# Elliptic Polylogarithms

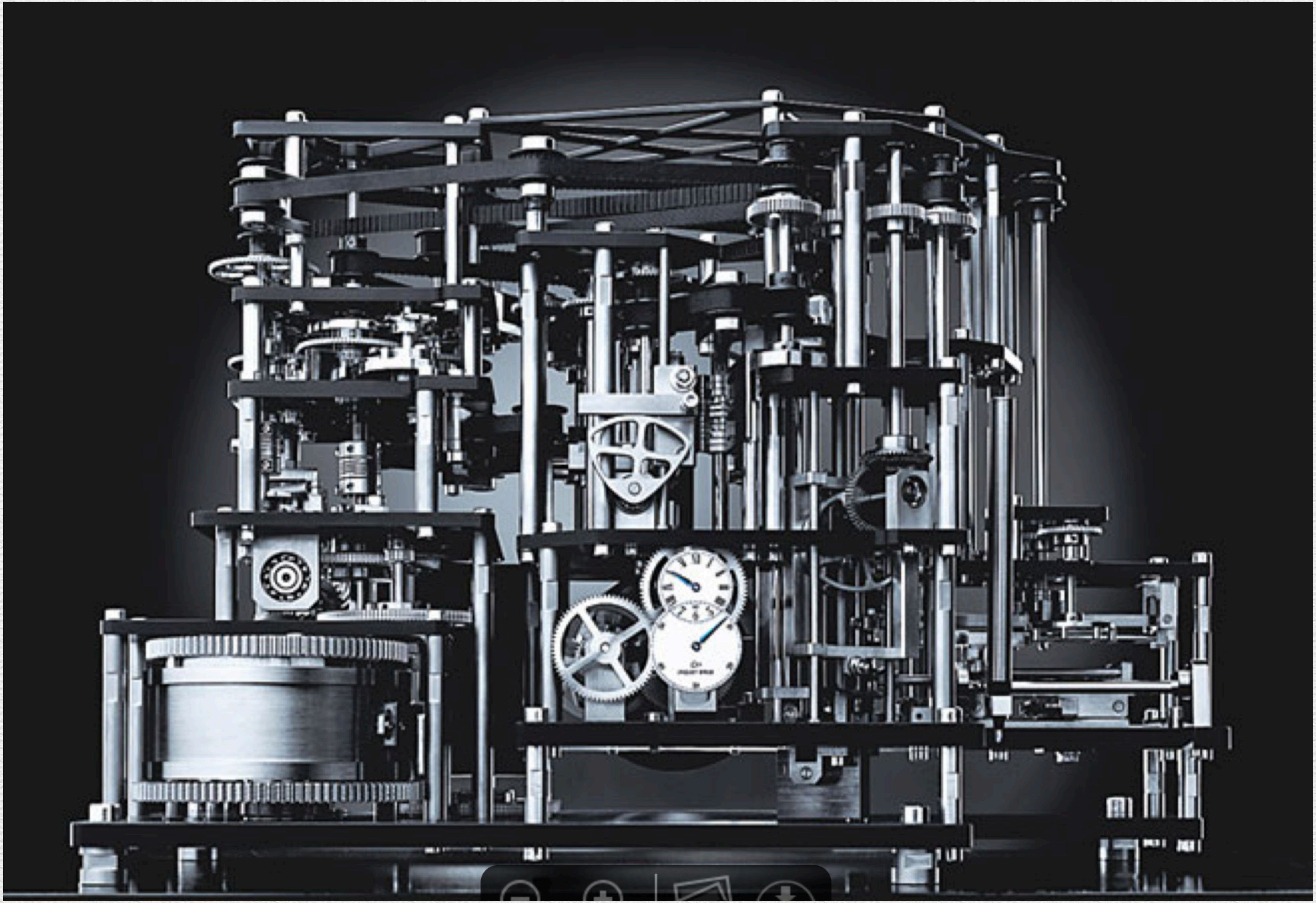
Brenda Penante

In collaboration with
J. Broedel, C. Duhr, F. Dulat, L. Tancredi

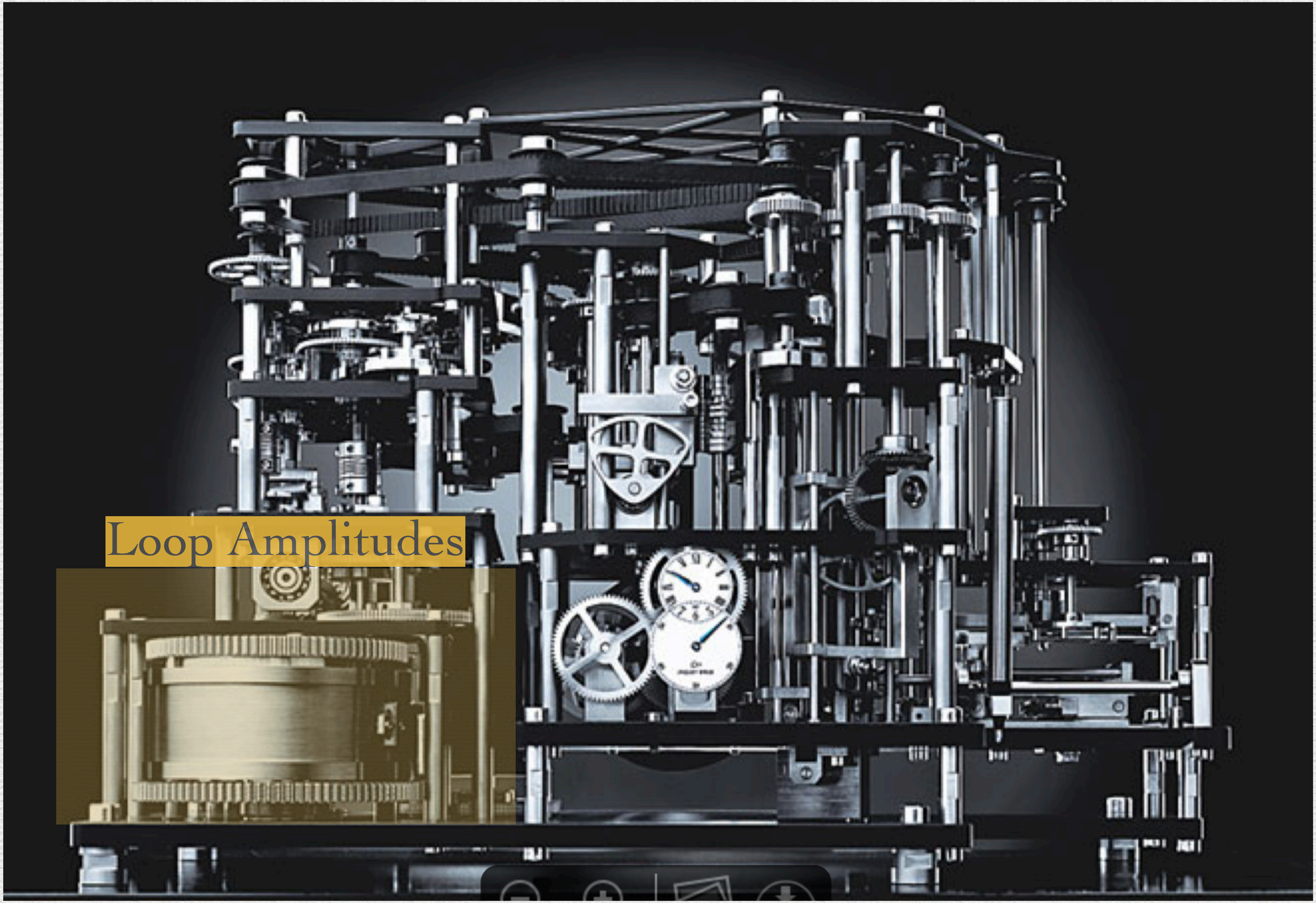
Amplitudes in the LHC era
Galileu Galilei Institute, Florence

Oct 30th, 2018

# Bernhard's Cross Section Machine

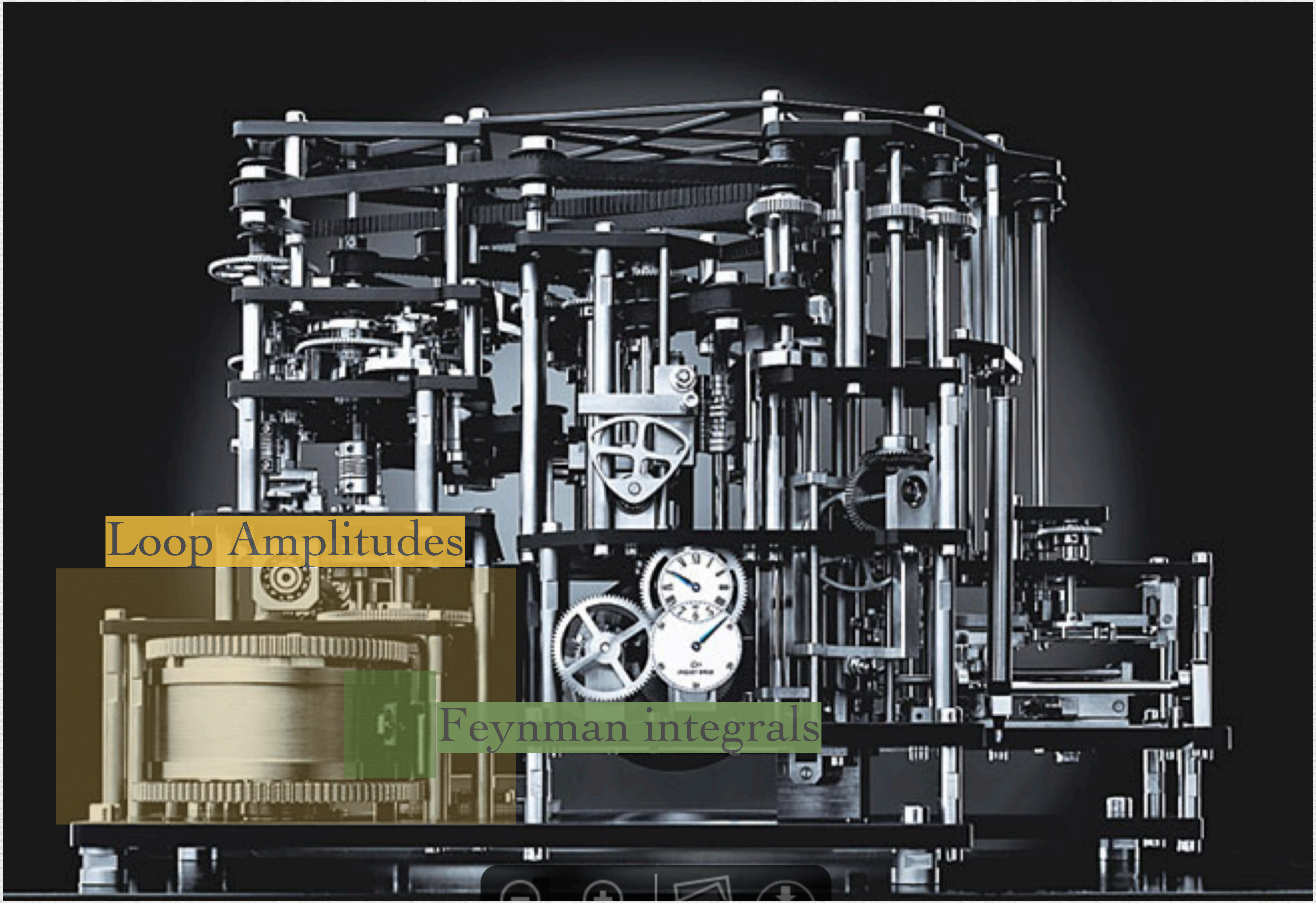


# Bernhard's Cross Section Machine



Loop Amplitudes

# Bernhard's Cross Section Machine



# Intro

- Feynman integrals are crucial ingredients of scattering amplitudes, which in turn enter cross sections
- They evaluate to "special functions" which contain the physics in their analytic structure
- Most well studied case: Multiple Polylogarithms (MPLs)

(all 1-loop examples and most 2-loop examples without internal masses)

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \qquad a_i \in \mathbb{C}$$

$$\mathrm{Li}_n(z) = G(0, \ldots, 0, 1; z) \qquad G(\underbrace{0, \ldots, 0}_{n \text{ times}}; z) = \frac{1}{n!} \log^n z$$

# Lots of nice properties:

Shuffle algebra: $G(a_1, \ldots, a_k; z)\, G(a_{k+1}, \ldots, a_{k+l}; z) = \sum_{\sigma \in \Sigma(k,l)} G(a_{\sigma(1)}, \ldots, a_{\sigma(k+l)}; z)$

Total differential: $dG(a_1, \ldots, a_n; z) \; = \sum_{i=1}^{n} G(a_1, \ldots, \hat{a}_i, \ldots, a_n; z)\, d\log \frac{a_{i-1} - a_i}{a_{i+1} - a_i}$

- **MPLs:** Weight = number of integrations

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t) \qquad a_i \in \mathbb{C}$$

$$G(\underbrace{0,\ldots,0}_{n \text{ times}};z) = \frac{1}{n!} \log^n z \qquad G(;z) \equiv 1$$

Weight $1$ $\quad i\pi = \log(-1)$

Weight $n$ $\quad G(a_1,\ldots,a_n;z)$

$$\zeta_n = -G(\vec{0}_{n-1},1;1)$$

# MPLs are *pure*

# What do you mean “Pure”?

- Definition based on total differential — Henn ’13 —

# of integrations

*A pure function of weight n is a function whose total derivative can be expressed in terms of pure functions of weight n-1 (times algebraic one-forms)*

algebraic

$$dG(a_1,\ldots,a_n;z) = \sum_{i=1}^{n} G(a_1,\ldots,\hat{a}_i,\ldots,a_n;z)\, d\log\frac{a_{i-1}-a_i}{a_{i+1}-a_i}$$

weight $n$

weight $n-1$

# What do you mean "Pure"?

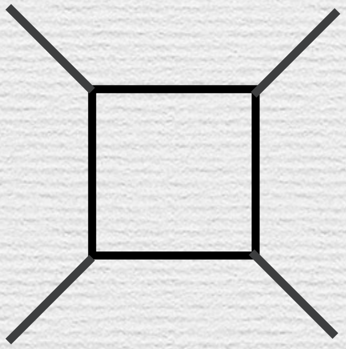
- Definition based on residues

— Arkani-Hamed, Bourjaily
Cachazo, Trnka '12 —

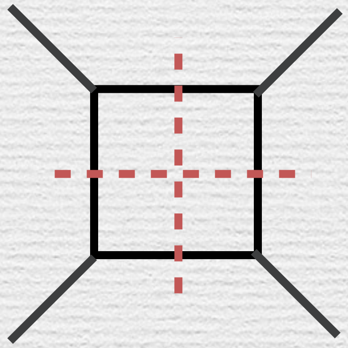
*An integral is pure if all of its non-vanishing residues are the same up to a sign*

"Integrals with unit leading singularity"

- Ex: 4-mass box



$$= \frac{2}{st}\left[\frac{1}{\epsilon^2} - \frac{\log(st)}{\epsilon} + \log(-s)\log(-t) - \frac{2\pi^2}{3}\right]$$



$$= \pm\frac{1}{st}$$

(weight of $\epsilon$ is -1: $q^\epsilon = e^{\epsilon\log(q)}$ )

# What do you mean "Pure"?

- Definition based on residues

— Arkani-Hamed, Bourjaily
Cachazo, Trnka '12 —

*An integral is pure if all of its non-vanishing residues are the same up to a sign*

"Integrals with unit leading singularity"

- Pure Feynman Integrals, when properly normalised
  - Are expressible in terms of pure functions
  - Satisfy a differential equation system in canonical form

# Pure integrals evaluate to pure functions

Differential equations in canonical form — Henn '13 —

Matrix of "dlog" forms

$$dF = \epsilon \, dA \, F$$

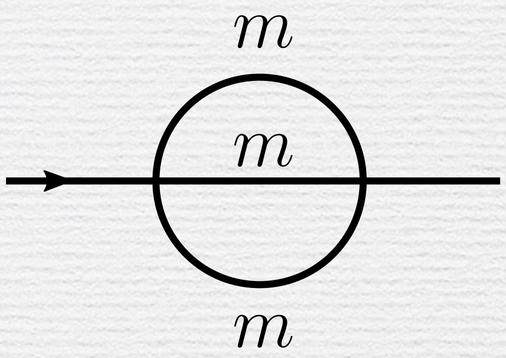
Vector of master integrals

For MPLs:
Natural solution in terms of
pure functions G as an expansion in $\epsilon$

$$F = P \exp\left[\epsilon \int^{x} dA\right]$$

## What to do when the integral cannot be evaluated in terms of MPLs?

## Ex: 2-loop massive sunrise in d=2



Two of the master integrals satisfy a coupled system

First master integral satisfies a 2nd order DE:

$$D\left(\frac{d^2}{da^2}, \frac{d}{da}\right) S_{111} = R(a) \qquad a = \frac{p^2}{m^2}$$

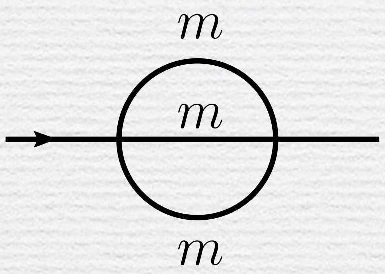
Homogeneous solution:

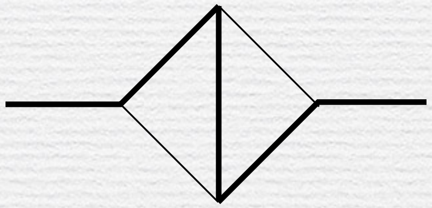
$$\mathrm{K}(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}$$

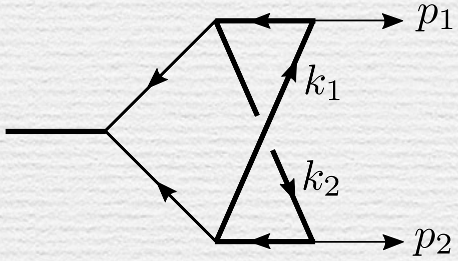
(complete elliptic integral
of the 1st kind)

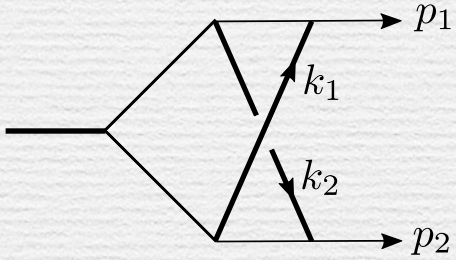
→ Sqrt of quartic polynomial

By now we know lots of examples that don't fit into the MPL framework: [see Jake's talk]









**Goal:** Develop a class of functions which is applicable in general for FI of the elliptic kind (next-to-simplest):

OUTLINE

- Elliptic generalisations of MPLs to functions on the elliptic curve w/ log singularities
- Well defined notion of weight / purity

# Purity: Why bother?

- Meaning not entirely understood even in the MPL case
- Nevertheless, shows underlying structure
  Eg. N=4 SYM:

anomalous dimensions, amplitudes,
certain form factors, etc

L-loops ↔ Weight 2L functions

"Uniform transcendentality"

- Organisational principle:
  functional identities among functions of fixed weight
- "Maximal transcendentality principle"

— Kotikov, Lipatov, Onishchenko, Velizhanin '04 —

# Purity: Why bother?

Total differential: $$dG(a_1,\ldots,a_n;z) = \sum_{i=1}^{n} G(a_1,\ldots,\hat{a}_i,\ldots,a_n;z)\, d\log\frac{a_{i-1}-a_i}{a_{i+1}-a_i}$$

Symbol: $$\mathcal{S}(G(a_1,\ldots,a_n;z)) = \sum_{i=1}^{n} \mathcal{S}(G(a_1,\ldots,\hat{a}_i,\ldots,a_n;z)) \otimes \frac{a_{i-1}-a_i}{a_{i+1}-a_i}$$

— Goncharov, Spradlin, Vergu, Volovich '10 —

Length $n$ → $n$ -fold tensor product

- Taming analytical expressions, functional identities
- Symbol bootstrap with MPL ansatz in N=4 SYM

— Caron-Huot, Dixon, Drummond, Duhr, Harrington, Henn, McLeod, Papathanaseou, Pennington, Spradlin, von Hippel —

# Purity: Why bother?

Differential equations in canonical form

Matrix of "dlog" forms

$$dF = \epsilon\, dA\, F$$

Vector of master integrals

For MPLs: natural solution in terms of pure functions G

To-do: develop a general framework also for elliptic integrals

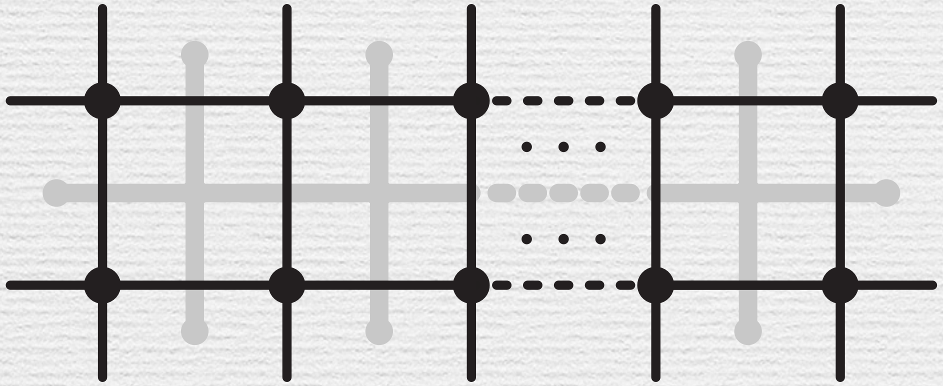
— Adams, Weinzierl / Adams, Chaubey, Weinzierl '18 —

# The real world: N=4 Super Yang-Mills

Conjecturally of uniform (maximal) weight

Elliptic integrals (and beyond) are known to appear:

— Caron-Huot, Larsen '12 / Nandan, Paulos, Spradlin, Volovich '14 /
Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm '17 —



The elliptic double box,
and more generally traintracks

— Bourjaily, He, McLeod, von Hippel, Wilhelm '18 —

We'd like to give an elliptic meaning to these statements!

# Define *pure* elliptic MPLs (eMPLs)

- We seek to generalise the following to the elliptic case:

*A function is called pure if it is unipotent and its total differential involves only pure functions and one-forms with at most logarithmic singularities.*

(Unipotent: total diff has no homogeneous term)

Log singularities

$$dG(a_1,\ldots,a_n;z) = \sum_{i=1}^{n} G(a_1,\ldots,\hat{a}_i,\ldots,a_n;z)\, d\log\frac{a_{i-1}-a_i}{a_{i+1}-a_i}$$

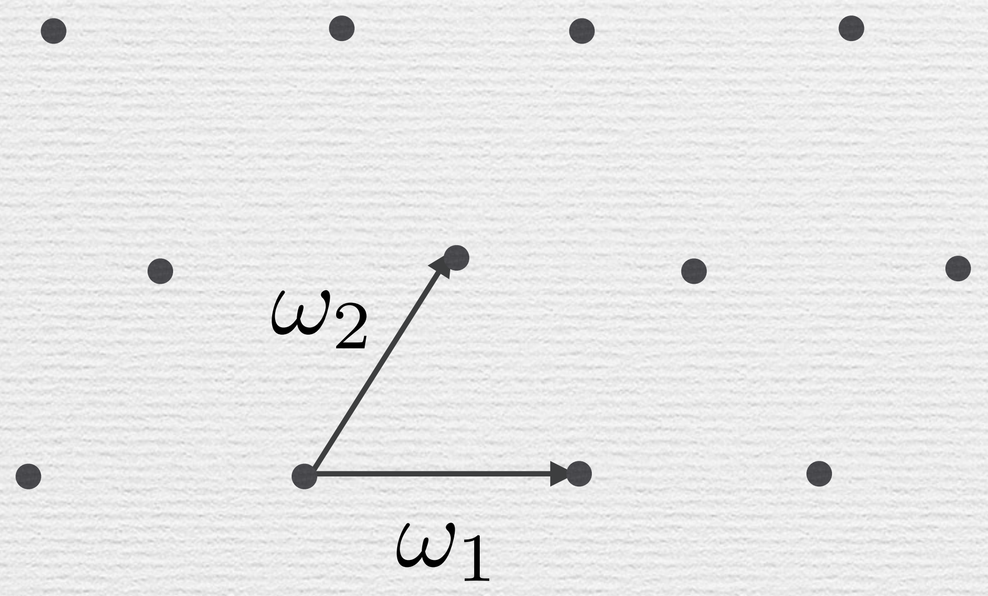
Pure Unipotent

# Elliptic Polylogarithms on the torus

— Brown, Levin '11, Broedel, Mafra, Matthes, Schlotterer '14 —

torus: $\mathbb{C}/\Lambda$

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$



Modular group: $\mathrm{SL}(2,\mathbb{Z})$

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$$

$$\tau = \frac{\omega_2}{\omega_1} \to \frac{a\tau + b}{c\tau + d}$$

# Elliptic Polylogarithms on the torus

— Brown, Levin '11, Broedel, Mafra, Matthes, Schlotterer '14 —

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z\right) = \int_0^z dz' g^{(n_1)}(z' - z_1) \tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z\right)$$

$n_i \in \mathbb{N}$

$z_i \in \mathbb{C} \cup \{\infty\}$

Kernels defined through generating function:

$$F(z, \alpha, \tau) = \frac{1}{\alpha} \sum_{n \geq 0} g^{(n)}(z, \tau) \alpha^n = \frac{\theta_1'(0, \tau)\theta_1(z + \alpha, \tau)}{\theta_1(z, \tau)\theta_1(\alpha, \tau)}$$

Odd Jacobi theta function

Kernels have at most simple poles at lattice points

$\tilde{\Gamma}$ form a basis for all eMPLs

# Elliptic Polylogarithms on the torus

— Brown, Levin ’11, Broedel, Mafra, Matthes, Schlotterer ’14 —

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z\right) = \int_0^z dz' g^{(n_1)}(z'-z_1)\tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z\right)$$

$n_i \in \mathbb{N}$

$z_i \in \mathbb{C} \cup \{\infty\}$

Kernels defined through generating function:

$$F(z,\alpha,\tau) = \frac{1}{\alpha}\sum_{n\geq 0} g^{(n)}(z,\tau)\alpha^n = \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)}$$

[

$$g^{(n)}(z,\tau)$$

holomorphic,
~~double periodic~~

$$\frac{1}{\alpha}\sum_{n\geq 0} f^{(n)}(z,\tau)\alpha^n = \exp\left[2\pi i\alpha\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right] F(z,\alpha,\tau)$$

~~holomorphic~~,
double periodic

]

# Like MPLs, $\tilde{\Gamma}$ satisfy nice properties

Total differential without homogeneous term (= unipotent)

— Broedel, Duhr, Dulat, Penante, Tancredi, 2018 —

$$d\widetilde{\Gamma}\left(A_1 \cdots A_k; z, \tau\right) = \sum_{p=1}^{k-1} (-1)^{n_{p+1}}\, \widetilde{\Gamma}\left(A_1 \cdots A_{p-1} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} A_{p+2} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p + n_{p+1})}$$

$$+ \sum_{p=1}^{k} \sum_{r=0}^{n_p+1} \left[ \binom{n_{p-1} + r - 1}{n_{p-1} - 1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1}^{[r]}\, \hat{A}_p\, A_{p+1} \cdots A_k; z, \tau\right) \omega_{p,p-1}^{(n_p - r)}\right.$$

$$\left. - \binom{n_{p+1} + r - 1}{n_{p+1} - 1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1}\, \hat{A}_p\, A_{p+1}^{[r]} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p - r)} \right]$$

one-forms w/
log singularities

$$A_i^{[r]} \equiv \left(\begin{smallmatrix} n_i + r \\ z_i \end{smallmatrix}\right) \qquad A_i^{[0]} \equiv A_i$$

$$\omega_{ij}^{(n)} = (dz_j - dz_i)\, g^{(n)}(z_j - z_i, \tau) + \frac{n\, d\tau}{2\pi i}\, g^{(n+1)}(z_j - z_i, \tau)$$

**Important:** $g^{(n)}(z, \tau)$ have at most simple poles for $z = m + n\tau,\ m, n \in \mathbb{Z}$

# Like MPLs, $\tilde{\Gamma}$ satisfy nice properties

Total differential without homogeneous term (= unipotent)

— Broedel, Duhr, Dulat, Penante, Tancredi, 2018—

$$d\widetilde{\Gamma}\left(A_1\cdots A_k;z,\tau\right)=\sum_{p=1}^{k-1}(-1)^{n_{p+1}}\,\widetilde{\Gamma}\left(A_1\cdots A_{p-1}\,\begin{smallmatrix}0\\0\end{smallmatrix}\,A_{p+2}\cdots A_k;z,\tau\right)\,\omega_{p,p+1}^{(n_p+n_{p+1})}$$

$$+\sum_{p=1}^{k}\sum_{r=0}^{n_p+1}\Bigg[\binom{n_{p-1}+r-1}{n_{p-1}-1}\widetilde{\Gamma}\left(A_1\cdots A_{p-1}^{[r]}\,\hat{A}_p\,A_{p+1}\cdots A_k;z,\tau\right)\,\omega_{p,p-1}^{(n_p-r)}$$

$$-\binom{n_{p+1}+r-1}{n_{p+1}-1}\widetilde{\Gamma}\left(A_1\cdots A_{p-1}\,\hat{A}_p\,A_{p+1}^{[r]}\cdots A_k;z,\tau\right)\,\omega_{p,p+1}^{(n_p-r)}\Bigg]$$

one-forms w/
log singularities

*A function is called pure if it is unipotent and it has at most logarithmic singularities.*

$\tilde{\Gamma}$ are pure!

So, we can use as guiding principle

*An elliptic Feynman integral is pure if it is pure when expressed in terms of* $\tilde{\Gamma}$

$$=$$

*Linear combination of* $\tilde{\Gamma}$ *with coefficients being rational numbers*

Why bother defining another version of eMPLs?

# Elliptic curves

$$y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4) \equiv P_4(x)$$

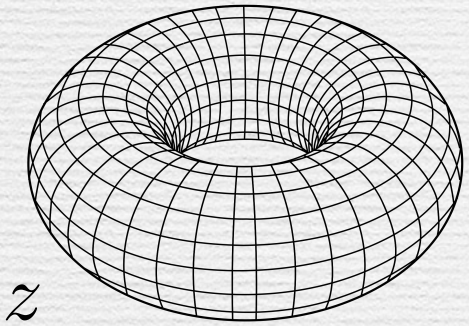
Vector of branch points of $y$ : $\vec{a} = (a_1, a_2, a_3, a_4)$

Periods:

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2\,\mathrm{K}(\lambda) \qquad \omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2\,i\,\mathrm{K}(1-\lambda)$$

$$\lambda = \frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)} \qquad c_4 = \frac{1}{2}\sqrt{(a_1 - a_3)(a_2 - a_4)}$$

# Elliptic Curves and Tori



vs. $$y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4) \equiv P_4(x)$$

$$\tau = \frac{\omega_2}{\omega_1}$$

Kappa function $\kappa(.,\vec{a}) : \mathbb{C}/\Lambda_\tau \to \mathbb{C}$

$$(c_4\kappa'(z))^2 = (\kappa(z)-a_1)(\kappa(z)-a_2)(\kappa(z)-a_3)(\kappa(z)-a_4)$$

$$y^2 = P_4(x)$$

$$(x,y) = (\kappa(z), c_4\kappa'(z))$$

Abel's map

$$(x,y) \mapsto z \equiv \frac{c_4}{\omega_1}\int_{a_1}^{x}\frac{dx}{y} \mod \Lambda$$

# Desired properties for eMPLs:

## 1. Pure eMPLs on the elliptic curve

Feynman integrals are more naturally studied on the elliptic curve (simpler functions of kinematic dof)

## 2. Definite Parity

Integrands are rational functions, result should not depend on choice of branch for the square root $y^2 = P_4(x)$

$$(x,y) \to (x,-y) \qquad \longleftrightarrow \qquad z \to -z$$

Basis of $\tilde{\Gamma}$ does not have definite parity

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z\right) = \int_0^z dz' g^{(n_1)}(z'-z_1)\tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z\right) \qquad g^{(n)}(-z,\tau) = (-1)^n g^{(n)}(z,\tau)$$

# To summarise:

We define a basis of eMPLs on the elliptic curve such that

1. They form a basis for all eMPLs
2. They are pure
3. They have definite parity
4. They manifestly contain ordinary MPLs

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \ldots & n_k \\ c_2 & \ldots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$n_i \in \mathbb{Z}$ is a label

$c_i \in \mathbb{C}$ indicate punctures (for $n_i \neq 0$)

Infinitely many kernels, $\Psi_n$ but only $|n| \leq 2$
typically appear in explicit problems

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\, \Psi_{\pm n}(c, x, \vec{a}) &= dz_x\, \Big[ g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) \\ &- \delta_{\pm n,1} \left( g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big] \end{aligned}$$

Recall: $g^{(i)}(z, \tau)$ are the kernels of the eMPLs $\tilde{\Gamma}$

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\, \Psi_{\pm n}(c, x, \vec{a}) &= dz_x \Big[ g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) \\ &- \delta_{\pm n, 1} \left( g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big] \end{aligned}$$

1. They form a basis for all eMPLs ✓

(one-to-one correspondence with basis of $\tilde{\Gamma}$ )

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\, \Psi_{\pm n}(c, x, \vec{a}) \; &= dz_x \, \Big[ g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) \\ &- \delta_{\pm n,1} \, \Big( g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \Big) \Big] \end{aligned}$$

2. They are pure ✓

(Linear combination of $\tilde{\Gamma}$ with numeric coefficients)

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\, \Psi_{\boxed{\pm n}}(c, x, \vec{a}) &= dz_x \Big[ g^{(n)}(z_x - z_c, \tau) \boxed{\pm} g^{(n)}(z_x + z_c, \tau) \\ &\quad - \delta_{\pm n, 1} \left( g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big] \end{aligned}$$

3. They have definite parity ✓

(Recall $g^{(n)}(-z, \tau) = (-1)^n g^{(n)}(z, \tau)$)

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\, \Psi_{\pm n}(c, x, \vec{a}) \;&= dz_x \Big[ g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) \\ &\quad \boxed{- \delta_{\pm n,1} \left( g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big]} \end{aligned}$$

$$\Big\downarrow$$

$$dx\, \Psi_1(c, x, \vec{a}) = \frac{dx}{x - c}, \qquad c \neq \infty$$

4. They manifestly contain ordinary MPLs

# Making it explicit

$$
\begin{aligned}
\Psi_0(0,x,\vec{a}) &= \frac{c_4}{\omega_1\, y} \\
\Psi_1(c,x,\vec{a}) &= \frac{1}{x-c}\,, \\
\Psi_{-1}(c,x,\vec{a}) &= \frac{y_c}{y(x-c)} + Z_4(c,\vec{a})\,\frac{c_4}{y}\,, \\
\Psi_1(\infty,x,\vec{a}) &= -Z_4(x,\vec{a})\,\frac{c_4}{y}\,, \\
\Psi_{-1}(\infty,x,\vec{a}) &= \frac{x}{y} - \frac{1}{y}\left[a_1 + 2c_4\, G_*(\vec{a})\right]
\end{aligned}
$$

$$y_c = \sqrt{P_4(c)}$$

Nothing comes without a price — explicit dependence on

$$G_*(\vec{a}) \equiv \frac{1}{\omega_1}\, g^{(1)}(z_*,\tau)$$

and

$$Z_4(c,\vec{a})$$

Image of $-\infty$ on the torus

Transcendental function
with pole at $c \to \infty$

In general transcendental, but simplify in specific applications

# Length and weight

For MPLs, notion of weight and length are straightforward

Length = weight = # of integrations (except for $i\pi$)

For eMPLs, they are not the same!

# Semi-simple vs. unipotent

Unipotent: total differential has no homogeneous term

$\omega_i$ : periods

$\eta_i$ : quasi-periods

$\omega_1\eta_2 - \omega_2\eta_1 = -i\pi$

(Legendre)

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} \omega_1 & 0 \\ \eta_1 & -i\pi/\omega_1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

$$\tau = \frac{\omega_2}{\omega_1}$$

semi-simple

unipotent

$$\omega_1, \eta_1, i\pi/\omega_1$$

## Roughly speaking:

Semi-simple periods have length 0

Unipotent periods have length = # of integrations

# Weight:

Empirically, by requiring relations between uniform weight functions, we postulate:

$$\begin{aligned} \omega_1 &= 2\mathrm{K}(\lambda) \\ \omega_2 &= 2i\,\mathrm{K}(1-\lambda) \end{aligned} \quad \longrightarrow \quad 1$$

Recall: $\lim\limits_{x\to 0} \mathrm{K}(x) = \dfrac{\pi}{2}$

$$\tau = \frac{\omega_2}{\omega_1} \quad \longrightarrow \quad 0$$

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z, \tau\right) \quad \longrightarrow \quad \sum_i n_i$$

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) \quad \longrightarrow \quad \sum_i |n_i|$$

We'll see in applications that using these definitions, results are of uniform weight

# Other properties that work the same way as for MPLs:

- Shuffle

$$\mathcal{E}_4(A_1\cdots A_k;x,\vec{a})\,\mathcal{E}_4(A_{k+1}\cdots A_{k+l};x,\vec{a})=\sum_{\sigma\in\Sigma(k,l)}\mathcal{E}_4(A_{\sigma(1)}\cdots A_{\sigma(k+l)};x,\vec{a})$$

- Unipotent — Symbol

- Shuffle preserving regularisation of $\mathcal{E}_4(\cdots\,{}^{\pm1}_{0},x,\vec{a})$

Analogue of $G(\underbrace{0,\ldots,0}_{n\text{ times}};z)=\frac{1}{n!}\log^n z$

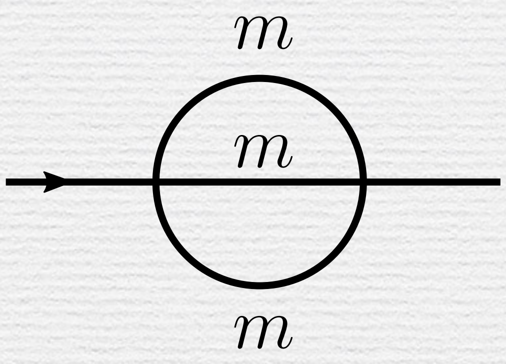
# THE COMPLETE LIST

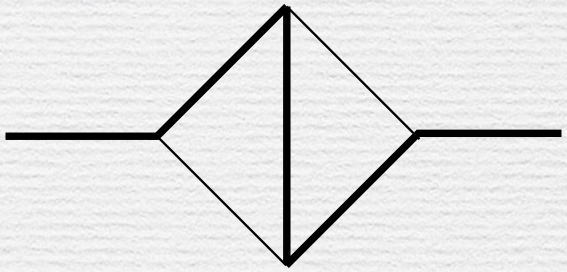
| Name | Unipotent | Length | Weight |
|---|---|---|---|
| Rational Functions | No | 0 | 0 |
| Algebraic Functions | No | 0 | 0 |
| $i\pi$ | No | 0 | 1 |
| $\zeta_{2n}$ | No | 0 | $2n$ |
| $\zeta_{2n+1}$ | Yes | 0 | $2n+1$ |
| $\log x$ | Yes | 1 | 1 |
| $\mathrm{Li}_n(x)$ | Yes | $n$ | $n$ |
| $G(c_1, \ldots, c_k; x)$ | Yes | $k$ | $k$ |
| $\omega_1$ | No | 0 | 1 |
| $\eta_1$ | No | 0 | 1 |
| $\tau$ | Yes | 1 | 0 |
| $g^{(n)}(z, \tau)$ | No | 0 | $n$ |
| $h^{(n)}_{N,r,s}(\tau)$ | No | 0 | $n$ |
| $Z_4(c, \vec{a})$ | No | 0 | 0 |
| $G_*(\vec{a})$ | No | 0 | 0 |
| $\mathcal{E}_4(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix}; x, \vec{a})$ | Yes | $k$ | $\sum_i \lvert n_i \rvert$ |
| $\widetilde{\Gamma}(\begin{smallmatrix} n_1 & \ldots & n_k \\ z_1 & \ldots & z_k \end{smallmatrix}; z, \tau)$ | Yes | $k$ | $\sum_i n_i$ |
| $I\left(\begin{smallmatrix} n_1 & N_1 \\ r_1 & s_1 \end{smallmatrix} \middle\vert \ldots \middle\vert \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau\right)$ | Yes | $k$ | $\sum_i n_i$ |

# Applications

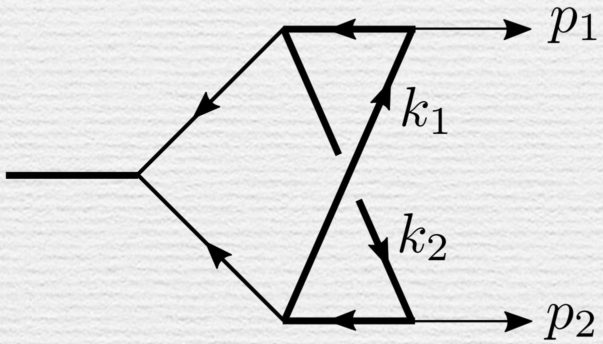
— Broedel, Duhr, Dulat, Penante, Tancredi (to appear) —

- 2-point functions

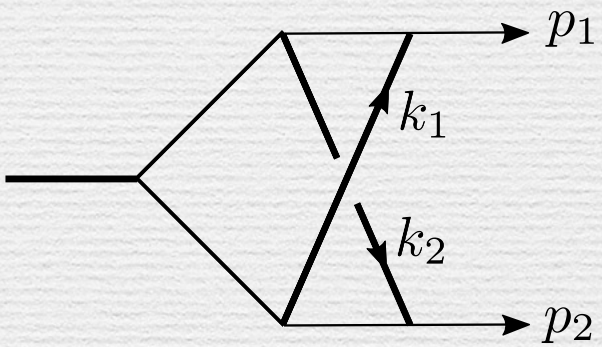




- 3-point functions

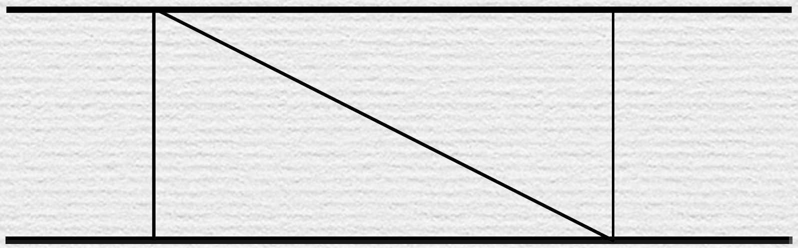


— Tancredi, von Manteuffel '17 —



— Aglietti, Bonciani '07 —

- 4-point functions

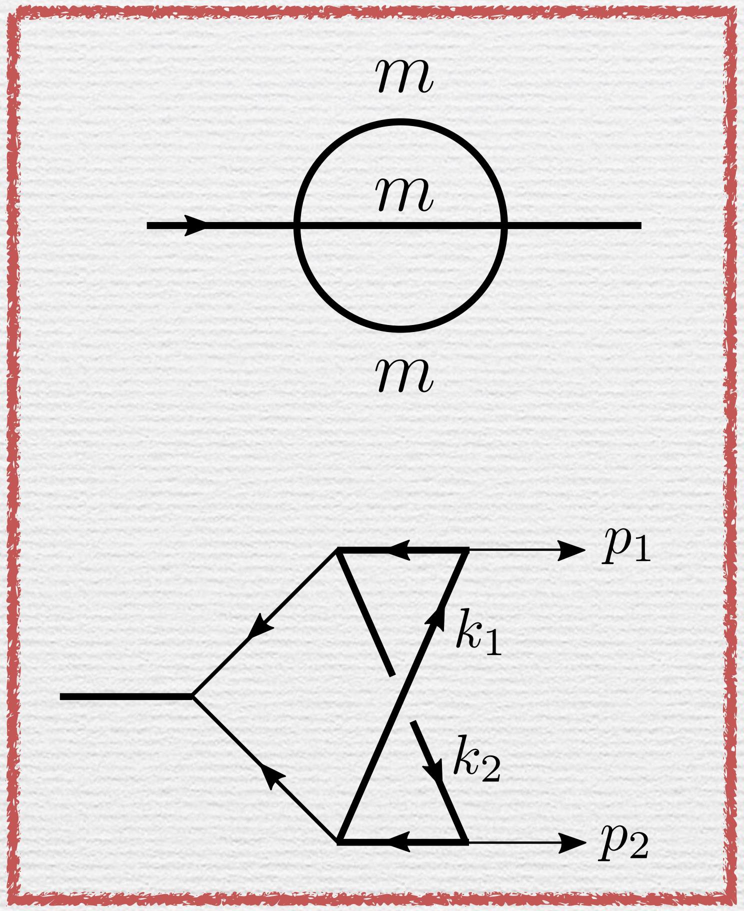


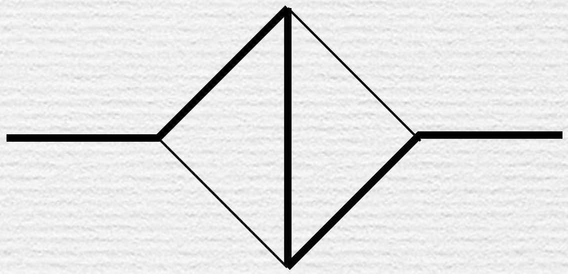
— Henn, Smirnov '13 —

# Applications

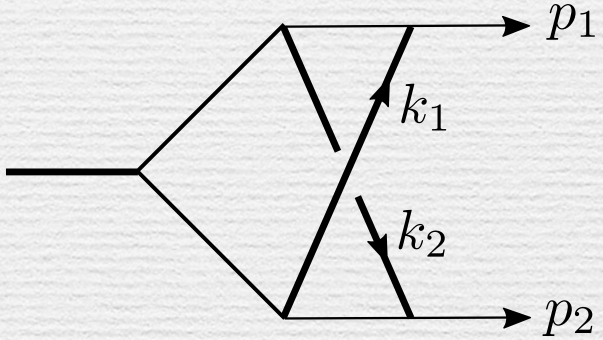
— Broedel, Duhr, Dulat, Penante, Tancredi (to appear) —

- 2-point functions



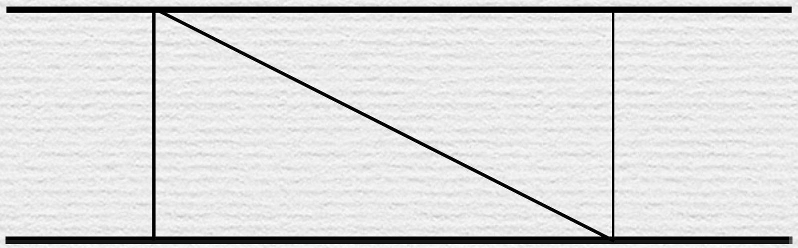


- 3-point functions



— Tancredi, von Manteuffel '17 — — Aglietti, Bonciani '07 —

- 4-point functions



— Henn, Smirnov '13 —

# Step by step

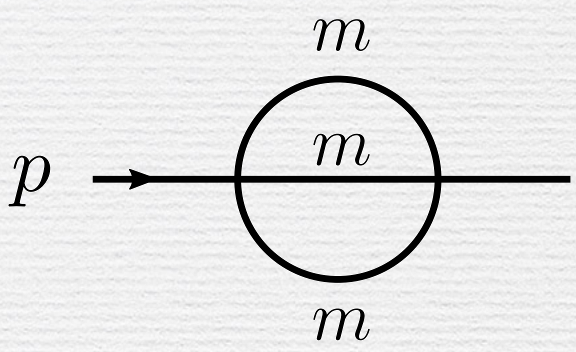
1. Start from Feynman parametric integral
2. Do as many integrals as possible in terms of MPLs G

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

3. Reach a representation of the type $I = \int_0^1 \frac{dx}{y} \times \text{(bunch of Gs)}$
4. Rewrite $\text{(bunch of Gs)}$ as $\Psi_n(\ldots,x,\vec{a})\mathcal{E}_4(\ldots;x,\vec{a})$
5. Integrate in terms of eMPLs

$$\mathcal{E}_4(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix};x,\vec{a}) = \int_0^x dt\, \Psi_{n_1}(c_1,t,\vec{a})\, \mathcal{E}_4(\begin{smallmatrix} n_2 & \ldots & n_k \\ c_2 & \ldots & c_k \end{smallmatrix};t,\vec{a})$$

# Ex 1: Sunrise first master



Semi-simple

Unipotent

We can write: $$S_1(p^2,m^2) = -\frac{\omega_1}{(p^2+m^2)\,c_4}\,T_1(p^2,m^2)$$

$$\mathrm{Cut}[S_1(p^2,m^2)_{|D=2}] = -\frac{\omega_1}{(p^2+m^2)\,c_4}$$

$$\boxed{S_1(p^2,m^2) = \mathrm{Cut}[S_1(p^2,m^2)_{|D=2}] \times T_1(p^2,m^2)}$$

Just like
non-elliptic
case

$$T_1(p^2,m^2) = \left(\frac{m^2}{-p^2}\right)^{-2\epsilon}\left[T_1^{(0)} + \epsilon\,T_1^{(1)} + \mathcal{O}(\epsilon^2)\right]$$

$$T_1(p^2,m^2) = \left(\frac{m^2}{-p^2}\right)^{-2\epsilon} \left[T_1^{(0)} + \epsilon\, T_1^{(1)} + \mathcal{O}(\epsilon^2)\right]$$

$$T_1^{(0)} = 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}; 1, \vec{a}\right)$$

$$\begin{aligned} T_1^{(1)} = & -4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_2 & \infty \end{smallmatrix}; 1, \vec{a}\right) \\ & - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\ & - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_2 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_2 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\ & + 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 0 \end{smallmatrix}; 1, \vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix}; 1, \vec{a}\right) + 6\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 0 & \infty \end{smallmatrix}; 1, \vec{a}\right) + 6\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & \infty \end{smallmatrix}; 1, \vec{a}\right) \\ & - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\ & + 6i\pi\mathcal{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 6i\pi\mathcal{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\ & + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) + \zeta_2\, \mathcal{E}_4\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; 1, \vec{a}\right) . \end{aligned}$$

$$\vec{a} = \left(\frac{1}{2}(1+\sqrt{1+\rho}), \frac{1}{2}(1+\sqrt{1+\overline{\rho}}), \frac{1}{2}(1-\sqrt{1+\rho}), \frac{1}{2}(1-\sqrt{1+\overline{\rho}})\right)$$

$$\rho = -\frac{4m^2}{(m+\sqrt{-p^2})^2} \quad \text{and} \quad \overline{\rho} = -\frac{4m^2}{(m-\sqrt{-p^2})^2} .$$

$$T_1(p^2,m^2) = \left(\frac{m^2}{-p^2}\right)^{-2\epsilon}\left[T_1^{(0)} + \epsilon\, T_1^{(1)} + \mathcal{O}(\epsilon^2)\right]$$

$$T_1^{(0)} = 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & \infty\end{smallmatrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 1\end{smallmatrix};1,\vec{a}\right)$$

$$\begin{aligned}T_1^{(1)} &= -4\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_3 & \infty\end{smallmatrix};1,\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_1 & \infty\end{smallmatrix};1,\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_4 & \infty\end{smallmatrix};1,\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_2 & \infty\end{smallmatrix};1,\vec{a}\right)\\
&- 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_3 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_3 & 1\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_1 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_1 & 1\end{smallmatrix};1,\vec{a}\right)\\
&- 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_4 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_4 & 1\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ & & \end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ & & \end{smallmatrix};1,\vec{a}\right)\\
&+ 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & \infty & 0\end{smallmatrix};1,\vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & \infty & 1\end{smallmatrix};1,\vec{a}\right)\\
&- 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & 0 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & 0 & 1\end{smallmatrix};1,\vec{a}\right)\\
&+ 6i\pi\mathcal{E}_4\left(\begin{smallmatrix}0 & 0 & 1\\ 0 & 0 & 0\end{smallmatrix};1,\vec{a}\right) + 6i\pi\mathcal{E}_4\left(\begin{smallmatrix}0 & 0 & 1\\ 0 & 0 & 1\end{smallmatrix};1,\vec{a}\right)\\
&+ 3\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & 1 & 0\end{smallmatrix};1,\vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & 1 & 1\end{smallmatrix};1,\vec{a}\right)\end{aligned}$$

$$\vec{a} = \left(\frac{1}{2}(1+\sqrt{1+\rho}), \frac{1}{2}(1+\sqrt{1+\overline{\rho}}), \frac{1}{2}(1-\sqrt{1+\rho}), \frac{1}{2}(1-\sqrt{1+\overline{\rho}})\right)$$

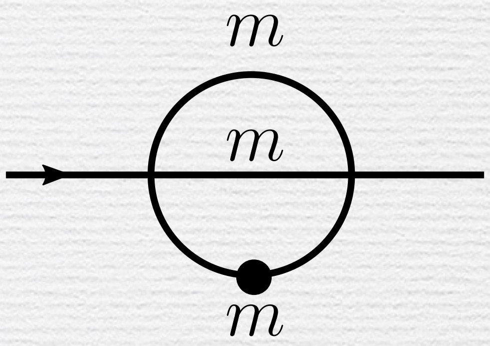
Recall weights:

$$\mathcal{E}_4\left(\begin{smallmatrix}n_1 & \cdots & n_k\\ c_1 & \cdots & c_k\end{smallmatrix};x\right) \;\to\; \sum_i |n_i|$$

$$\omega_1, \pi \;\to\; 1$$

Manifestly pure of weight 2! 🎉

# The second master



Semi-simple

Unipotent

$$\begin{pmatrix} S_1(p^2,m^2) \\ S_2(p^2,m^2) \end{pmatrix} = \begin{pmatrix} \Omega_1 & 0 \\ H_1 & -\frac{2}{m^2(p^2+m^2)(p^2+9m^2)\,\Omega_1} \end{pmatrix} \begin{pmatrix} T_1(p^2,m^2) \\ T_2(p^2,m^2) \end{pmatrix}$$

$$\begin{aligned} \Omega_1 &= -\frac{\omega_1}{c_4\left(m^2+p^2\right)}\,, \\ H_1 &= -\frac{4c_4\eta_1}{m^2\left(9m^2+p^2\right)} - \frac{\omega_1\left(15m^4+12m^2p^2+p^4\right)}{6c_4m^2\left(m^2+p^2\right)^2\left(9m^2+p^2\right)} \end{aligned}$$

$\mathrm{Cut}[S_2(p^2,m^2)_{|D=2}]$

$$T_2(p^2,m^2) = \left(\frac{m^2}{-p^2}\right)^{-2\epsilon}\left[T_2^{(0)} + \epsilon\,T_2^{(1)} + \mathcal{O}(\epsilon^2)\right]$$

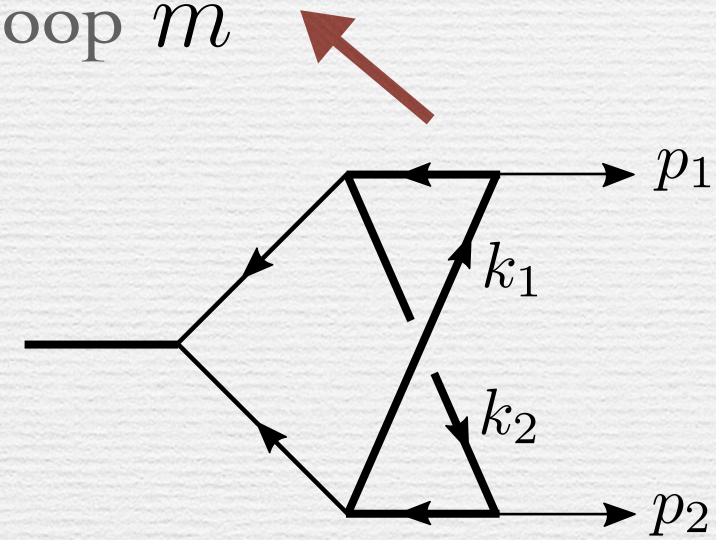
$$T_2^{(0)} = 2\mathcal{E}_4\left(\begin{smallmatrix}-2\\ \infty\end{smallmatrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix}-2\\ 0\end{smallmatrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix}-2\\ 1\end{smallmatrix};1,\vec{a}\right)$$

Uniform weight 2!

# Ex 2: $t\bar{t}$ production

— Tancredi, von Manteuffel ’17 —

Massive loop $m$



$$a = m^2/S$$

$$I = \int \frac{d^dk_1 d^dk_2}{(i\pi)^4} \frac{1}{\prod_{i=1}^{6} D_i}$$

$$p_1^2 = p_2^2 = 0$$

$$S = -2(p_1 \cdot p_2)$$

$$D_1 = k_1^2 - m^2, \quad D_3 = (k_1 - p_1)^2 - m^2, \quad D_5 = (k_1 - k_2 - p_1)^2,$$

$$D_2 = k_2^2 - m^2, \quad D_4 = (k_2 - p_2)^2 - m^2, \quad D_6 = (k_2 - k_1 - p_2)^2$$

# In terms of pure eMPLs $\mathcal{E}_4$ :

$$I = \frac{32\omega_1}{q^2(1+\sqrt{1-16a})}\left[T_0(a) + 3T_-(a) + 5T_+(a) + \mathcal{O}(\epsilon)\right]$$

$$\begin{aligned} T_a = & -\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 0 & 0\end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 0 & 1\end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 1 & 0\end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1 & 1\\ 0 & \infty & 1 & 1\end{smallmatrix};1\right) + \\ & \log(a)\left[\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & \infty & 0\end{smallmatrix};1\right) + \mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & \infty & 1\end{smallmatrix};1\right)\right] + \frac{1}{2}\mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & \infty\end{smallmatrix};1\right)\left(\zeta_2 - \log^2(a)\right) \end{aligned}$$

$$\begin{aligned} T_- = & -\frac{3}{2}\zeta_2\,\mathcal{E}_4\left(\begin{smallmatrix}-1\\ \infty\end{smallmatrix};r_-\right) + \zeta_2\mathcal{E}_4\left(\begin{smallmatrix}-1 & 0\\ \infty & 0\end{smallmatrix};r_-\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}-1 & -1\\ \infty & \infty\end{smallmatrix};r_-\right)\mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & \infty\end{smallmatrix};1\right) \\ & + \mathcal{E}_4\left(\begin{smallmatrix}-1 & 0 & 1 & 1\\ \infty & 0 & 0,0 & \end{smallmatrix};r_-\right) + \mathcal{E}_4\left(\begin{smallmatrix}-1 & 0 & 1 & 1\\ \infty & 0 & 0 & 1\end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix}-1 & 0 & 1 & 1\\ \infty & 0 & 1 & 0\end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix}-1 & 0 & 1 & 1\\ \infty & 0 & 1 & 1\end{smallmatrix};r_-\right) \\ & + \mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 0 & 1\\ \infty & 0 & 0 & 1\end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix}-1 & 1 & 0 & 1\\ \infty & 1 & 0 & 0\end{smallmatrix};r_-\right) + \mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & 0 & 1\\ 0 & \infty & 0 & 1\end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & 0 & 1\\ 1 & \infty & 0 & 0\end{smallmatrix};r_-\right) \\ & - \mathcal{E}_4\left(\begin{smallmatrix}-1 & 0 & 1\\ \infty & 0 & 1\end{smallmatrix};r_-\right)\log(r_-) + \mathcal{E}_4\left(\begin{smallmatrix}-1 & 0 & 1\\ \infty & 0 & 0\end{smallmatrix};r_-\right)\log(1-r_-) \end{aligned}$$

$$\begin{aligned} T_+ = & \frac{i\pi}{4}\left(\mathcal{E}_4\left(\begin{smallmatrix}1 & -1\\ 0 & \infty\end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix}1 & -1\\ 1 & \infty\end{smallmatrix};r_+\right) - 4\left(\mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & 0\\ 0 & \infty & 0\end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & 0\\ 1 & \infty & 0\end{smallmatrix};r_+\right)\right)\right) \\ & - \mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & 0 & 1\\ 0 & \infty & 1 & 0\end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & 0 & 1\\ 0 & \infty & 0 & 1\end{smallmatrix};r_+\right) - \mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & 0 & 1\\ 1 & \infty & 1 & 0\end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & 0 & 1\\ 1 & \infty & 0 & 1\end{smallmatrix};r_+\right) \end{aligned}$$

$$\vec{b} = \left(0, \frac{1}{2}(1-\sqrt{1-16a}), \frac{1}{2}(1+\sqrt{1-16a}), 1\right) \qquad r_\pm = \frac{1}{2}(1-\sqrt{1\pm 4a}) \qquad a = m^2/(-q^2)$$

# In terms of pure eMPLs $\mathcal{E}_4$ :

$$I = \frac{32\omega_1}{q^2(1+\sqrt{1-16a})}\left[T_0(a) + 3T_-(a) + 5T_+(a) + \mathcal{O}(\epsilon)\right]$$

$$\begin{aligned}
T_a \; = & -\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1\\ 0 & \infty & 0 & 0\end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1\\ 0 & \infty & 0 & 1\end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1\\ 0 & \infty & 1 & 0\end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1\\ 0 & \infty & 1 & 1\end{smallmatrix};1\right) + \\
& \log(a)\left[\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1\\ 0 & \infty & 0\end{smallmatrix};1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1\\ 0 & \infty & 1\end{smallmatrix};1\right)\right] + \frac{1}{2}\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1\\ 0 & \infty\end{smallmatrix};1\right)\left(\zeta_2 - \log^2(a)\right)
\end{aligned}$$

$$\begin{aligned}
T_- \; = & -\frac{3}{2}\zeta_2\,\mathcal{E}_4\left(\begin{smallmatrix} -1\\ \infty\end{smallmatrix};r_-\right) + \zeta_2\mathcal{E}_4\left(\begin{smallmatrix} -1 & 0\\ \infty & 0\end{smallmatrix};r_-\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} -1 & -1\\ \infty & \infty\end{smallmatrix};r_-\right)\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1\\ 0 & \infty\end{smallmatrix};1\right) \\
& + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1\\ \infty & 0 & 0{,}0 & \end{smallmatrix};r_-\right) + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1\\ \infty & 0 & 0 & 1\end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1\\ \infty & 0 & 1 & 0\end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1\\ \infty & 0 & 1 & 1\end{smallmatrix};r_-\right) \\
& + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & 0 & 1\\ \infty & 0 & 0 & 1\end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & 0 & 1\\ \infty & 1 & 0 & 0\end{smallmatrix};r_-\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1\\ 0 & \infty & 0 & 1\end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1\\ 1 & \infty & 0 & 0\end{smallmatrix};r_-\right) \\
& - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1\\ \infty & 0 & 1\end{smallmatrix};r_-\right)\log(r_-) + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1\\ \infty & 0 & 0\end{smallmatrix};r_-\right)\log(1-r_-)
\end{aligned}$$

$$\begin{aligned}
T_+ \; = & \frac{i\pi}{4}\left(\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1\\ 0 & \infty\end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1\\ 1 & \infty\end{smallmatrix};r_+\right) - 4\left(\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0\\ 0 & \infty & 0\end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0\\ 1 & \infty & 0\end{smallmatrix};r_+\right)\right)\right) \\
& - \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1\\ 0 & \infty & 1 & 0\end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1\\ 0 & \infty & 0 & 1\end{smallmatrix};r_+\right) - \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1\\ 1 & \infty & 1 & 0\end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1\\ 1 & \infty & 0 & 1\end{smallmatrix};r_+\right)
\end{aligned}$$

Uniform weight 4!

# Back to the real world

## The elliptic double box of N=4 SYM

— Caron-Huot, Larsen ’12 / Nandan, Paulos, Spradlin, Volovich ’14 /
Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm ’17 —

$$I_{\text{db}}^{\text{ell}} \sim \int \frac{d\alpha}{\sqrt{Q(\alpha)}}\, \mathcal{G}_3(\alpha)$$

Quartic polynomial

Pure combination of MPLs
(with quadradic sqrts)

Using

$$\Psi_0(0,x,\vec{a}) = \frac{c_4}{\omega_1 y}$$

Weight 1

Weight 3

$$I_{\text{db}}^{\text{ell}} \sim \frac{\omega_1}{c_4}\, T_{\text{db}}^{\text{ell}} \qquad T_{\text{db}}^{\text{ell}} = \int d\alpha\, \Psi_0(0,\alpha)\, \mathcal{G}_3(\alpha)$$

Uniform weight 4, as expected!

# Conclusions

- First step into defining a concept of purity and uniform weight in the elliptic case, worked out several examples

- Both conceptual and practical relevance — in the end we are interested in computing amplitudes and obtaining reliable analytical expressions

- Purity is of great relevance in the MPL case (differential equations), hopefully soon we will have a similar understanding for elliptic Feynman integrals too

- Not the end, integrals with multiple elliptic curves, more complicated geometries, etc. — Adams, Chaubey, Weinzierl '18 — [Jake's talk]

- Lots to do still, but we are definitely moving forward!

# Conclusions

- First step into defining a concept of purity and uniform weight in the elliptic case, worked out several examples

- Both conceptual and practical relevance — in the end we are interested in computing amplitudes and obtaining reliable analytical expressions

- Purity is of great relevance in the MPL case (differential equations), hopefully soon we will have a similar understanding for elliptic Feynman integrals too

- Not the end, integrals with multiple elliptic curves, more complicated geometries, etc. — Adams, Chaubey, Weinzierl '18 —

[Jake's talk]

- Lots to do still, but we are definitely moving forward!

Grazie! 🍕