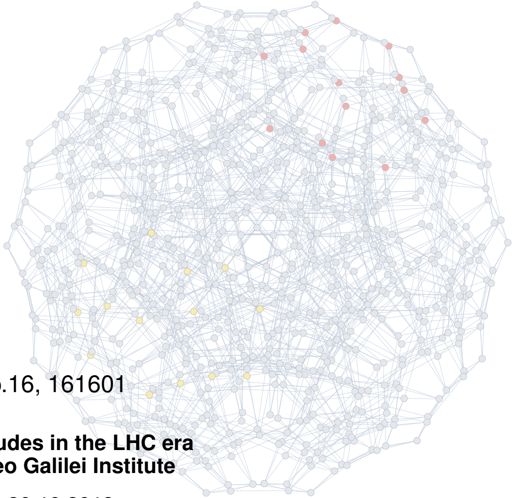
Cluster Adjacency



Ömer Gürdoğan
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in collaboration with

James Drummond
Jack Foster

- arXiv:`181x.xxxxx`
- arXiv:`1810.08149`
- Phys.Rev.Lett. **120** (2018) no.16, 161601

Amplitudes in the LHC era
Galileo Galilei Institute

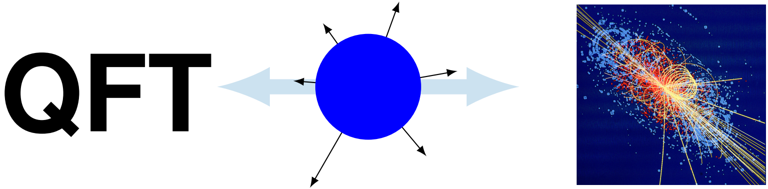
30.10.2018

UNIVERSITY OF Southampton

erc

Scattering amplitudes

QFT is the framework in which we understand the microscopic universe.



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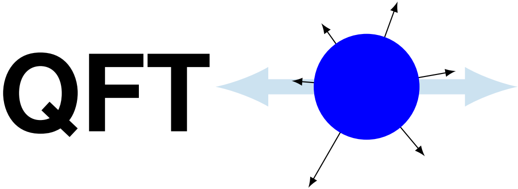
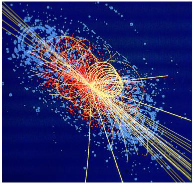




- Amplitudes relate Quantum Field Theory to observations in colliders.
- Higher-order computations notoriously difficult
  bottleneck: **loop integration**
- **Either:** QFT is intrinsically hard **Or:** An ideal formulation is lacking

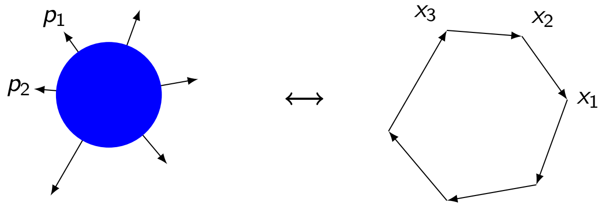
The more we compute $\rightsquigarrow$ the more amazing mathematical structures.

Goal: Use these to develop an understanding of scattering amplitudes and QFT free of redundancies

Scattering in $\mathcal{N} = 4$ super Yang-Mills

- Duality with Wilson loops

  [Alday, Maldacena; Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini]



$p_i = x_i - x_{i+1}$

  - Profound consequences for the scattering amplitude: **dual conformal symmetry**

- The "BDS-like" normalised amplitude

$$\mathcal{A} = \mathcal{A}_{\text{BDS-like}}\, E$$

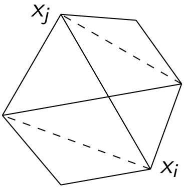
  - $\mathcal{A}_{\text{BDS-like}} \sim \exp\left[\mathcal{A}^{(1-\text{loop})}\right]$,
  - $\mathcal{A}_{\text{BDS-like}}$ unique solution to dual-conformal Ward identities

The "BDS-like"-normalised amplitude

$$\mathcal{A} = \mathcal{A}_{\text{BDS-like}}\, E$$

- Finite : $\mathcal{A}_{\text{BDS-like}}$ removes the IR divergences to all loops
- Depends on $n(n-5)/2$ dual-conformal cross ratios

$$u_{ij} = \frac{x_{i\,j+1}^2 x_{i+1\,j}^2}{x_{i+1\,j+1}^2 x_{i\,j}^2} =$$



- Inherits the analytic structure of $\mathcal{A}$
- (Lore) MHV and NMHV: $E =$ linear combination of (generalised) polylogarithms of weight $w = 2L$ at L loops

$$G(a_w, a_{w-1}, \ldots, a_1; z) = \int_0^z \frac{\mathrm{d}z'}{z' - a_w} G(a_{w-1}, \ldots, a_1; z'), \quad G(z) = 1$$

The symbol map

- A map from polylogarithms of weight w to a w-fold tensor space:

$$\mathcal{S}: \quad \mathcal{G}_w \longrightarrow \bigotimes^{w} \mathbb{S}$$

$\mathbb{S}$: "alphabet"

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$$G(\vec{0}_w; z) \propto \ln^w z \mapsto z \otimes z \ldots \otimes z$$

$$G(\vec{0}_{w-1}, 1, z) = -\mathrm{Li}_w(z) \mapsto (1-z) \otimes z \otimes z \otimes \cdots$$

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- **Discontinuities** from the symbol:
  - Initial entries $\leftrightarrow$ branch points
  - Tails $\leftrightarrow$ symbol of the discontinuity

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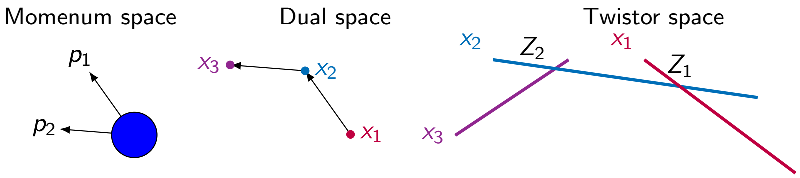
Symbol $\rightarrow$ function much easier than loop integration

Kinematics in twistor space

Twistor space $\mathbb{CP}_3$The natural parametrisation of light-like kinematics

- Points in dual space $\leftrightarrow$ lines in twistor space
- Intersecting lines in $\mathbb{CP}_3 \leftrightarrow$ light-like separated dual points



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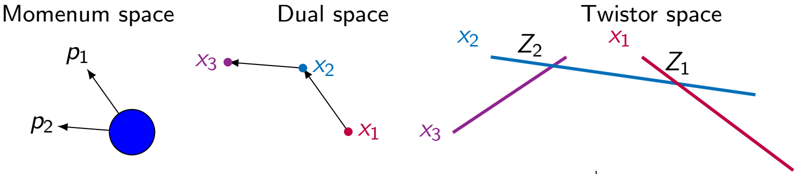


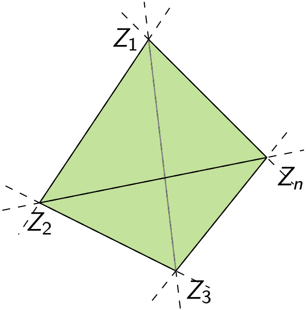
Basic $SL(4)$ invariants:

Plücker coordinates

$$s_{12} = (p_1 + p_2)^2 = x_{13}^2 \sim \langle n123 \rangle$$

where

$$\langle n123 \rangle := \epsilon_{ABCD} Z_n^A Z_1^B Z_2^C Z_3^D$$



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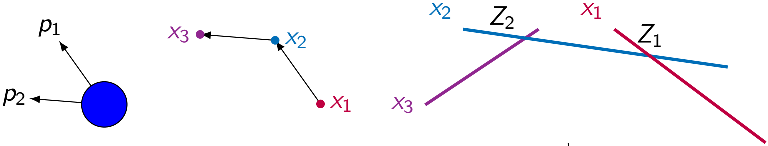
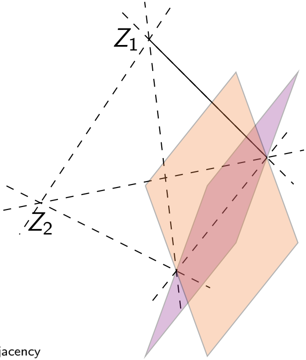
Momenum space Dual space Twistor space



Other invariants:

Eg constructed with two points and two planes $\eta_1/$ & η_2.
$\eta_1 = Z_2 \wedge Z_3 \wedge Z_4$, $\eta_2 = Z_5 \wedge Z_6 \wedge Z_7$

$$\langle n1\, \eta_1 \cap \eta_2 \rangle$$
$$= \langle n234 \rangle \langle 1567 \rangle - \langle n567 \rangle \langle 1234 \rangle$$



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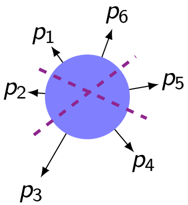

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Bootstrap strategy: Construct the symbol of E from all possible constraints on an **Ansatz**.

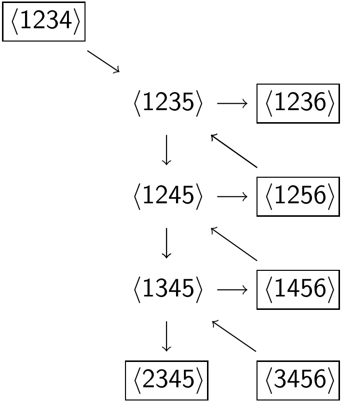
[Caron-Huot, Dixon, Drummond, Harrington, Henn, McLeod, Papathanasiou, Pennington, Spradlin, von Hippel]

Where are the branch points? $\rightsquigarrow$ **Cluster Algebras**

Branch points from Gr(4, n) cluster algebras

[Fomin, Zelevinsky; Golden, Goncharov, Spradlin, Vergu, Volovich]

Gr(4, 6) **example:** Six-particle scattering



Cluster: Quiver diagram with nodes in $\mathcal{A}$ and adjacency matrix b_{ij}.

Nodes: All-adjacent Plückers: **frozen** , others **active**

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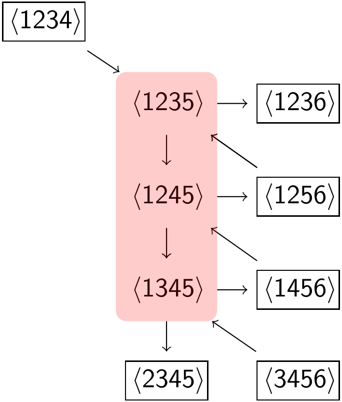

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A_3-type: The active nodes are connected in an A_3 Dynkin diageam

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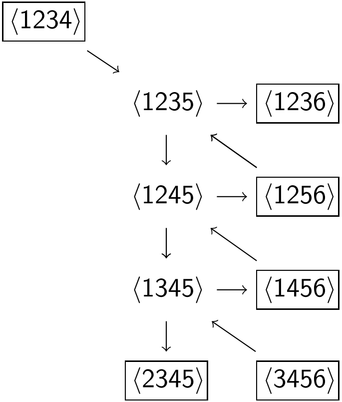

Mutations “mutation on an active node” transform the cluster to a new one:

$$b'_{ij} = \begin{cases} -b_{ij} & k \in \{i, j\} \\ b_{ij} & b_{ik} b_{kj} \leq 0 \\ b_{ij} + b_{ik} b_{kj} & b_{ik} b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & b_{ik} b_{kj} < 0 \end{cases}$$

$$a'_k = \frac{1}{a_k} \left[\prod_{i \,|\, b_{ik} > 0} a_i^{b_{ik}} + \prod_{i \,|\, b_{ik} < 0} a_i^{-b_{ik}} \right]$$

Branch points from $\mathrm{Gr}(4,n)$ cluster algebras

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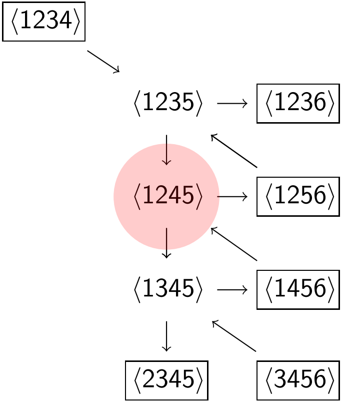

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Example mutate the initial cluster on $\langle 1245\rangle$:

$$\langle 1245\rangle \mapsto \frac{\langle 1235\rangle\langle 1456\rangle - \langle 1256\rangle\langle 1345\rangle}{\langle 1245\rangle} = \langle 1356\rangle$$

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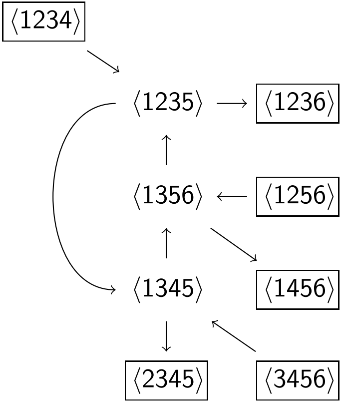

$$a_k' = \frac{1}{a_k}\left[\prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}}\right]$$

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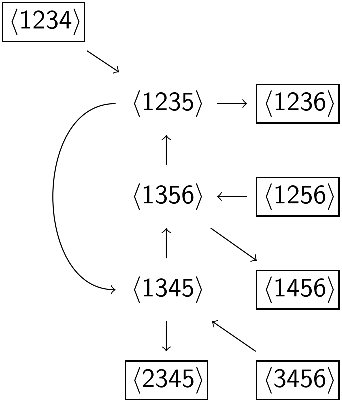

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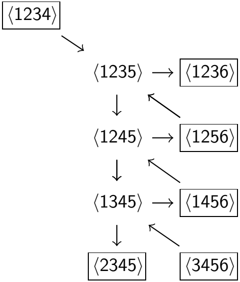
- Mut's generate $\binom{6}{4} = 15$ branch points
- $15 - 6 = 9$ homogeneous combinations, eg $\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$

The $\mathrm{Gr}(4,6) \cong A_3$ / Stasheff polytope



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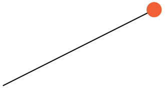

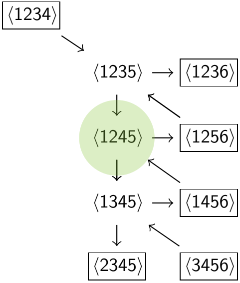

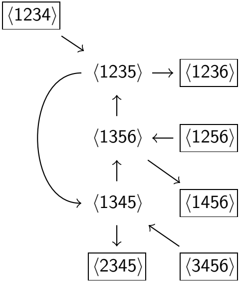

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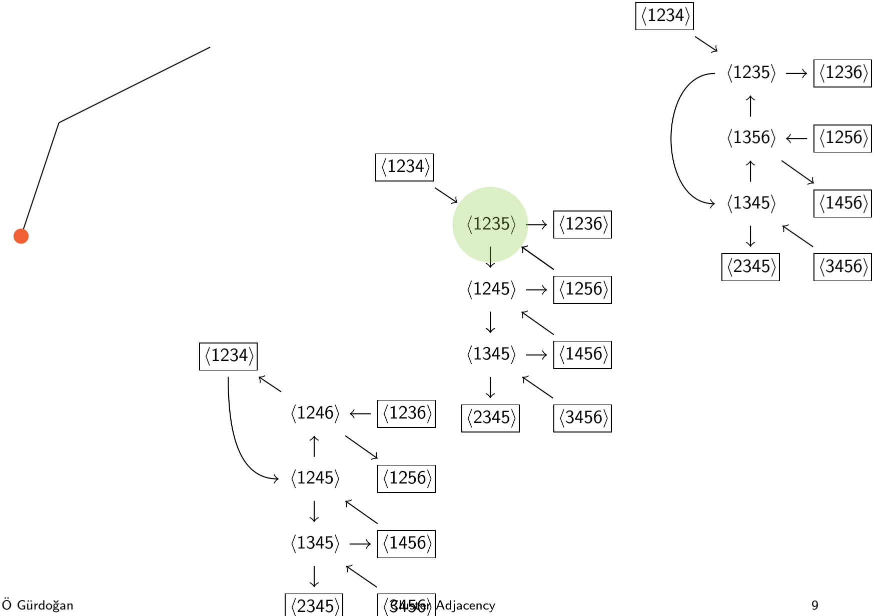

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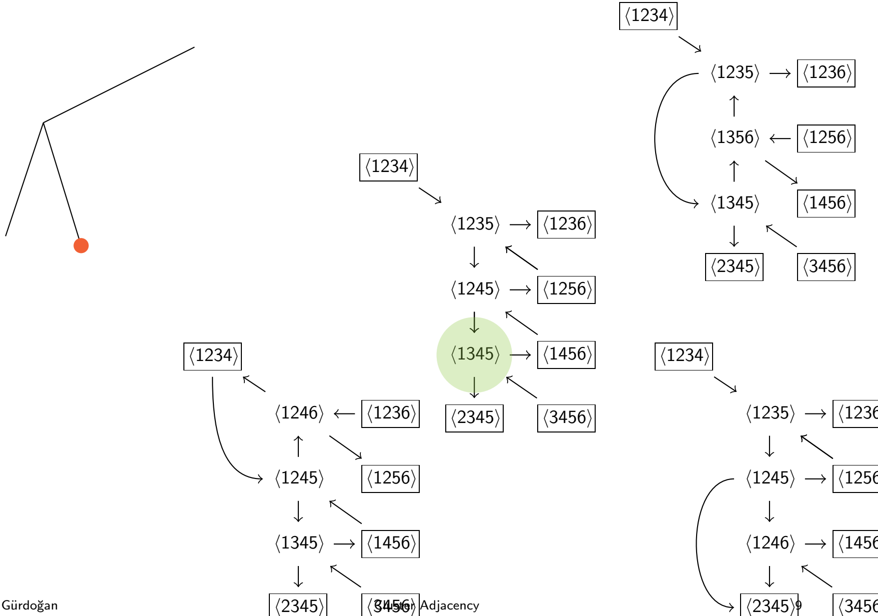

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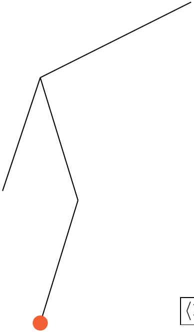

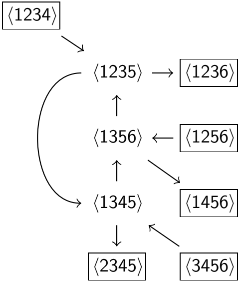

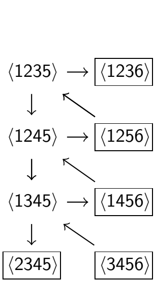

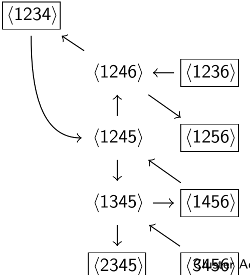

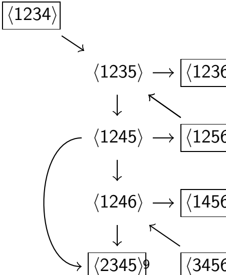

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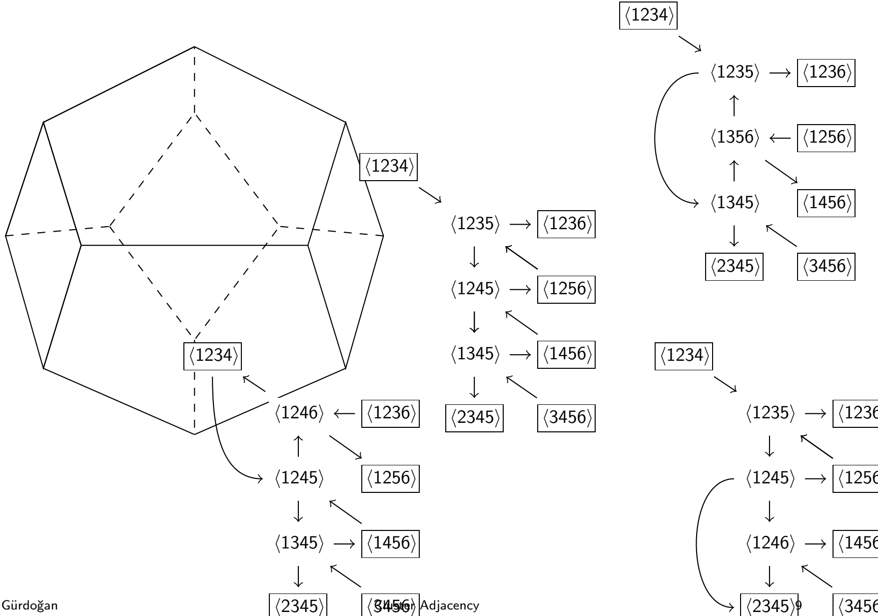

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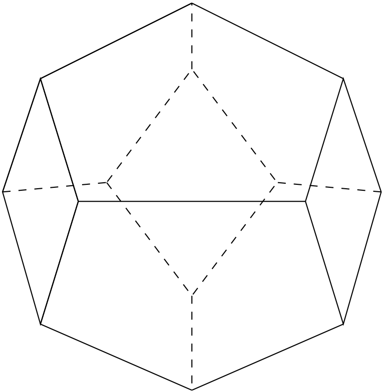

The associahedron

The mutations of clusters close on a polytope.

Status of (Steinmann) Cluster bootstrap

LOOPS	BDS	MHV	NMHV	MHV	NMHV
7					
6		✓	✓		
5		✓	✓		
4		✓	✓	✓ (symbol)	
3		✓	✓	✓ (symbol)	✓ (symbol)
2		✓	✓	✓	✓ (symbol)
1		✓	✓	✓	✓
n	4,5	6		7	

Very high in loop level

but

constrained by linear algebra technology

✓: Function, ✓: Symbol comp.

[Caron-Huot, Dixon, Drummond, Harrington, Henn, McLeod, Papathanasiou, Pennington, Spradlin, von Hippel]

High-loop computations essential for uncovering structures and patterns with important lessons

Status of (Steinmann) Cluster bootstrap

LOOPS	n = 4,5	n = 6 MHV	n = 6 NMHV	n = 7 MHV	n = 7 NMHV
7	BDS				
6	BDS	✓	✓		
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3	BDS	✓	✓	✓ (symbol)	✓ (symbol)
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Aim: Write down amplitudes without any calculation.

Good start: Knowledge of initial ($\sim$ branch cuts) & final entries

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	7					
	6		✓	✓		
	5		✓	✓		
	4	BDS	✓	✓	✓	
	3		✓	✓	✓	✓
	2		✓	✓	✓	✓
	1		✓	✓	✓	✓
			MHV	NMHV	MHV	NMHV
	n	4,5	6		7	

Very high in loop level

but

constrained by linear algebra technology

✓: Function, ✓: Symbol comp.

[Caron-Huot, Dixon, Drummond, Harrington, Henn, McLeod, Papathanasiou, Pennington, Spradlin, von Hippel]

High-loop computations essential for uncovering structures and patterns with important lessons

Aim: Write down amplitudes without any calculation.

Good start: Knowledge of initial ($\sim$ branch cuts) & final entries

Useful: Knowing what we cannot write $\rightsquigarrow$ **Cluster Adjacency**

Cluster Adjacency

Consecutive branch points of scattering amplitudes must appear in the same cluster.

[Drummond, Foster, ÖCG]

A geometric principle governing the analytic structure of scattering amplitudes

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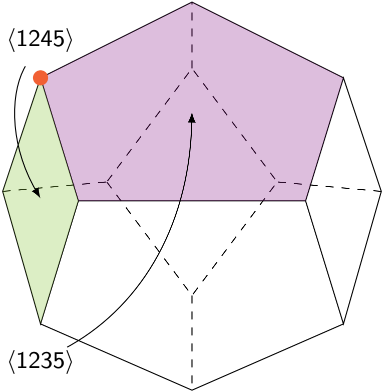

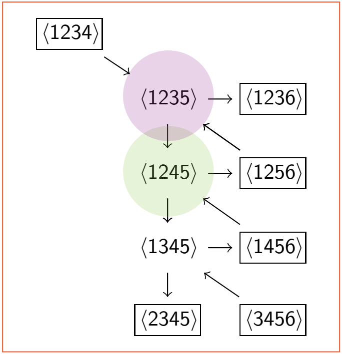

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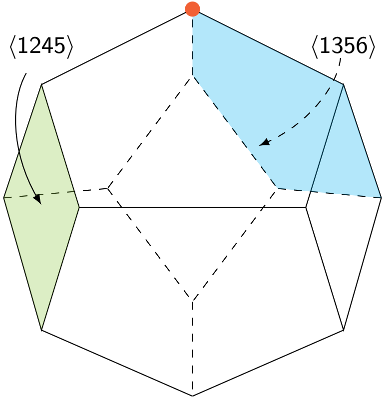

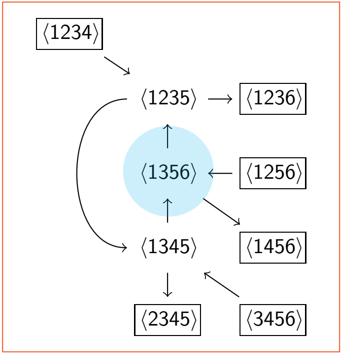

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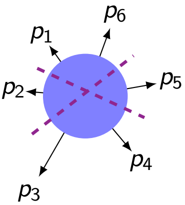

Brackets in overlapping invariants mutate to each other:

$$\langle 1245\rangle \sim (p_2+p_3+p_4)^2 \mapsto \langle 2356\rangle \sim (p_2+p_6+p_1)^2$$

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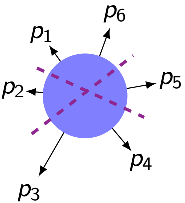

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- Various **further relations**

shorthand:
$(12) = \langle 3456\rangle$
etc

(12) :	Frozen, adjacent to anything
(13) :	$\{(13),(14),(15),(35),(36),\&\text{ frozen}\}$
(14) :	$\{(13),(14),(15),(24),\&\text{ frozen}\}$

The heptagon: $\mathrm{Gr}(4,7) \cong E_6$

odd n: homogenise active nodes with frozen-adjacent Plückers and ignore latter

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle} \quad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle} \quad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle}$$

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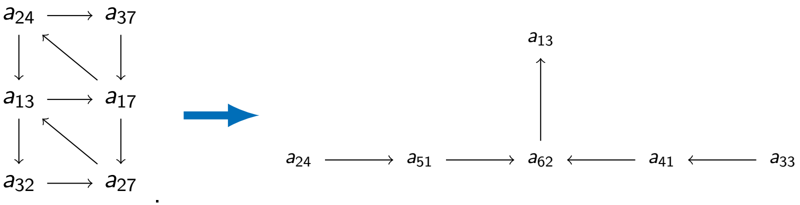

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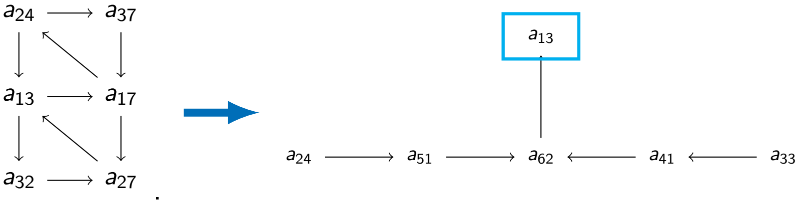

A_5 subalgebra: $20 + 1$ neighbours

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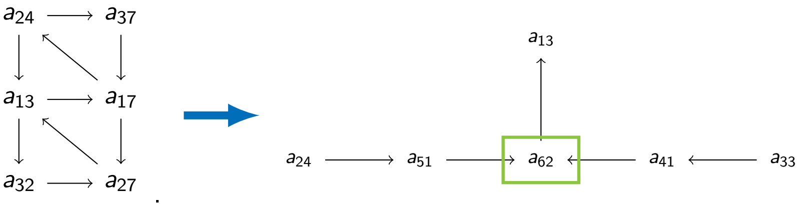

$A_2 \times A_2 \times A_1$ subalgebra: $(5+5+2)+1$ neighbours

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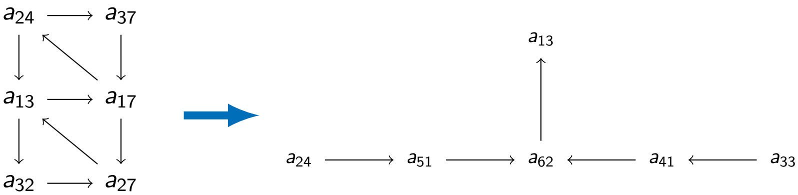

	a_{1i}	a_{2i}	a_{3i}	a_{4i}	a_{5i}	a_{6i}
a_{11}	●○○◆◆○○	◆◆○●◆●○	◆○●◆●○◆	●○◆○○◆○	●○◆○○◆○	◇◆○○○○◆
a_{21}	◆○●◆●○◆	●○●◆◆●○	◆○◆◆○◆●	◆○◆○●◆○	○◆●○◆○◆	○◆○●○◆○
a_{31}	◆◆○●◆●○	◆●◆○◆◆○	●○●◆◆●○	◆○◆●○◆○	○◆○◆○●◆	○○◆○●○◆
a_{41}	●○◆○○◆○	◆○◆●○◆○	◆○◆○●◆○	●◇◆○○◆◇	●○○○○○○	◇◆◇○○◇◆
a_{51}	●○◆○○◆○	○◆○◆○●◆	○◆●○◆○◆	●○○○○○○	●◇◆○○◆◇	◇◆◇○○◇◆
a_{61}	◇◆○○○○◆	○○◆○●○◆	○◆○●○◆○	◇◆◇○○◇◆	◇◆◇○○◇◆	●◇○◇◇○◇

The adjacency table for $\mathrm{Gr}(4,7)$ branch points
◇: non-neighbour, ○: mutation non-neighbour,
◆: connected neighbour, ●: disconnected neighbour

Some consequences of Cluster Adjacency

- **Smaller function spaces**

Weight	2	3	4	5	6	7	8	9	10	11	12	13	14
	Hexagon:												
CA	6	13	26	51	98	184	340	613	1085	1887	3224	5431	9014
St	6	13	27	54	106	207	405	796	1572	3117	6199	12354	24654
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CA	28	97	308	911	2555	6826							
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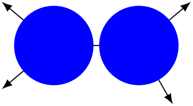
- **Smaller Ansätze** for functions of weight-k:

$$f^{(k)} = \sum c_{i\alpha} f_i^{(k-1)} \otimes \phi_\alpha, \qquad \phi_\alpha \in \mathcal{A} \tag{1}$$

Only $f_i^{(k-1)}$ ending with neighbours of ϕ_α needed in the coproduct.

Tree amplitudes



BCFW recursion of tree amplitudes via factorisation on physical poles.

NMHV: $\mathcal{A}^{\text{NMHV}} = \sum_{2<i<j\leq n} \left[1\, i-1\, i\, j-1\, j\right]$, where

$$[abcde] = \frac{\delta(\langle abcd\rangle \chi_e + \text{cyclic})}{\langle abcd\rangle\langle abce\rangle\langle abde\rangle\langle acde\rangle\langle bcde\rangle} \quad \ni \quad \text{physical and spurious poles}.$$

N$^{>1}$MHV: More complicated terms e.g. $[12345][56781] \in \mathcal{A}_{8,2}$

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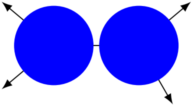

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Poles of BCFW terms are cluster neighbours!

[Drummond, Foster, ÖCG]

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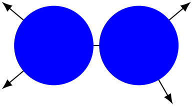

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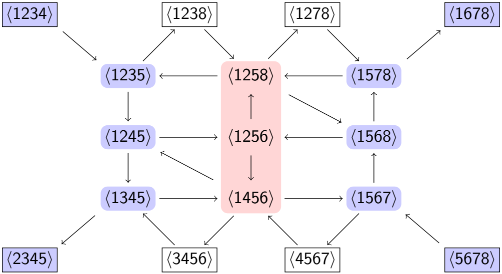

NMHV loop amplitudes

- Expansion in R-invariants: $E = \sum \; [abcde] \quad f_{abcde}$

Rational R-invariants CA poles in Plückers

Polylog coefficient functions CA branch cuts

Dual superconformal symmetry:
$\bar{Q}$ equation $\Rightarrow$ final entries of $f \leftrightarrow [\cdots]$

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$\bar{Q}$ final entries of NMHV coefficient functions are cluster compatible with the R-invariants.

NMHV Heptagon up to 4 loops

[Drummond, Foster, ÖCG, Papathanasiou]

Three types of R-invariants for 7-particles

$$E_7 = (12)\ f_{(12)} + (13)\ f_{(13)} + (14)\ f_{(14)} + \text{cyclic}, \qquad (12) = [34567] \text{ etc}$$

$\bar{Q}$ - compatible final entries in CA form:

$$f_{(12)} = \cdots \otimes \{a_{15}, a_{21}, a_{26}, a_{32}, a_{34}, a_{53}, a_{57}\} \subset \text{neighbour set of } (12)$$
$$f_{(13)} = \cdots \otimes \{a_{21}, a_{23}, a_{31}, a_{33}, a_{41}, a_{43}, a_{62}\} \subset \text{neighbour set of } (13)$$
$$f_{(14)} = \cdots \otimes \{a_{11}, a_{14}, a_{21}, a_{24}, a_{31}, a_{34}, a_{46}\} \subset \text{neighbour set of } (14)$$

- Neighbour set of $(ij) = \bigcap_{a \in \text{poles of } (ij)}$ neighbour sets of a.

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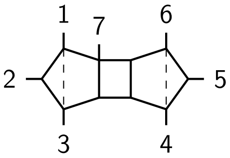
Cluster-adjacent Ansatz for the $\{k-1, 1\}$ coproduct of $f_{(ij)}$: (17,490 coeffs @ 4 loops with partial reflection symmetry implemented)

- Integrability of $f_{(ij)}$ (⤷ 56 coefficients @ 4 loops)
- E_7 free of spurious poles (⤷ 5 coefficients @ 4 loops)
- E_7 finite in the collinear limit

✓Fixes the 4-loop NMHV Heptagon

Beyond $\mathcal{N} = 4$ super Yang-Mills

Individual integrals satisfy cluster adjacency [Drummond, Foster, ÖCG]



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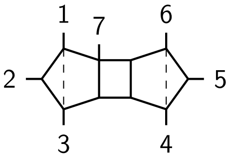

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- Other integrals known to obey Steinmann relations and $\exists$ evidence for more. $\rightsquigarrow$ **cluster interpretation?**

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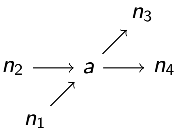
Can **only** come in the form $\cdots a \otimes X \otimes a' \cdots,$ where

$$X = \prod_{\phi \in \text{nodes}} \phi^{b_{\phi a}}$$

e.g.



$$\Rightarrow X(a) = \frac{n_1 n_2}{n_3 n_4}$$

NB: X(a) is independent of the cluster.

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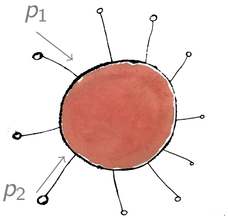
- Better understanding of Cluster Adjacency for A_n functions enumeration of all CA functions, those with special initial/final entries



Thank You!