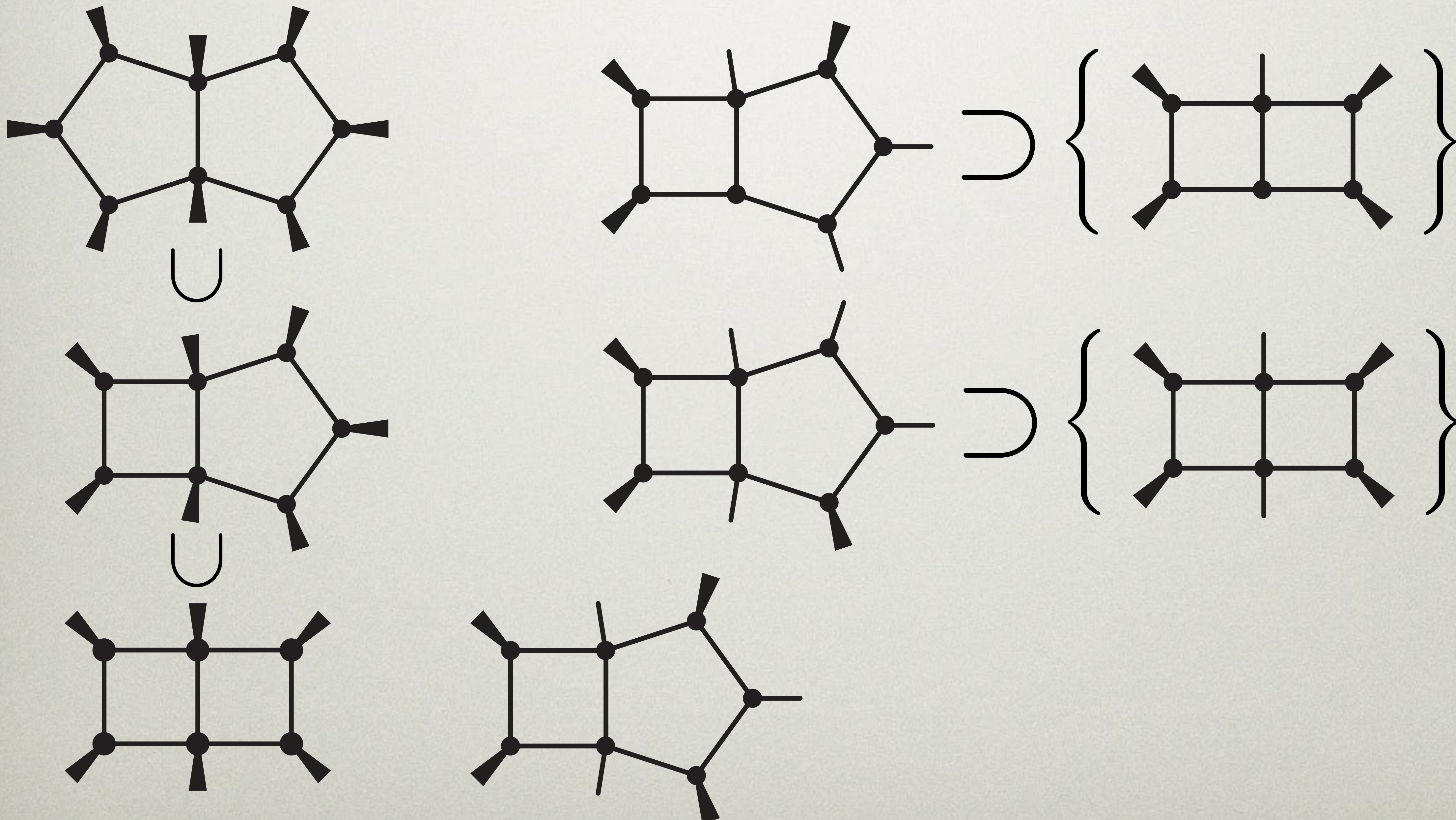

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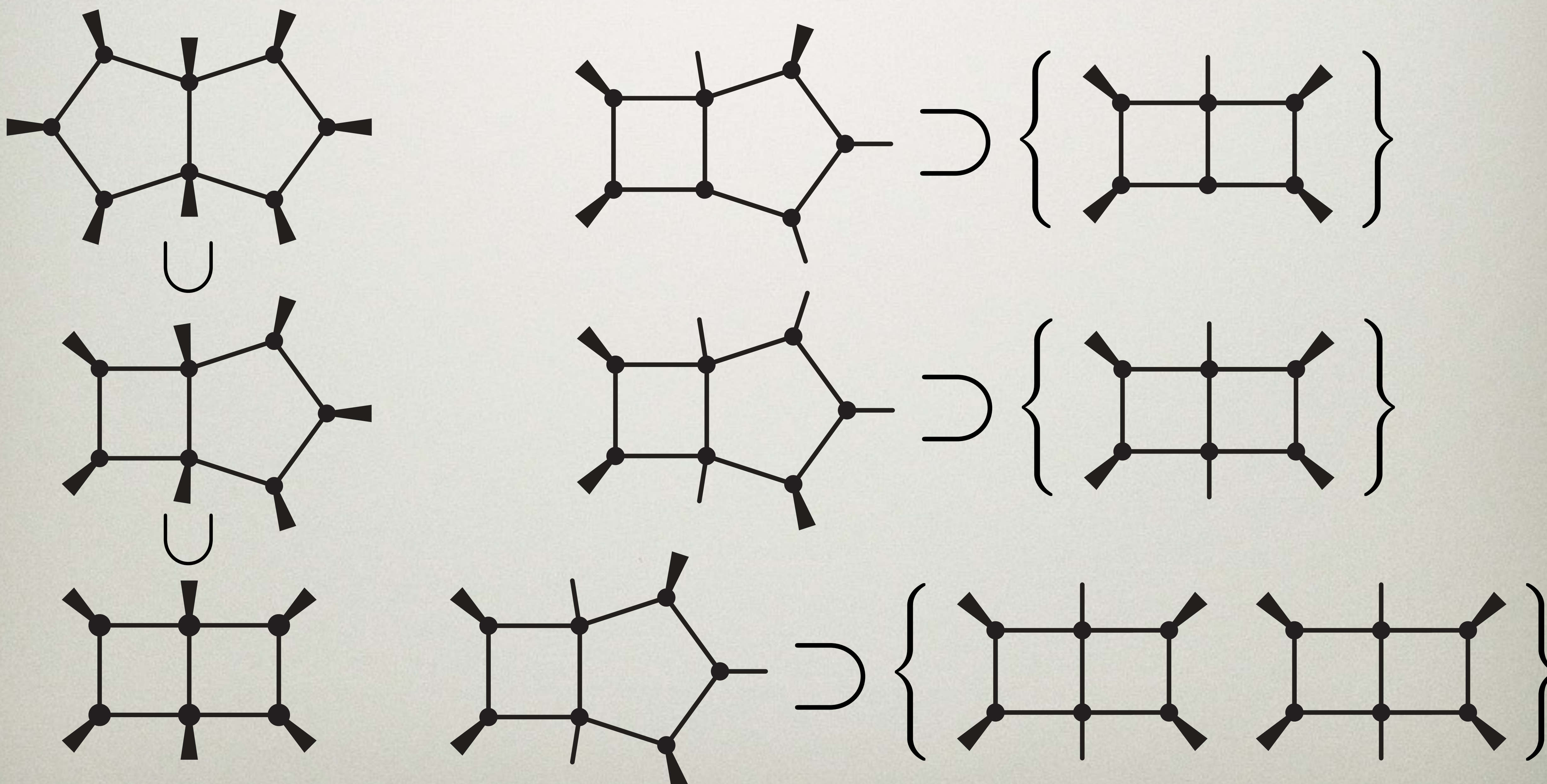

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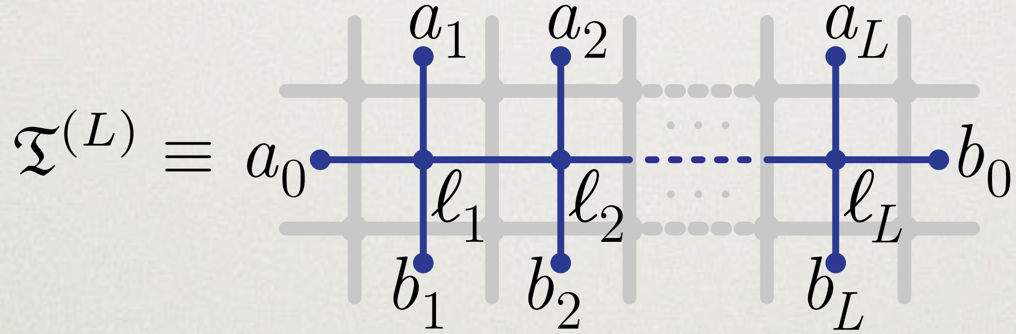
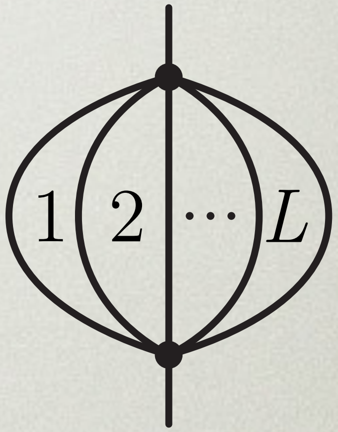
Traintracks Past Polylogarithms

- Despite their ubiquity at low multiplicity and low loop orders, iterated polylogarithms are far from the only class of integrals that are needed in QFT

[**JB**, He, McLeod, von Hippel, Wilhelm (2018)]
[Bloch, Kerr, Vanhove; Broadhurst;…]





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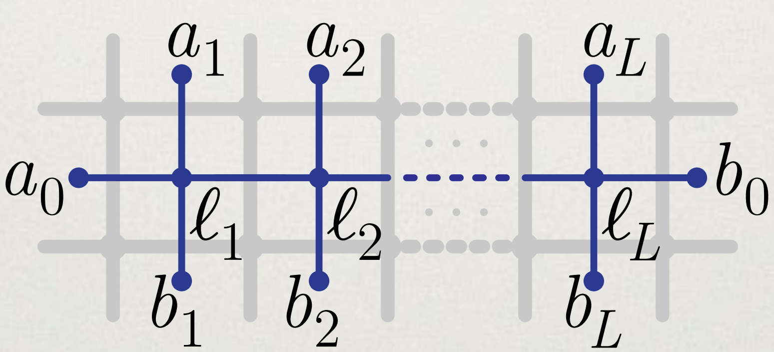
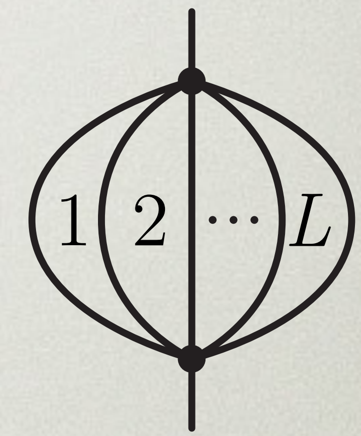
$\mathfrak{T}^{(L)} \equiv$ 



$$\mathfrak{T}^{(L)} = \int_0^\infty [d^L\vec{\alpha}]\, d^L\vec{\beta}\, \frac{1}{(f_1\cdots f_L)\, g_L}$$

$$\begin{aligned} f_k \equiv\, & (a_0 a_{k-1}; a_k b_{k-1})(a_{k-1} b_k; b_{k-1} a_0)(a_k b_k; a_{k-1} b_{k-1}) f_{k-1} \\ & + \alpha_0(\alpha_k + \beta_k) + \alpha_k \beta_k + \sum_{j=1}^{k-1} \Big[\alpha_j \alpha_k (b_j a_0; a_j a_k) \\ & + \alpha_j \beta_k (b_j a_0; a_j b_k) + \alpha_k \beta_j (a_0 a_j; a_k b_j) + \beta_j \beta_k (a_0 a_j; b_k b_j)\Big], \end{aligned}$$

$$g_L \equiv \alpha_0 + \sum_{j=1}^{L} \Big[\alpha_j (b_j a_0; a_j b_0) + \beta_j (a_0 a_j; b_0 b_j)\Big].$$

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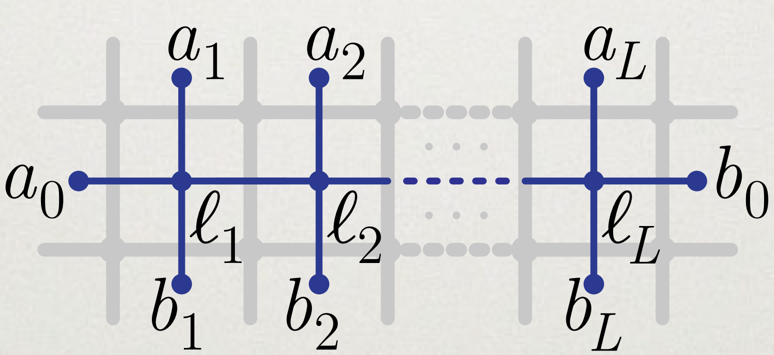
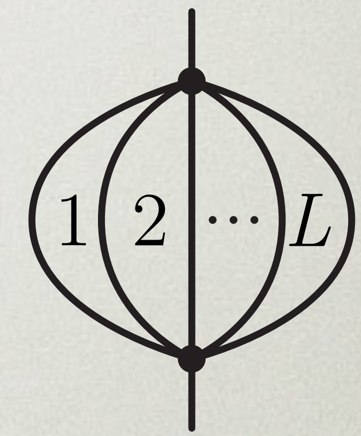
$\mathfrak{T}^{(L)} \equiv$ 



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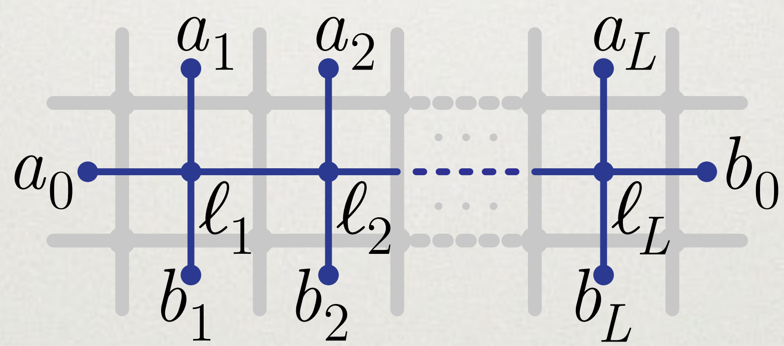
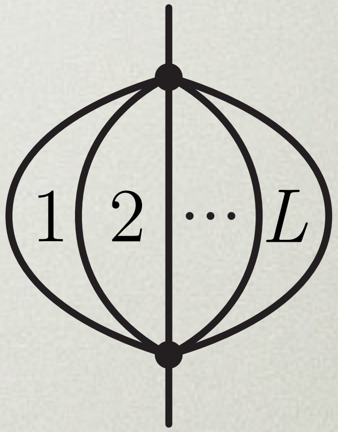
$$\mathcal{A}(\overbrace{\varphi_{12},\ldots,\varphi_{12}}^{L+1},\varphi_{13},\varphi_{13},\overbrace{\varphi_{34},\ldots,\varphi_{34}}^{L+1},\varphi_{24},\varphi_{24})$$

$\mathfrak{T}^{(L)} \equiv$ 



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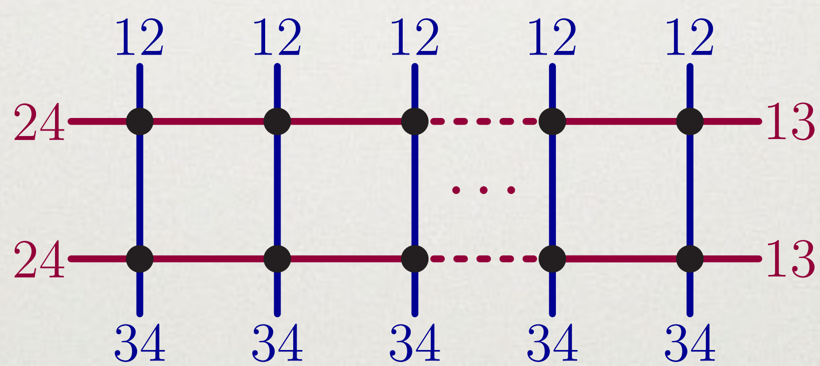
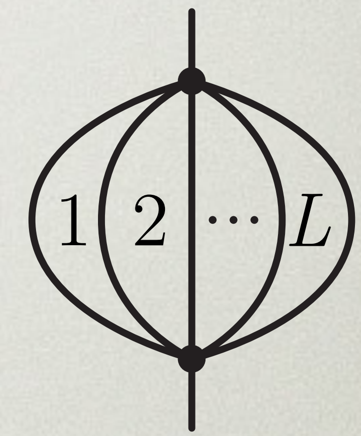
$\mathfrak{T}^{(L)} \equiv$ 



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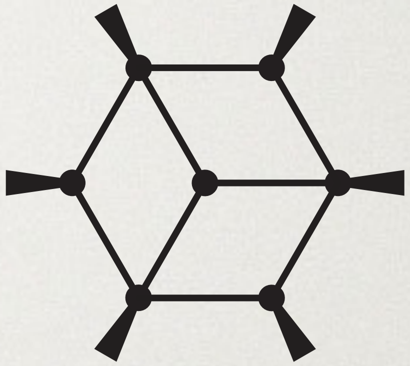
A Three-Loop Calabi-Yau 3-Fold

- Consider the simplest finite 3-loop wheel integral:

[**JB**, McLeod, von Hippel, Wilhelm (*in prep*.)]



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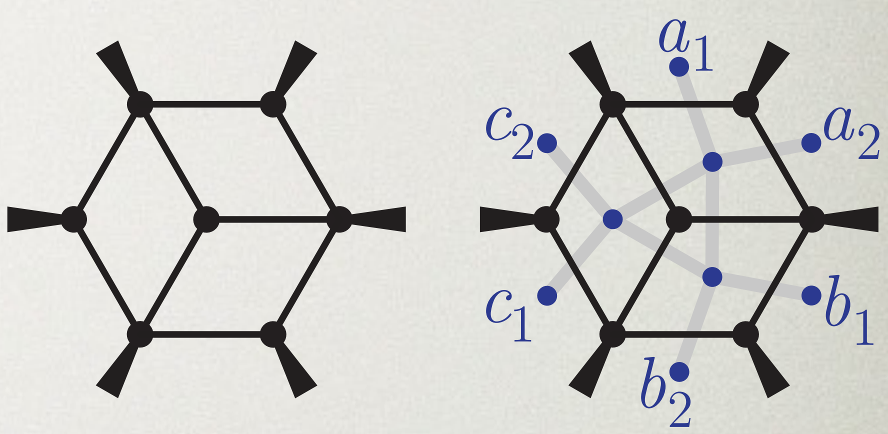

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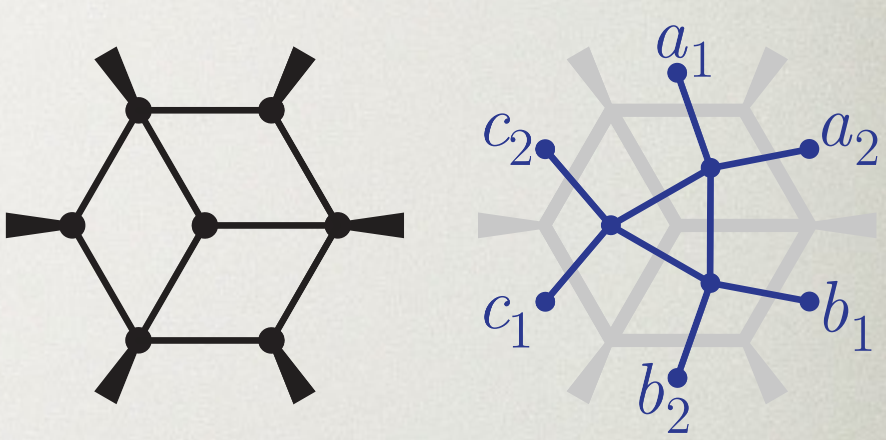

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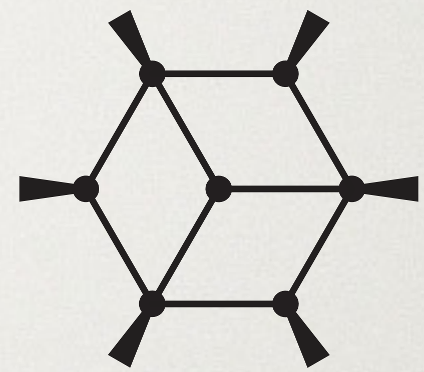
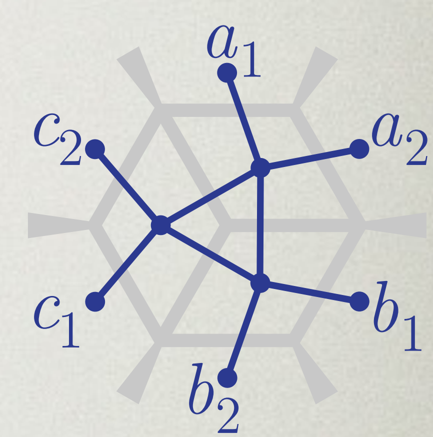
$$\int \frac{d^4x_A\, d^4x_B\, d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(A,a_1)(A,a_2)(A,B)(B,b_1)(B,b_2)(B,C)(C,c_1)(C,c_2)(C,A)}$$





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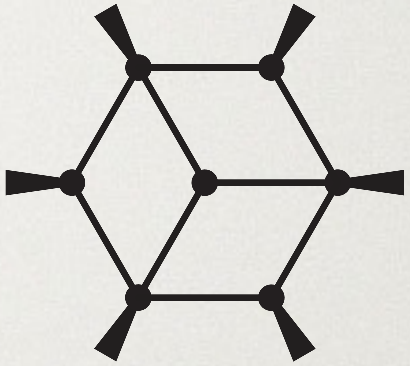
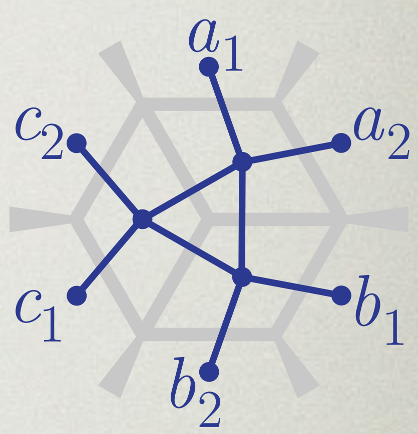




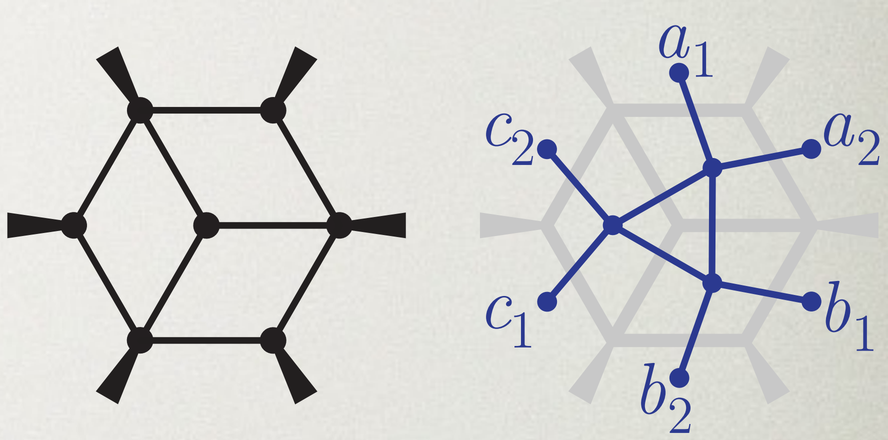
$$|Y_A) \coloneqq |a_1)\alpha_1 + |a_2)\alpha_2 + |C)\alpha_3 + |B)\eta_1 \eqqcolon |Q_A) + |B)\eta_1$$

A Three-Loop Calabi-Yau 3-Fold

- Consider the simplest finite 3-loop wheel integral:

[**JB**, McLeod, von Hippel, Wilhelm (*in prep.*)]

$$\int \frac{d^4x_A\, d^4x_B\, d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(A,a_1)(A,a_2)(A,B)(B,b_1)(B,b_2)(B,C)(C,c_1)(C,c_2)(C,A)}$$



$$|Y_A) \coloneqq |a_1)\alpha_1 + |a_2)\alpha_2 + |C)\alpha_3 + |B)\eta_1 \eqqcolon |Q_A) + |B)\eta_1$$

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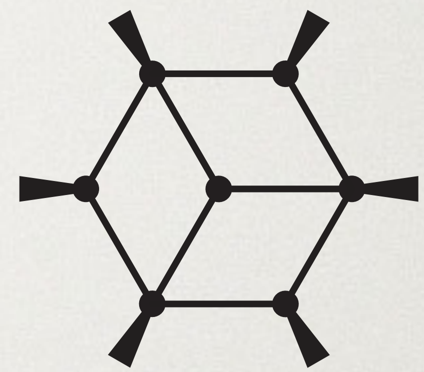

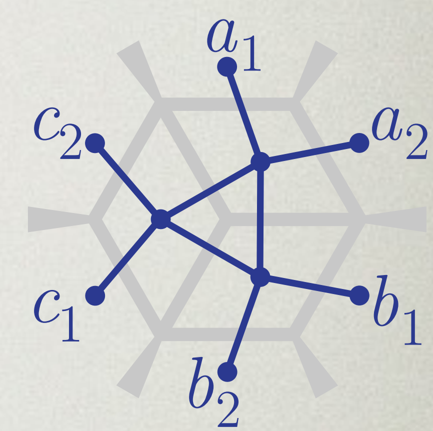

$$|Y_A) \coloneqq |a_1)\alpha_1 + |a_2)\alpha_2 + |C)\alpha_3 + |B)\eta_1 \eqqcolon |Q_A) + |B)\eta_1$$

$$|Y_B) \coloneqq |b_1)\beta_1 + |b_2)\beta_2 + |Q_A)\beta_3 + |C)\eta_2 \eqqcolon |Q_B) + |C)\eta_2$$

$$= \int_0^\infty [d^2\vec{\alpha}][d^2\vec{\beta}] \int \frac{d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(Q_A,Q_A)(Q_B,Q_B)(Q_B,C)(C,c_1)(C,c_2)}$$

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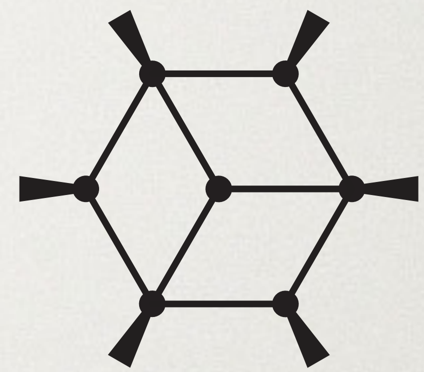

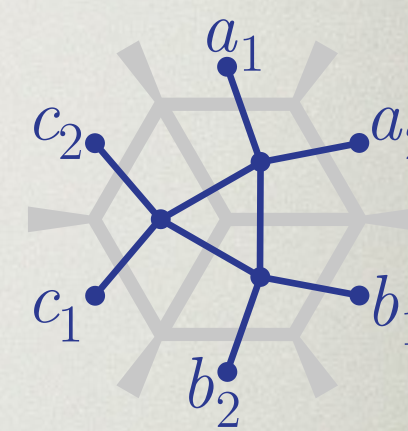

$$|Y_A) \coloneqq |a_1)\alpha_1 + |a_2)\alpha_2 + |C)\alpha_3 + |B)\eta_1 \eqqcolon |Q_A) + |B)\eta_1$$

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$$\alpha_1 \mapsto \alpha_1 (C,a_2) \qquad \alpha_2 \mapsto \alpha_2 (C,a_1) \qquad \alpha_3 \mapsto (a_1,a_2)$$

$$\beta_1 \mapsto \beta_1 \frac{(C,a_1)(a_1,a_2)}{(a_1,b_1)} \quad \beta_2 \mapsto \beta_2 \frac{(C,a_1)(a_1,a_2)}{(a_1,b_2)} \quad \beta_3 \mapsto 1$$

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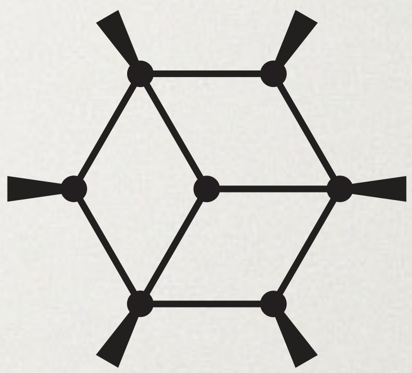

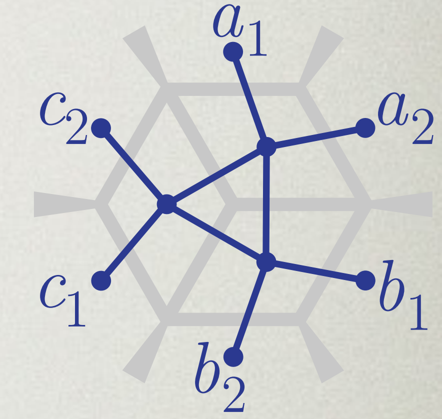

$$|Y_A) \coloneqq |a_1)\alpha_1 + |a_2)\alpha_2 + |C)\alpha_3 + |B)\eta_1 \eqqcolon |Q_A) + |B)\eta_1$$
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$$= \int_0^\infty \left[d^2\vec{\alpha}\right]\left[d^2\vec{\beta}\right] \int \frac{d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(Q_A,Q_A)(Q_B,Q_B)(Q_B,C)(C,c_1)(C,c_2)}$$

$$\alpha_1 \mapsto \alpha_1 (C,a_2) \qquad \alpha_2 \mapsto \alpha_2 (C,a_1) \qquad \alpha_3 \mapsto (a_1,a_2)$$
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$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta} \int \frac{d^4x_C\ (a_1,a_2)^2 (b_1,b_2)(c_1,c_2)/(a_1,b_1)}{(\alpha_1+\alpha_2+\alpha_1\alpha_2)(C,R)(C,S)(C,c_1)(C,c_2)}$$

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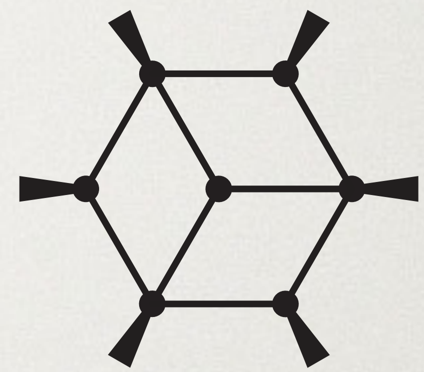

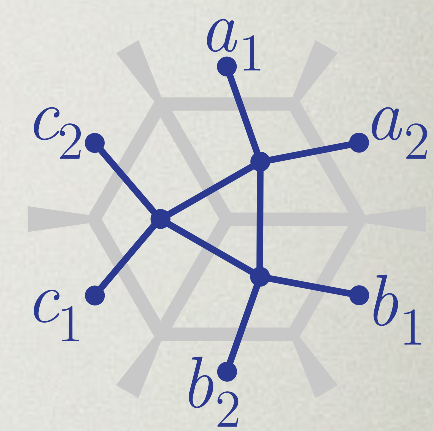

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$$|Y_C) \coloneqq |c_1)\gamma_1 + |R)\gamma_2 + |S)\gamma_3 + |c_2)\eta_3 \eqqcolon |Q) + |c_2)\eta_3$$

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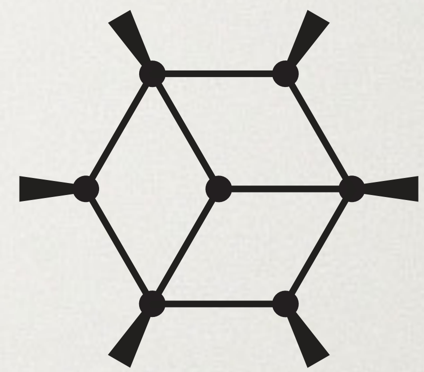

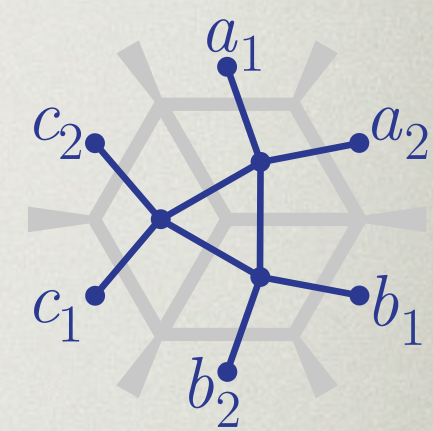

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$$|Y_C) \coloneqq |c_1)\gamma_1 + |R)\gamma_2 + |S)\gamma_3 + |c_2)\eta_3 \eqqcolon |Q) + |c_2)\eta_3$$

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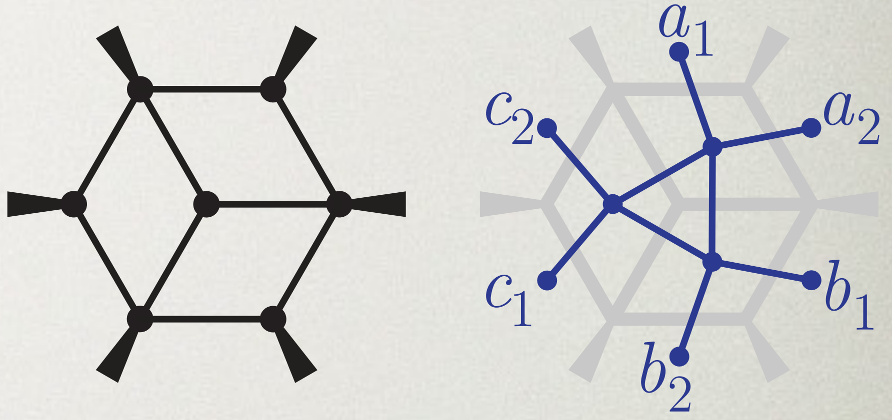

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta} \int \frac{d^4x_C\; (a_1,a_2)^2(b_1,b_2)(c_1,c_2)/(a_1,b_1)}{(\alpha_1+\alpha_2+\alpha_1\alpha_2)(C,R)(C,S)(C,c_1)(C,c_2)} = \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\; \frac{(a_1,a_2)^2(b_1,b_2)(c_1,c_2)/(a_1,b_1)}{(\alpha_1+\alpha_2+\alpha_1\alpha_2)(Q,c_1)(Q,Q)}$$

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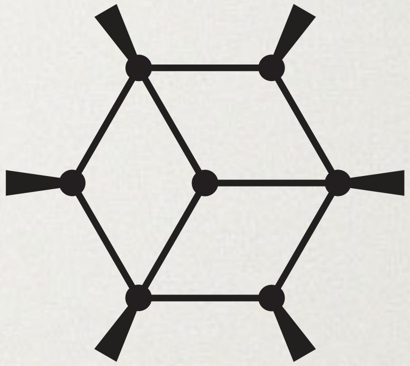

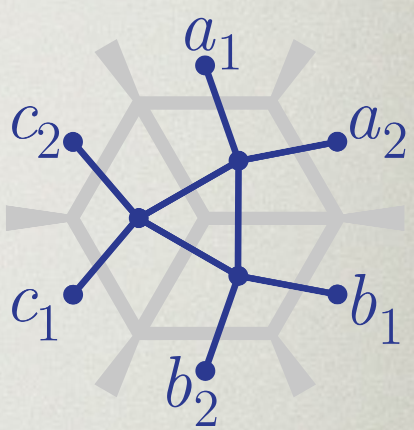

$$\int \frac{d^4x_A\, d^4x_B\, d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(A,a_1)(A,a_2)(A,B)(B,b_1)(B,b_2)(B,C)(C,c_1)(C,c_2)(C,A)}$$

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\; \frac{n_0}{f_1\, f_2\, f_3}$$

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta} \int \frac{d^4x_C\; (a_1,a_2)^2(b_1,b_2)(c_1,c_2)/(a_1,b_1)}{(\alpha_1+\alpha_2+\alpha_1\alpha_2)(C,R)(C,S)(C,c_1)(C,c_2)} = \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\; \frac{(a_1,a_2)^2(b_1,b_2)(c_1,c_2)/(a_1,b_1)}{(\alpha_1+\alpha_2+\alpha_1\alpha_2)(Q,c_1)(Q,Q)}$$

$$|Y_C) \coloneqq |c_1)\gamma_1 + |R)\gamma_2 + |S)\gamma_3 + |c_2)\eta_3 \eqqcolon |Q) + |c_2)\eta_3$$

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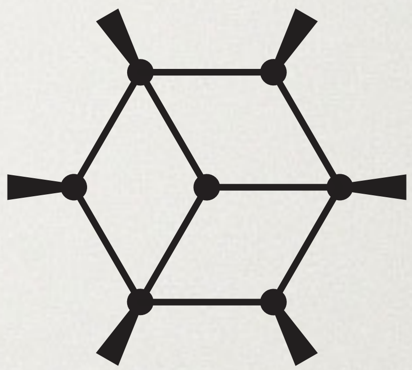

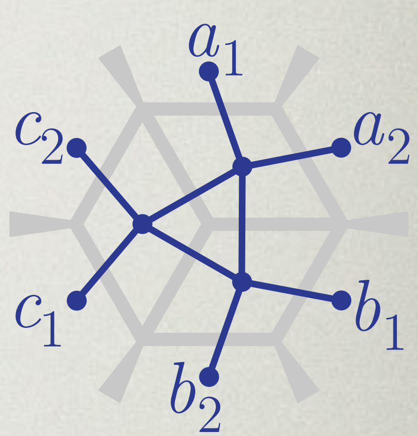

$$\int \frac{d^4x_A\, d^4x_B\, d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(A,a_1)(A,a_2)(A,B)(B,b_1)(B,b_2)(B,C)(C,c_1)(C,c_2)(C,A)}$$

$$= \int_0^\infty d^2\vec{\alpha}\; d^2\vec{\beta}\; d^2\vec{\gamma}\; \frac{n_0}{f_1\, f_2\, f_3}$$

$$x_1 \coloneqq (c_1a_1;a_2b_2) \quad x_2 \coloneqq (a_1b_1;b_2c_2) \quad x_3 \coloneqq (b_1c_1;c_2a_2)$$
$$y_1 \coloneqq (a_1a_2;b_1c_2) \quad y_2 \coloneqq (b_1b_2;c_1a_2) \quad y_3 \coloneqq (c_1c_2;a_1b_2)$$
$$z_1 \coloneqq (b_2c_1;c_2b_1) \quad z_2 \coloneqq (c_2a_1;a_2c_1) \quad z_3 \coloneqq (a_2b_1;b_2a_1)$$

$$= \int_0^\infty d^2\vec{\alpha}\; d^2\vec{\beta} \int \frac{d^4x_C\; (a_1,a_2)^2(b_1,b_2)(c_1,c_2)/(a_1,b_1)}{(\alpha_1+\alpha_2+\alpha_1\alpha_2)(C,R)(C,S)(C,c_1)(C,c_2)} = \int_0^\infty d^2\vec{\alpha}\; d^2\vec{\beta}\; d^2\vec{\gamma}\; \frac{(a_1,a_2)^2(b_1,b_2)(c_1,c_2)/(a_1,b_1)}{(\alpha_1+\alpha_2+\alpha_1\alpha_2)(Q,c_1)(Q,Q)}$$

$$|Y_C) \coloneqq |c_1)\gamma_1 + |R)\gamma_2 + |S)\gamma_3 + |c_2)\eta_3 \eqqcolon |Q) + |c_2)\eta_3$$

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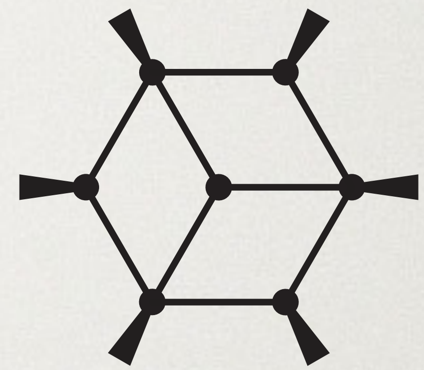

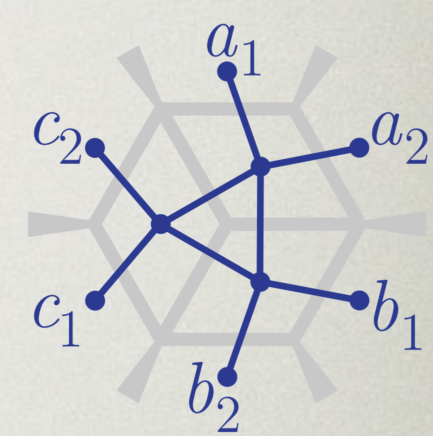

$$\int \frac{d^4x_A\, d^4x_B\, d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(A,a_1)(A,a_2)(A,B)(B,b_1)(B,b_2)(B,C)(C,c_1)(C,c_2)(C,A)}$$

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\ \frac{n_0}{f_1\, f_2\, f_3}$$

$$x_1 \coloneqq (c_1a_1;a_2b_2) \quad x_2 \coloneqq (a_1b_1;b_2c_2) \quad x_3 \coloneqq (b_1c_1;c_2a_2)$$
$$y_1 \coloneqq (a_1a_2;b_1c_2) \quad y_2 \coloneqq (b_1b_2;c_1a_2) \quad y_3 \coloneqq (c_1c_2;a_1b_2)$$
$$z_1 \coloneqq (b_2c_1;c_2b_1) \quad z_2 \coloneqq (c_2a_1;a_2c_1) \quad z_3 \coloneqq (a_2b_1;b_2a_1)$$

$$n_0 \coloneqq y_1(x_1x_2x_3y_1y_2y_3)$$

$$f_1 \coloneqq \alpha_1+\alpha_2+\alpha_1\alpha_2\,;$$

$$\begin{aligned} f_2 \coloneqq\ & \alpha_1(1+\alpha_2+\beta_1+\beta_2+\gamma_2)+\alpha_2(1+x_1z_2(z_3\beta_1+\beta_2)+\gamma_2) \\ & +\beta_1y_1(1+x_1x_3y_2z_2\beta_2+\gamma_2)+x_2y_1(x_1y_3\gamma_1+\beta_2(1+\gamma_2))\,; \end{aligned}$$

$$\begin{aligned} f_3 \coloneqq\ & \alpha_1(1+\alpha_2+\beta_1+\beta_2+\gamma_2)\Big[\gamma_1+\beta_2(1+\alpha_2+x_3y_1y_2\beta_1+\gamma_2) \\ & +z_3\beta_1(1+\alpha_2+\gamma_2)\Big]+\gamma_1\Big[\alpha_2(1+x_1(z_3\beta_1+\beta_2)+\gamma_2) \\ & +x_3y_1(x_2z_1\beta_2(1+\gamma_2)+\beta_1(1+x_1y_2\beta_2+\gamma_2))\Big] \\ & +(1+\gamma_2)(1+\alpha_2+\beta_1+\beta_2+\gamma_2)(z_3\alpha_2\beta_1+(\alpha_2+x_3y_1y_2\beta_1)\beta_2) \end{aligned}$$

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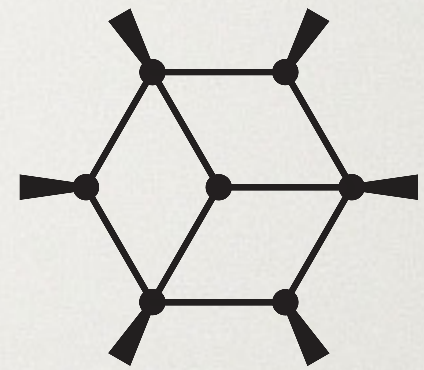
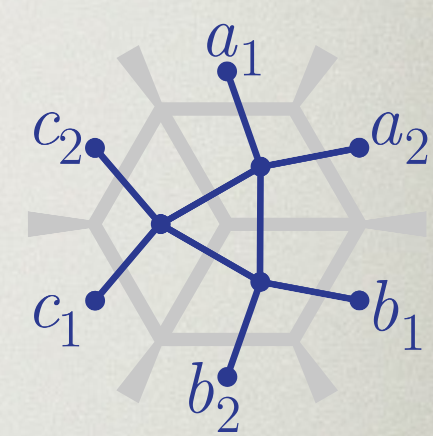
[**JB**, McLeod, von Hippel, Wilhelm (*in prep.*)]

$$\int \frac{d^4x_A\, d^4x_B\, d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(A,a_1)(A,a_2)(A,B)(B,b_1)(B,b_2)(B,C)(C,c_1)(C,c_2)(C,A)}$$





$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\ \frac{n_0}{f_1\, f_2\, f_3}$$

$$= \int \frac{d^3\vec{q}}{\sqrt{Q(\vec{q})}} H_3(\vec{q})$$

$$x_1 \coloneqq (c_1a_1;a_2b_2) \quad x_2 \coloneqq (a_1b_1;b_2c_2) \quad x_3 \coloneqq (b_1c_1;c_2a_2)$$

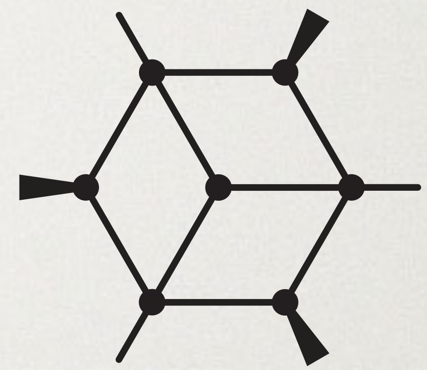
$$y_1 \coloneqq (a_1a_2;b_1c_2) \quad y_2 \coloneqq (b_1b_2;c_1a_2) \quad y_3 \coloneqq (c_1c_2;a_1b_2)$$

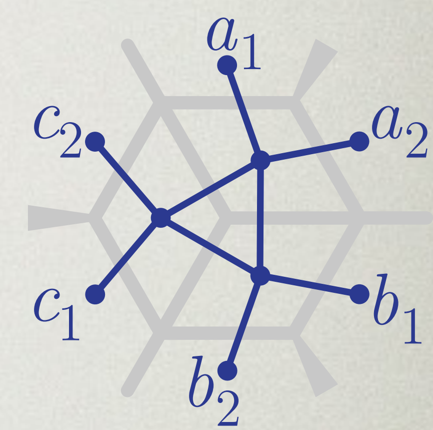
$$z_1 \coloneqq (b_2c_1;c_2b_1) \quad z_2 \coloneqq (c_2a_1;a_2c_1) \quad z_3 \coloneqq (a_2b_1;b_2a_1)$$

A Three-Loop Calabi-Yau 3-Fold

- Consider the simplest finite 3-loop wheel integral:
  [**JB**, McLeod, von Hippel, Wilhelm (*in prep*.)]

$$\int \frac{d^4x_A\, d^4x_B\, d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(A,a_1)(A,a_2)(A,B)(B,b_1)(B,b_2)(B,C)(C,c_1)(C,c_2)(C,A)}$$





$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\ \frac{n_0}{f_1\, f_2\, f_3}$$

$$= \int \frac{d^3\vec{q}}{\sqrt{Q(\vec{q})}} H_3(\vec{q})$$

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$$y_1 \coloneqq (a_1a_2;b_1c_2) \quad y_2 \coloneqq (b_1b_2;c_1a_2) \quad y_3 \coloneqq (c_1c_2;a_1b_2)$$
$$z_1 \coloneqq 0 \quad z_2 \coloneqq 0 \quad z_3 \coloneqq 0$$

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[**JB**, McLeod, von Hippel, Wilhelm (*in prep.*)]

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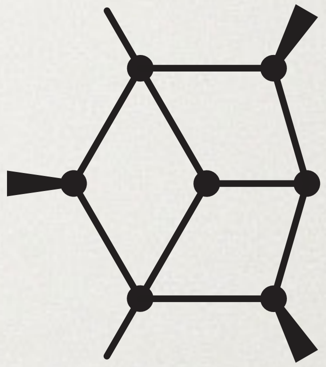

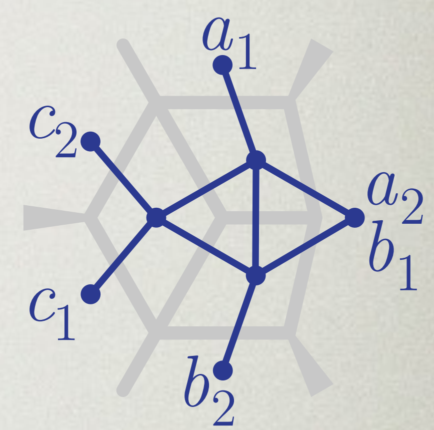

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\; \frac{n_0}{f_1\, f_2\, f_3}$$

$$= \int \frac{d^3\vec{q}}{\sqrt{Q(\vec{q})}} H_3(\vec{q})$$

$$x_1 \coloneqq (c_1a_1;a_2b_2) \quad x_2 \coloneqq (a_1b_1;b_2c_2) \quad x_3 \coloneqq 1$$

$$y_1 \coloneqq 1 \quad y_2 \coloneqq 1 \quad y_3 \coloneqq (c_1c_2;a_1b_2)$$

$$z_1 \coloneqq 0 \quad z_2 \coloneqq 0 \quad z_3 \coloneqq 0$$

A Three-Loop Calabi-Yau 3-Fold

- Consider the simplest finite 3-loop wheel integral:

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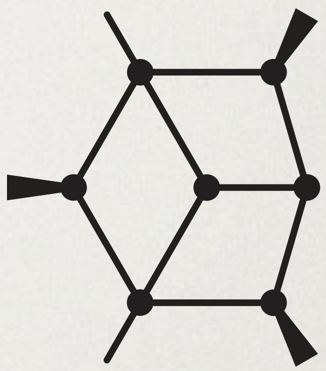

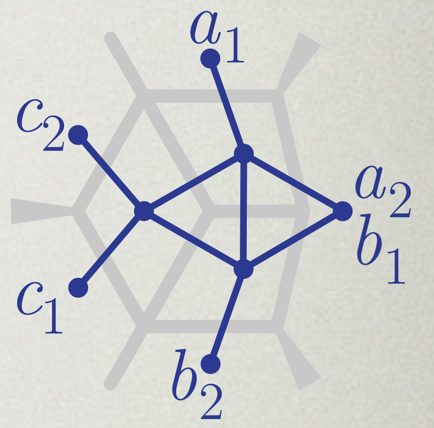

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\ \frac{n_0}{f_1\, f_2\, f_3}$$

$$= \int \frac{d^3\vec{q}}{G(\vec{q})} H_3(\vec{q})$$

$x_1 \coloneqq (c_1 a_1; a_2 b_2)$ $\quad x_2 \coloneqq (a_1 b_1; b_2 c_2)$ $\quad x_3 \coloneqq 1$

$y_1 \coloneqq 1$ $\quad y_2 \coloneqq 1$ $\quad y_3 \coloneqq (c_1 c_2; a_1 b_2)$

$z_1 \coloneqq 0$ $\quad z_2 \coloneqq 0$ $\quad z_3 \coloneqq 0$

A Three-Loop Calabi-Yau 3-Fold

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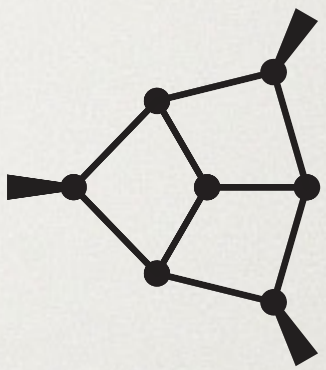

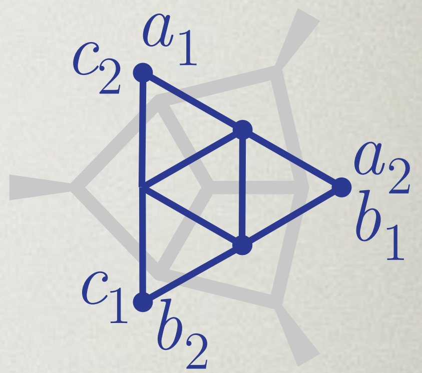

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\; \frac{n_0}{f_1\, f_2\, f_3}$$

$$= \int \frac{d^3\vec{q}}{G(\vec{q})} H_3(\vec{q})$$

$x_1 \coloneqq 1$	$x_2 \coloneqq 1$	$x_3 \coloneqq 1$
$y_1 \coloneqq 1$	$y_2 \coloneqq 1$	$y_3 \coloneqq 1$
$z_1 \coloneqq 0$	$z_2 \coloneqq 0$	$z_3 \coloneqq 0$

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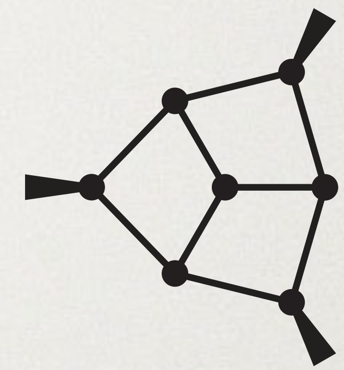

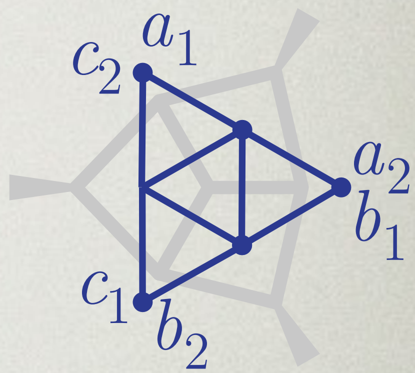

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\; \frac{n_0}{f_1\, f_2\, f_3}$$

$$= \int \frac{d^3\vec{q}}{G(\vec{q})} H_3(\vec{q}) \to 20\,\zeta_5$$

$$x_1 \coloneqq 1 \qquad x_2 \coloneqq 1 \qquad x_3 \coloneqq 1$$

$$y_1 \coloneqq 1 \qquad y_2 \coloneqq 1 \qquad y_3 \coloneqq 1$$

$$z_1 \coloneqq 0 \qquad z_2 \coloneqq 0 \qquad z_3 \coloneqq 0$$

A Three-Loop Calabi-Yau 3-Fold

- Consider the simplest finite 3-loop wheel integral:

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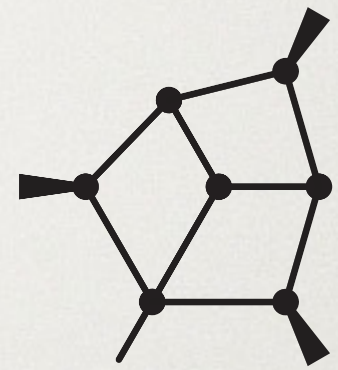

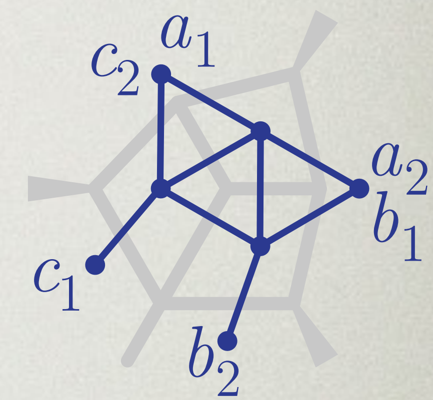

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\; \frac{n_0}{f_1\, f_2\, f_3}$$

$$= \int \frac{d^3\vec{q}}{G(\vec{q})} H_3(\vec{q})$$

$x_1 \coloneqq (c_1 a_1; a_2 b_2)$	$x_2 \coloneqq 1$	$x_3 \coloneqq 1$
$y_1 \coloneqq 1$	$y_2 \coloneqq 1$	$y_3 \coloneqq 1$
$z_1 \coloneqq 0$	$z_2 \coloneqq 0$	$z_3 \coloneqq 0$

A Three-Loop Calabi-Yau 3-Fold

✦ Consider the simplest finite 3-loop wheel integral:

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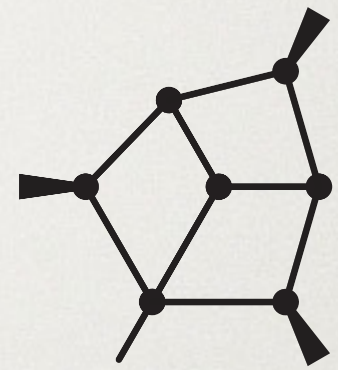

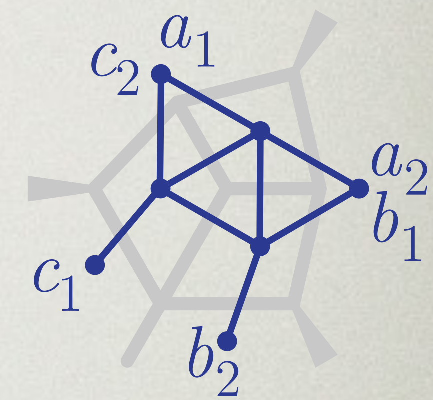

$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\; \frac{n_0}{f_1\, f_2\, f_3}$$

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$y_1 \coloneqq 1$ $\quad y_2 \coloneqq 1$ $\quad y_3 \coloneqq 1$

$z_1 \coloneqq 0$ $\quad z_2 \coloneqq 0$ $\quad z_3 \coloneqq 0$

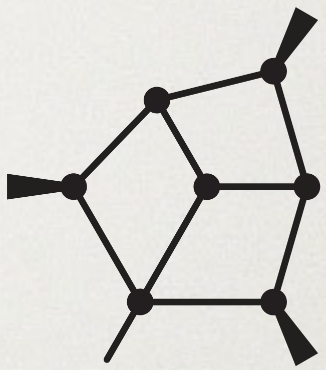
$$\begin{aligned} = \frac{x_1}{1-x_1}\Big[& G(\{0,0,0,1,0,0\},x_1) - G(\{0,0,1,0,0,0\},x_1) + G(\{0,1,1,0,0,0\},x_1) - G(\{0,0,0,1,1,0\},x_1) \\ & + G(\{0,0,1,0,1,0\},x_1) - G(\{0,1,0,1,0,0\},x_1) + G(\{0,1,0,1,1,0\},x_1) - G(\{0,1,1,0,1,0\},x_1) \\ & + \zeta_2 \Big[G(\{0,0,0,1\},x_1) - G(\{0,0,1,0\},x_1) + G(\{0,1,1,0\},x_1) - G(\{0,1,0,1\},x_1) \Big] \\ & + 2\zeta_3 \Big[G(\{0,1,0\},x_1) - G(\{0,0,1\},x_1) + G(\{0,1,1\},x_1) - G(\{0,0,0\},x_1) \Big] \\ & + 6\zeta_4 \Big[G(\{0,0\},x_1) - G(\{0,1\},x_1) \Big] - 2(5\zeta_5 + \zeta_2\zeta_3) G(\{0\},x_1) + 4(\zeta_2^4 - \zeta_3^2) + 3\zeta_6 \Big] \end{aligned}$$

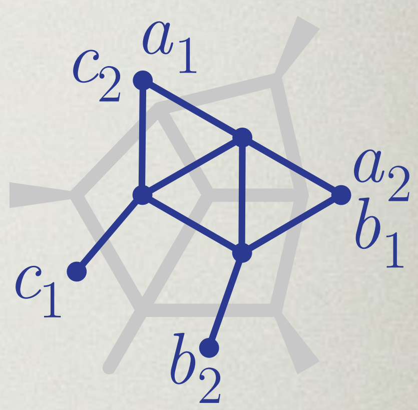
A Three-Loop Calabi-Yau 3-Fold

- Consider the simplest finite 3-loop wheel integral:

[**JB**, McLeod, von Hippel, Wilhelm (*in prep.*)]

$$\int \frac{d^4x_A\, d^4x_B\, d^4x_C \quad (a_1,a_2)(b_1,b_2)(c_1,c_2)}{(A,a_1)(A,a_2)(A,B)(B,b_1)(B,b_2)(B,C)(C,c_1)(C,c_2)(C,A)}$$





$$= \int_0^\infty d^2\vec{\alpha}\, d^2\vec{\beta}\, d^2\vec{\gamma}\ \frac{n_0}{f_1\, f_2\, f_3}$$

$$= \int \frac{d^3\vec{q}}{G(\vec{q})} H_3(\vec{q})$$

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$z_1 \coloneqq 0$ $\quad z_2 \coloneqq 0$ $\quad z_3 \coloneqq 0$

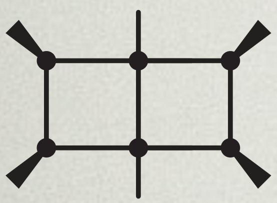
$$= \frac{x_1}{1-x_1}\Big[G(\{0,0,0,1,0,0\},x_1) - G(\{0,0,1,0,0,0\},x_1) + G(\{0,1,1,0,0,0\},x_1) - G(\{0,0,0,1,1,0\},x_1)$$
$$+ G(\{0,0,1,0,1,0\},x_1) - G(\{0,1,0,1,0,0\},x_1) + G(\{0,1,0,1,1,0\},x_1) - G(\{0,1,1,0,1,0\},x_1)$$
$$+ \zeta_2\Big[G(\{0,0,0,1\},x_1) - G(\{0,0,1,0\},x_1) + G(\{0,1,1,0\},x_1) - G(\{0,1,0,1\},x_1)\Big]$$
$$+ 2\zeta_3\Big[G(\{0,1,0\},x_1) - G(\{0,0,1\},x_1) + G(\{0,1,1\},x_1) - G(\{0,0,0\},x_1)\Big]$$
$$+ 6\zeta_4\Big[G(\{0,0\},x_1) - G(\{0,1\},x_1)\Big] - 2(5\zeta_5 + \zeta_2\zeta_3)G(\{0\},x_1) + 4(\zeta_2^4 - \zeta_3^2) + 3\zeta_6\Big]$$

$$\to 20\,\zeta_5$$

Bestiary of Loop Integral Geometry

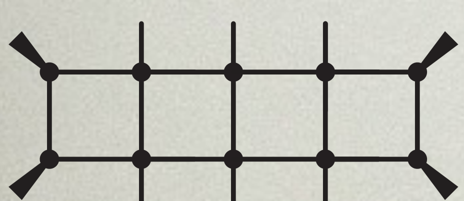
- The bad news is that even elliptic polylogarithms are *far* from sufficient for loop integrals in QFT

[**JB**, McLeod, Spradlin, von Hippel, Wilhelm (2018)]
[**JB**, He, McLeod, von Hippel, Wilhelm (2018)]

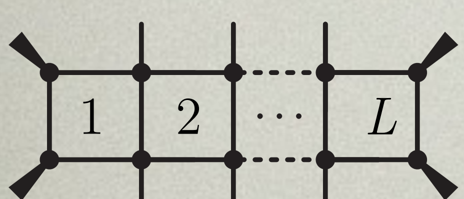


CY_1

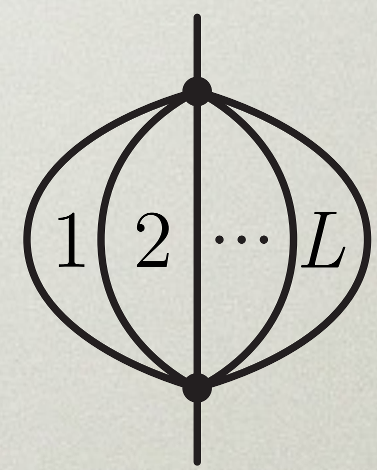
CY_2



CY_3 (?)



CY_{L-1}(?)



CY_{L-1}

[Bloch, Kerr, Vanhove; Broadhurst;…]

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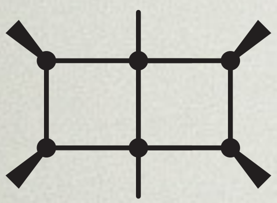
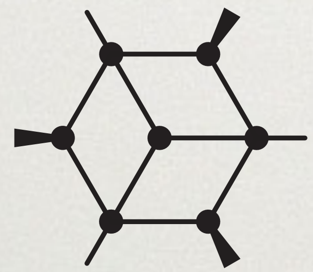
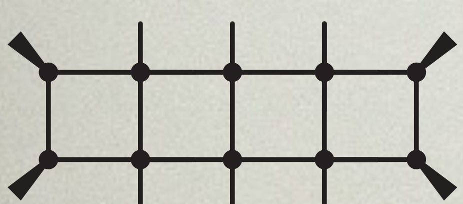


CY_1

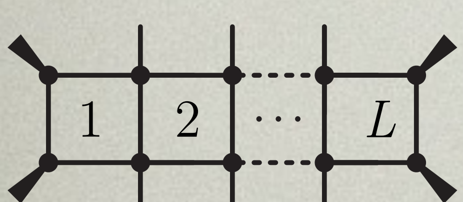
CY_2



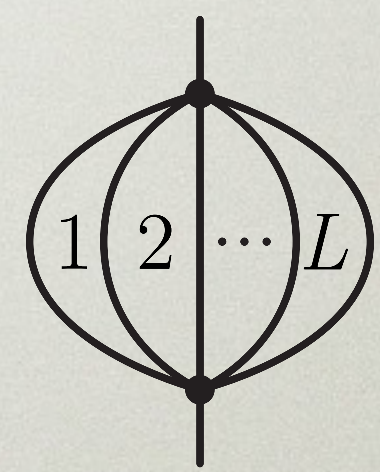
CY_3



CY_3 (?)



CY_{L-1}(?)



CY_{L-1}

[Bloch, Kerr, Vanhove; Broadhurst;…]

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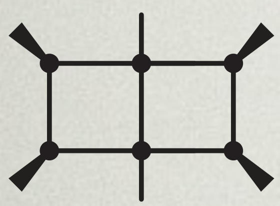
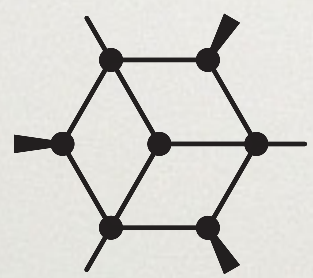
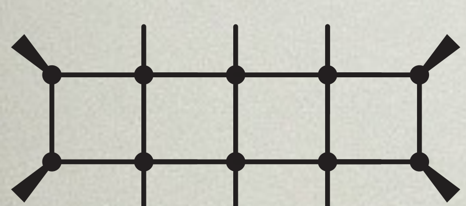


CY_1

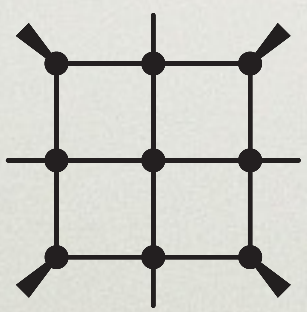
CY_2



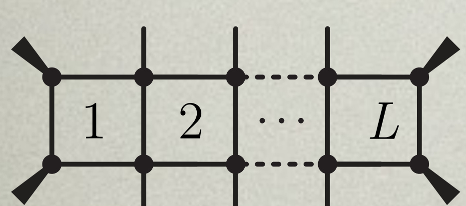
CY_3



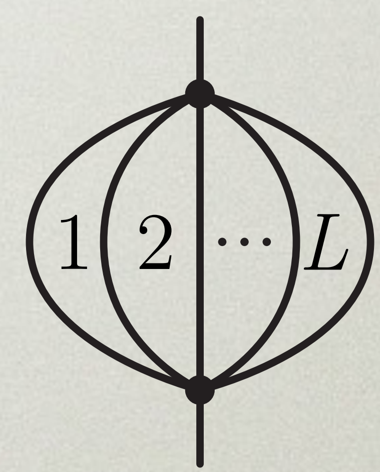
CY_3 (?)



CY_4 (¿?)



CY_{L-1}(?)



CY_{L-1}

[Bloch, Kerr, Vanhove; Broadhurst;…]

Bestiary of Loop Integral Geometry

- Exactly how ***rigid***—aka far from polylogarithmic—can Feynman integrals be?

[**JB**, McLeod, Spradlin, von Hippel, Wilhelm (2018)]
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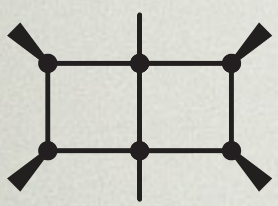
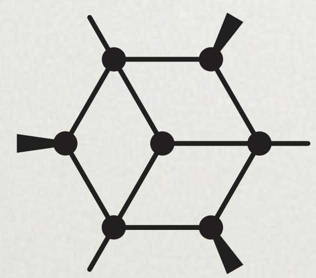
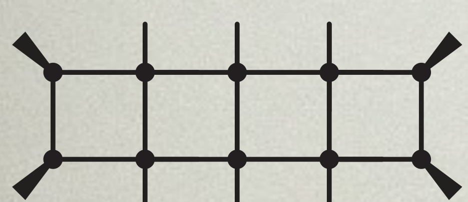


CY_1

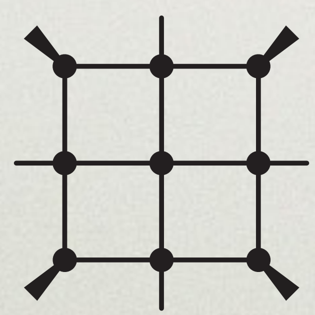
CY_2



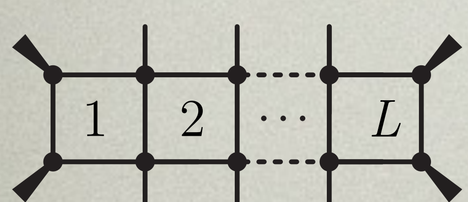
CY_3



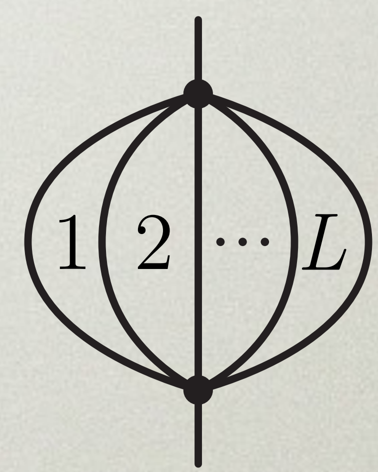
$\mathrm{CY}_3\,(?)$



$\mathrm{CY}_4\,(¿?)$



$\mathrm{CY}_{L-1}(?)$



CY_{L-1}

[Bloch, Kerr, Vanhove; Broadhurst;…]

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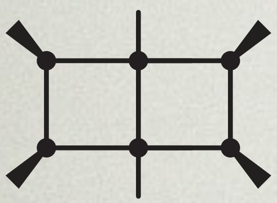
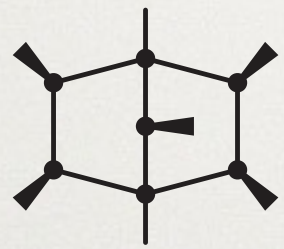
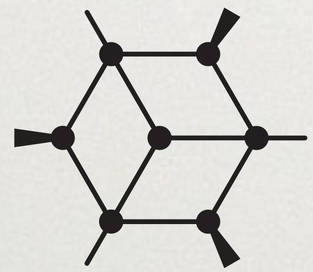


CY_1

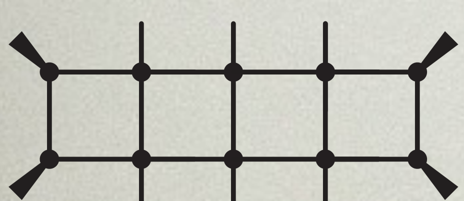


CY_2 (¿?)

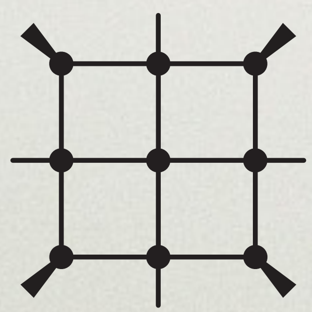
CY_2



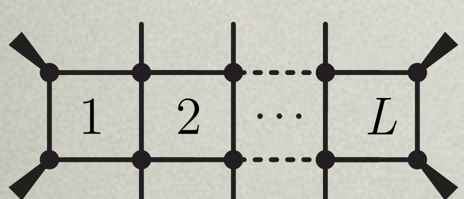
CY_3



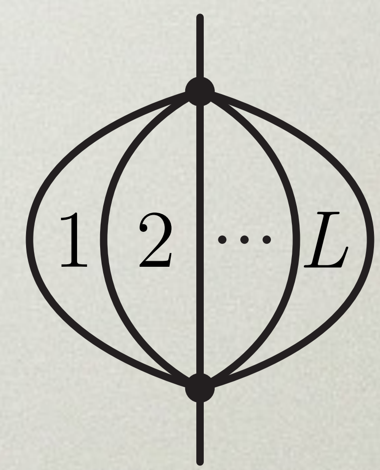
CY_3 (?)



CY_4 (¿?)



CY_{L-1}(?)



CY_{L-1}

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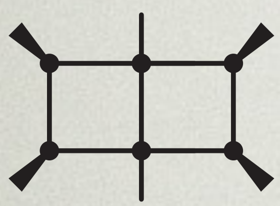
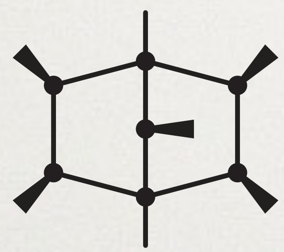
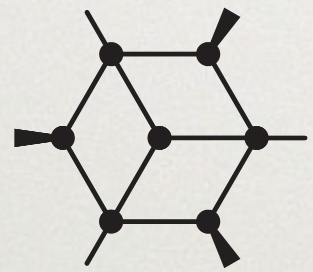


CY_1

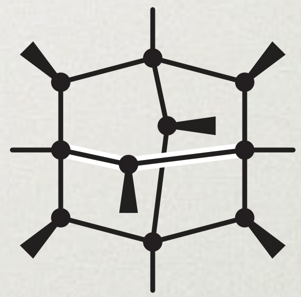


CY_2 (¿?)

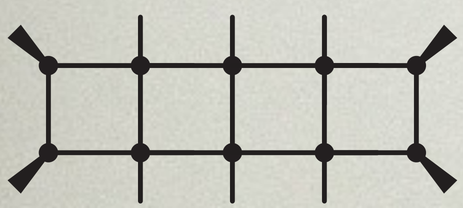
CY_2



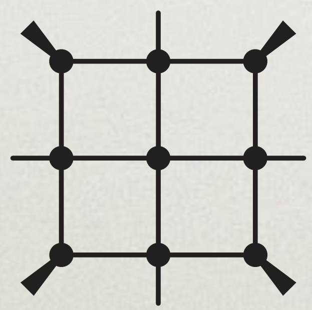
CY_3



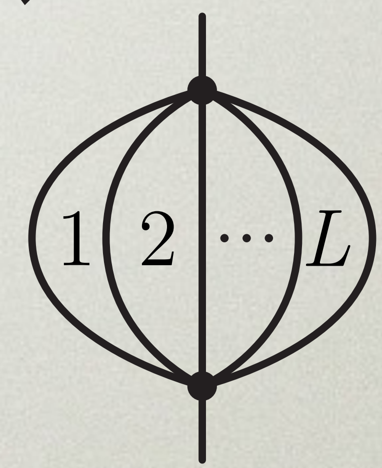
CY_4 (¿?)



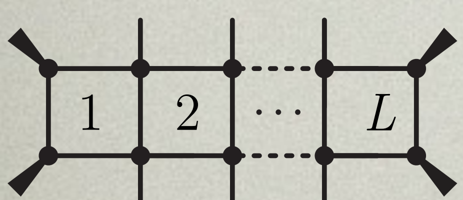
CY_3 (?)



CY_4 (¿?)



CY_{L-1}



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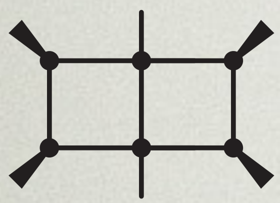
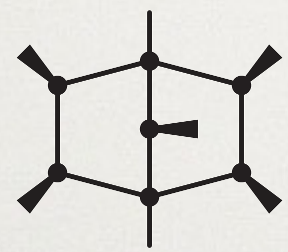
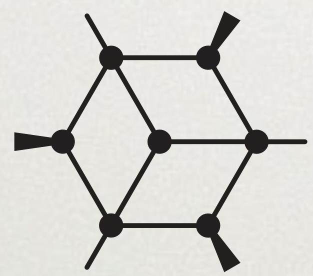


CY_1

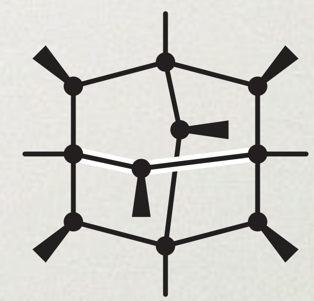


CY_2 (¿?)

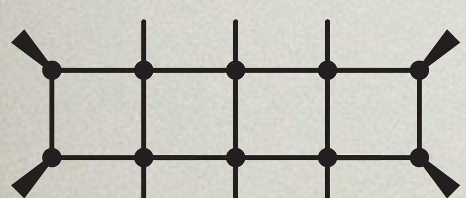
CY_2



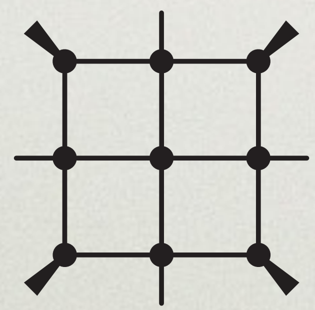
CY_3



CY_4 (¿?)



CY_3 (?)



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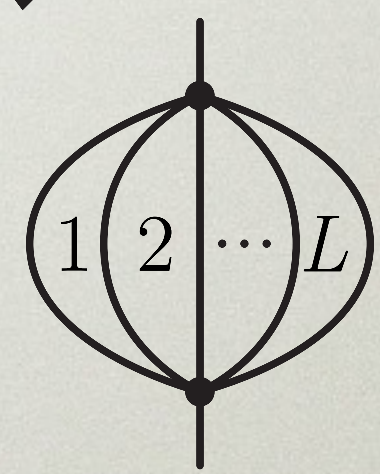
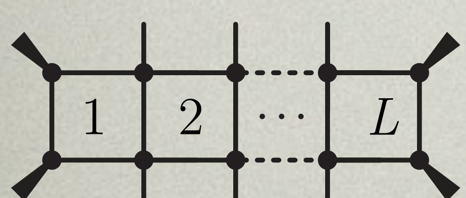


CY_{L-1}



CY_{L-1}(?)

[Bloch, Kerr, Vanhove; Broadhurst;…]

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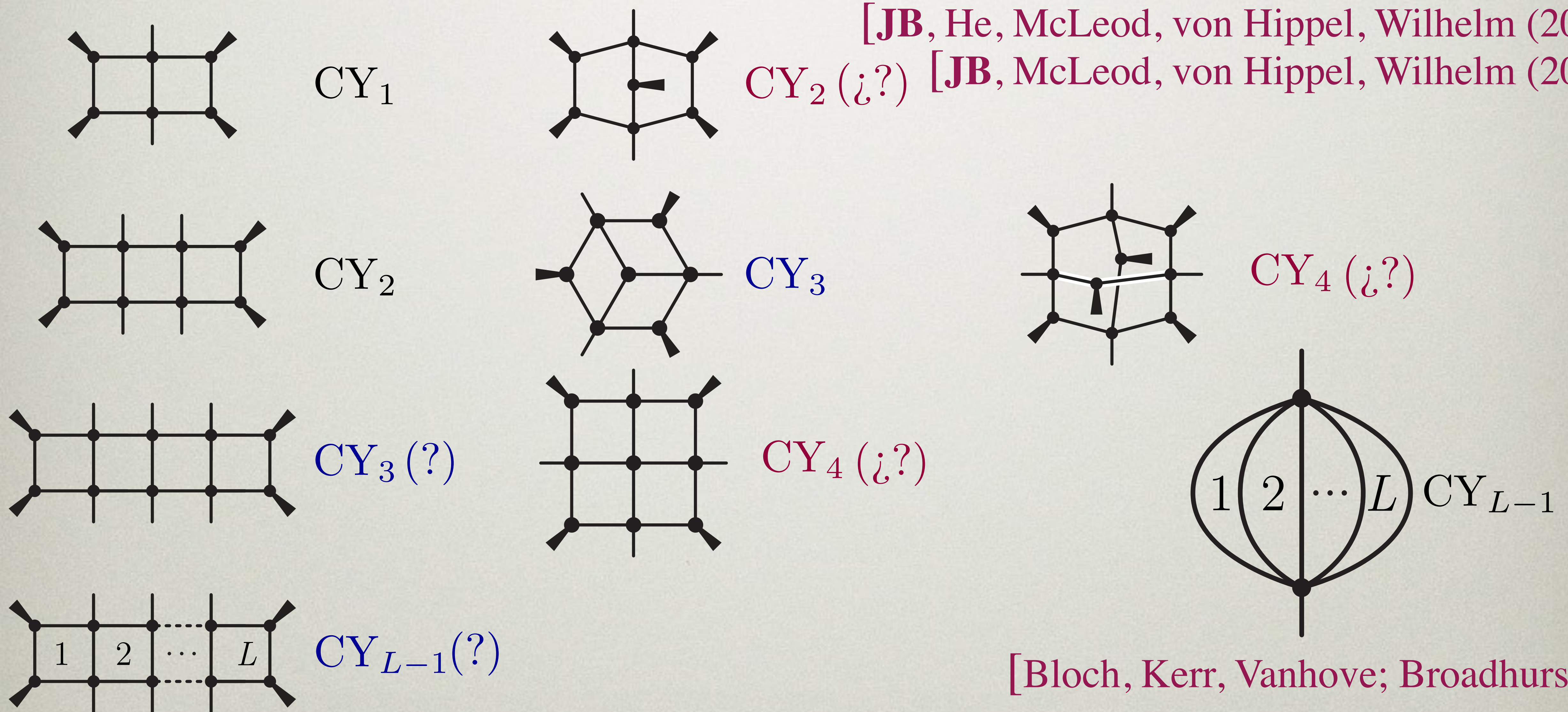

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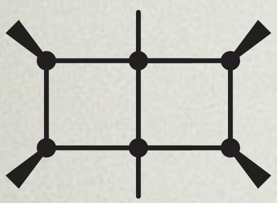
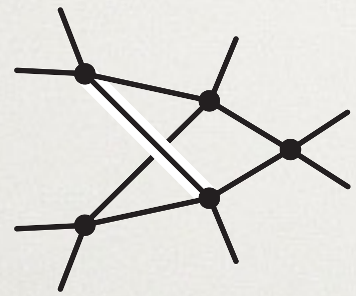
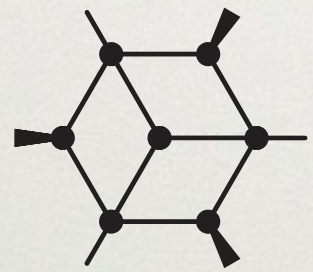


CY_1

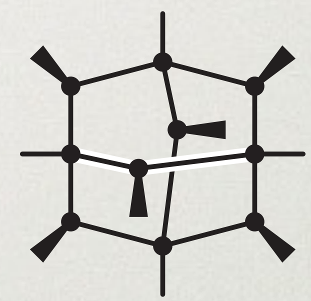


CY_2

CY_2



CY_3



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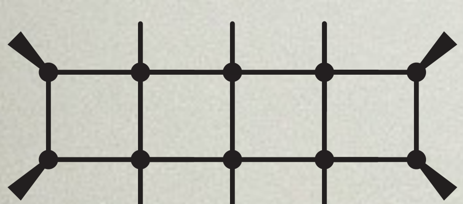
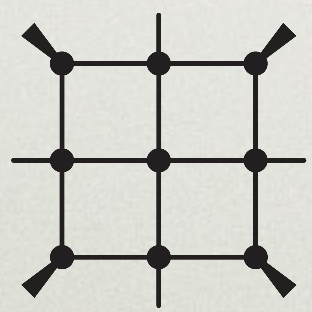


CY_3 (?)



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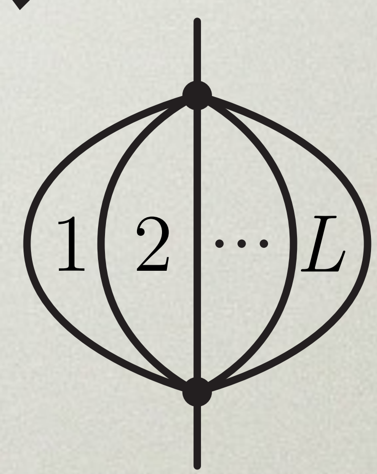
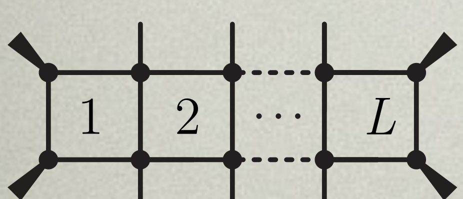


CY_{L-1}



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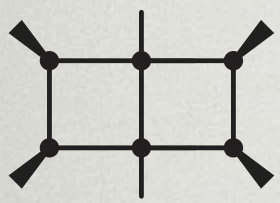
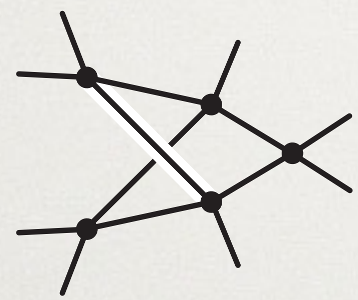
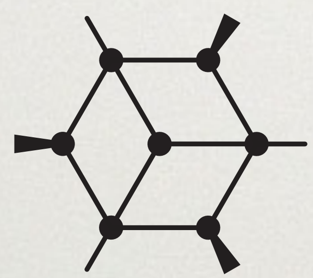
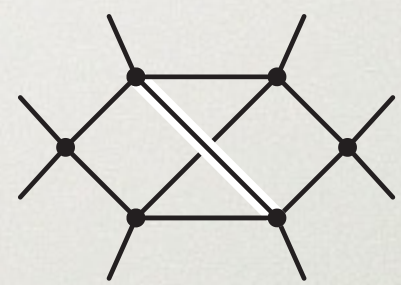
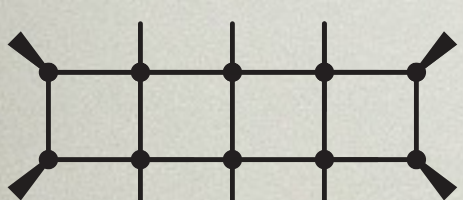

CY_1


CY_2


CY_2


CY_3


CY_4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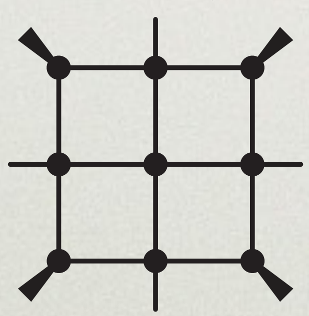
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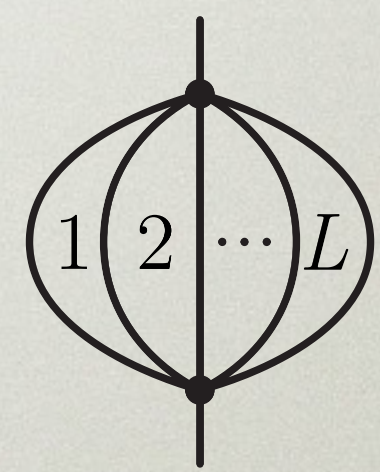
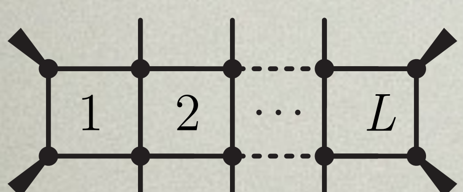


CY_{L-1}



CY_{L-1}(?)

[Bloch, Kerr, Vanhove; Broadhurst;…]

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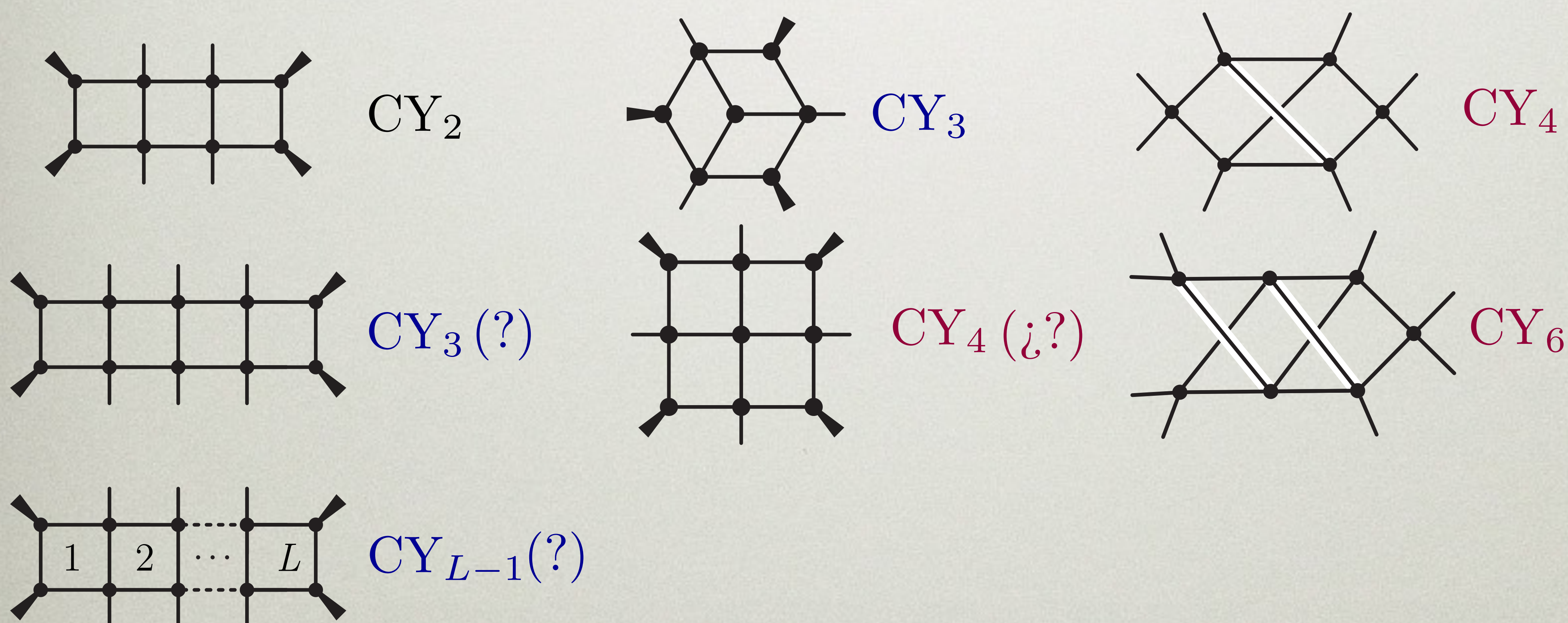

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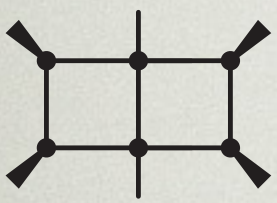
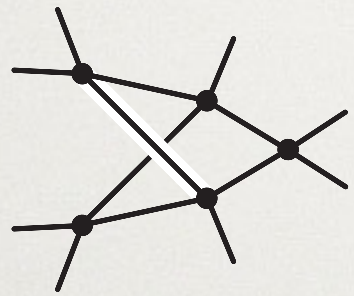
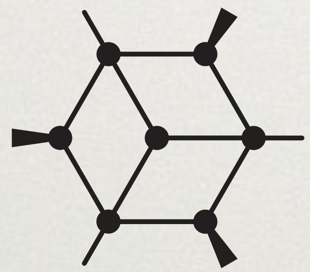


CY_1

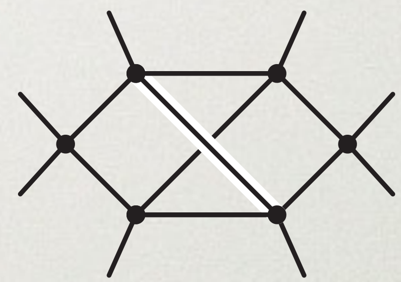


CY_2

CY_2



CY_3



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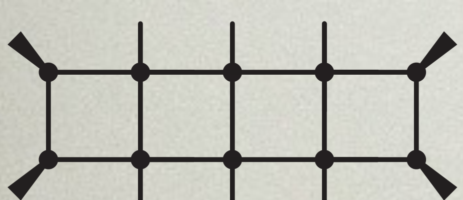

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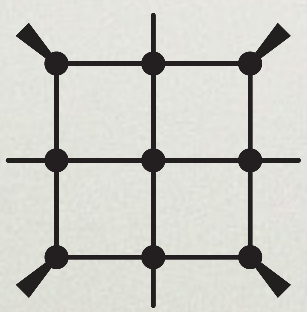
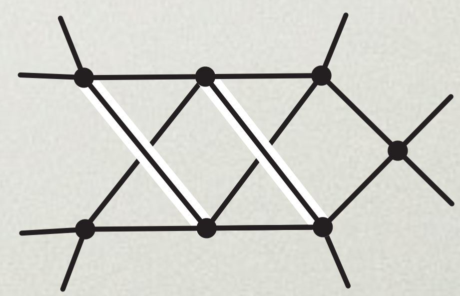
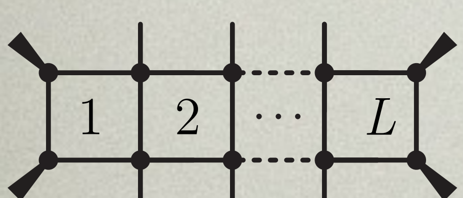


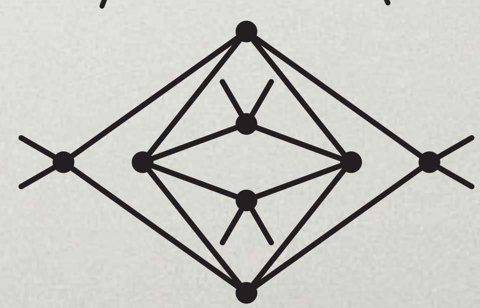
CY_4 (¿?)



CY_6



CY_{L-1}(?)



CY_8

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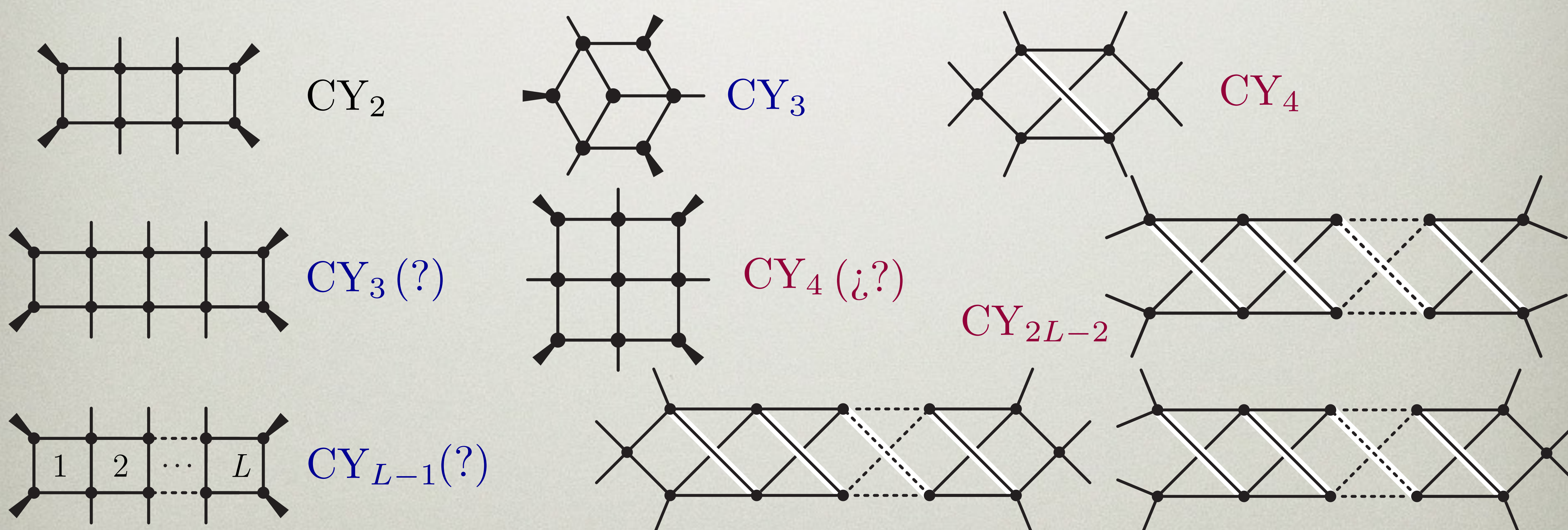

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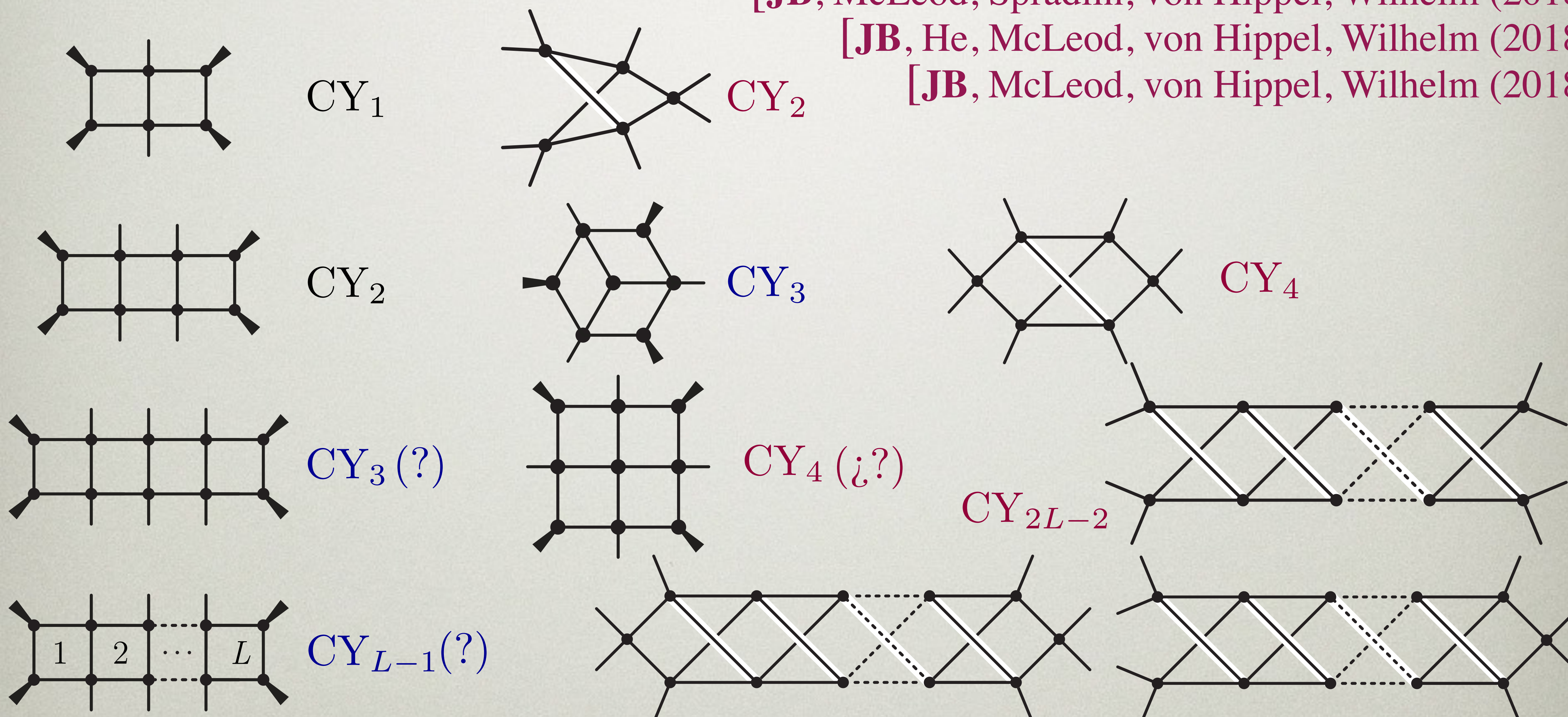

- The good news is that the relevant geometries are extremely special (and small in number!)

Questions?