Regge limit and the soft anomalous dimension

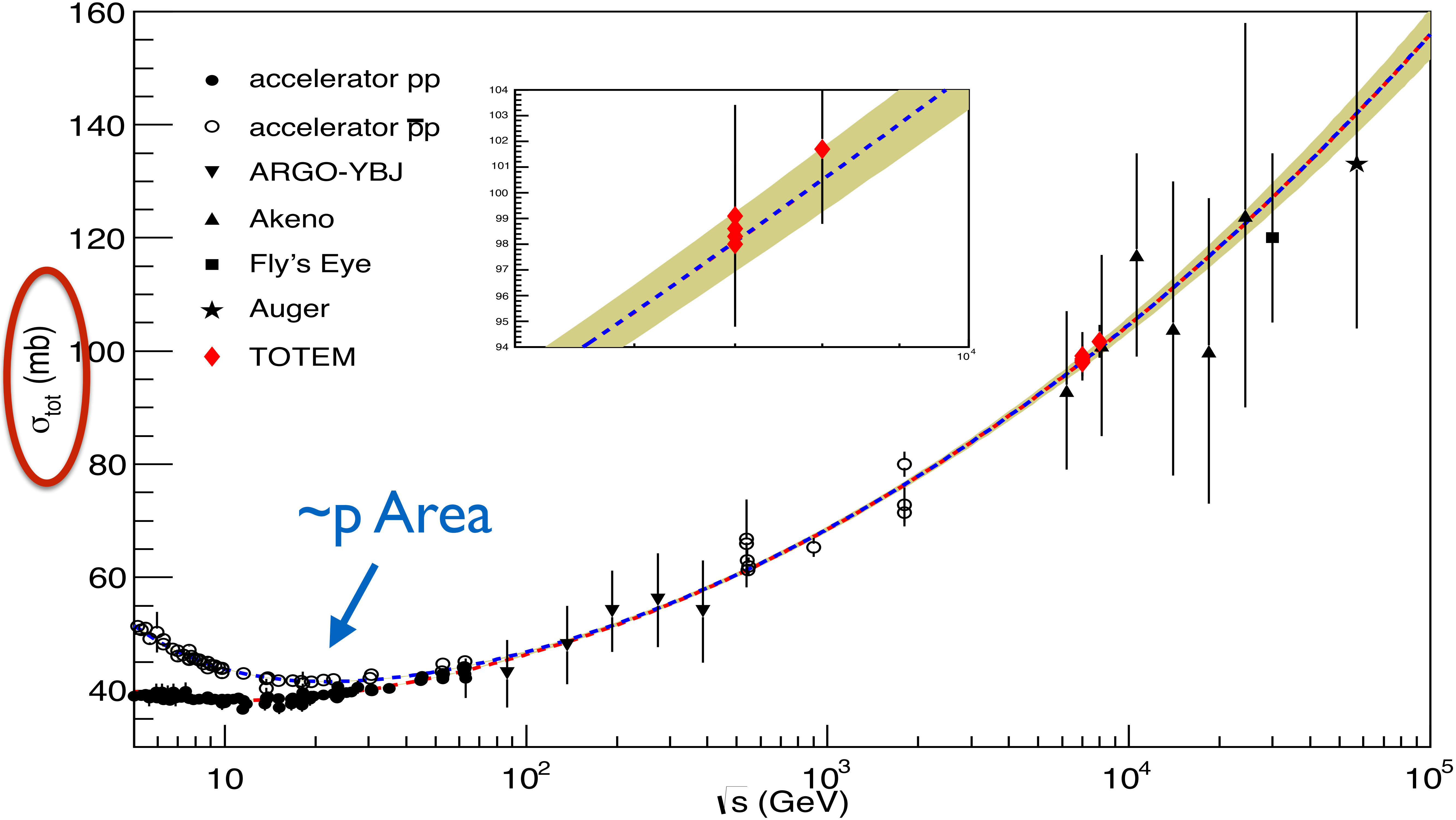
Simon Caron-Huot

(McGill University)

Based on: 1701.05241, 1711.04850 & in progress
with: Einan Gardi, Joscha Reichel & Leonardo Vernazza

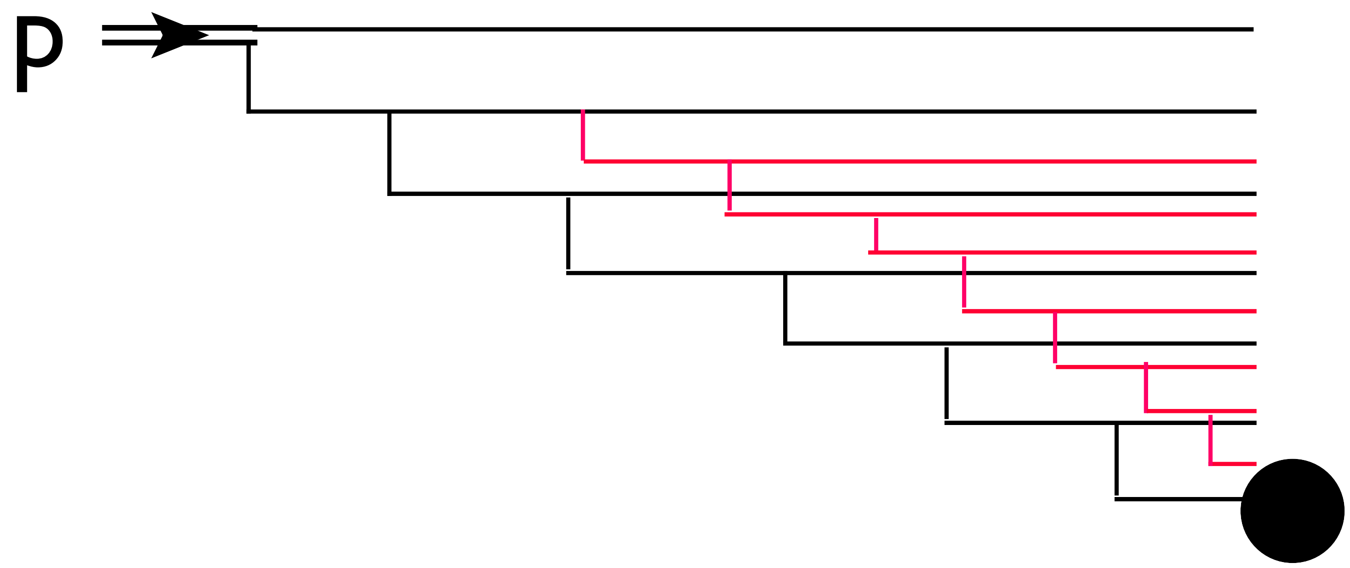
Galileo Galilei Institute: Amplitudes in the LHC era, Oct. 31th 2018

Nonperturbative forward physics: total pp cross-section:



[fig: Menon& Silva `13]

- Why does it grow?
- Relativity + quantum mechanics



- A cloud of virtual particles (pions, rho's,..) builds around the proton!

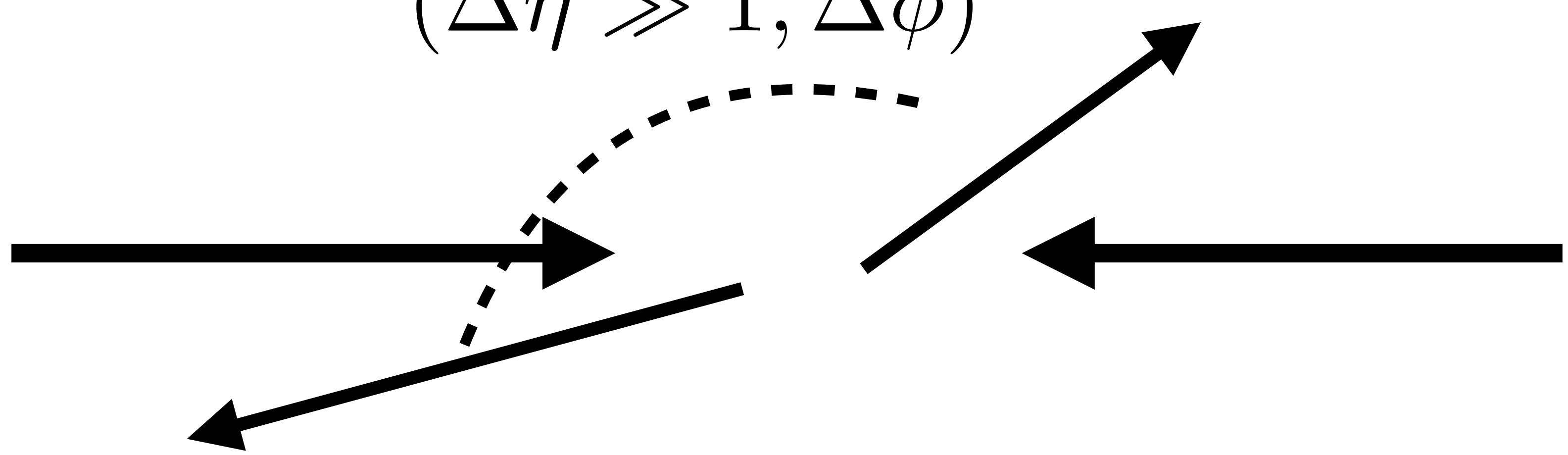
[cartoon: Gottsman, Level&Maor]

perturbative phenomenology of forward scattering:

- Deep inelastic scattering/saturation (HERA,heavy ions)
  small x = Regge, large $Q^2 \Rightarrow$ perturbative

- Mueller-Navelet: pp->X+2jets, forward&backward

$$(\Delta\eta \gg 1, \Delta\phi)$$



for review of these: see 1611.05079

theoretical motivations:

One of few limits where perturbation theory can be resumed

Retain rich dynamics in 2D transverse plane:
- -toy model for full amplitude
- -nontrivial function spaces
- -predicts amplitudes and other observables in overlapping limits

The (multi-)Regge limit at higher points has been extensively studied, especially in planar N=4 SYM.

It reveals an amazing integrable system (next talk?)

Here we'll focus on $A_{2\to 2}$, but in QCD at finite N_c.
It depends on:

- energy: $L \equiv \log|s/t| - i\pi/2$
- IR regulator: $1/\varepsilon \Leftrightarrow \log(-t/\mu^2)$
- color: $C_A, \mathbf{T}_s^2, \mathbf{T}_t^2, \ldots$

Nice variables

1. Coupling runs with transverse momenta, not CM energy

$$\Rightarrow \text{use } \alpha_s(-t)$$

2. Crossing symmetry relates large-s & large-u limits

$\Rightarrow$ use symmetrical combination

$$L \equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right)$$

$$\to \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}$$

Crossing symmetry: Project onto *signature* eigenstates:

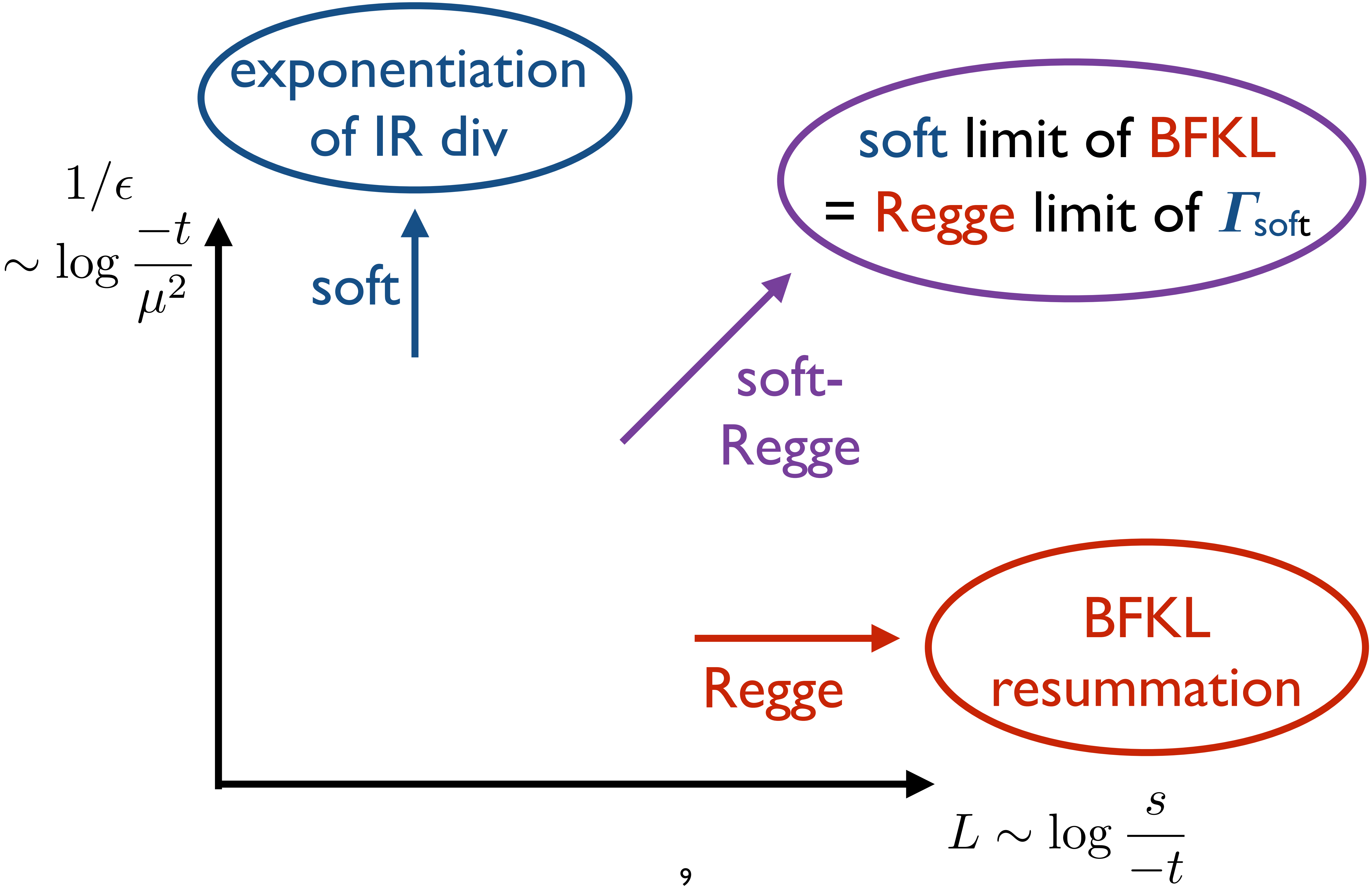
$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big)$$

These simple definitions remove all i\pi's.

The following have nice& real coefficients:

$$\mathcal{M}^{(-)}(L, \alpha_s(-t), \epsilon), \qquad \frac{1}{i\pi}\mathcal{M}^{(+)}(L, \alpha_s(-t), \epsilon)$$

2→2 kinematic limits

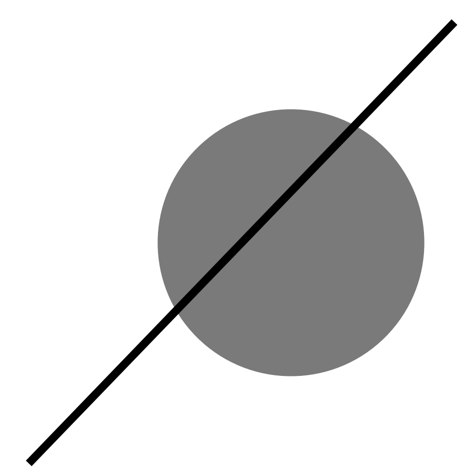


BFKL redux

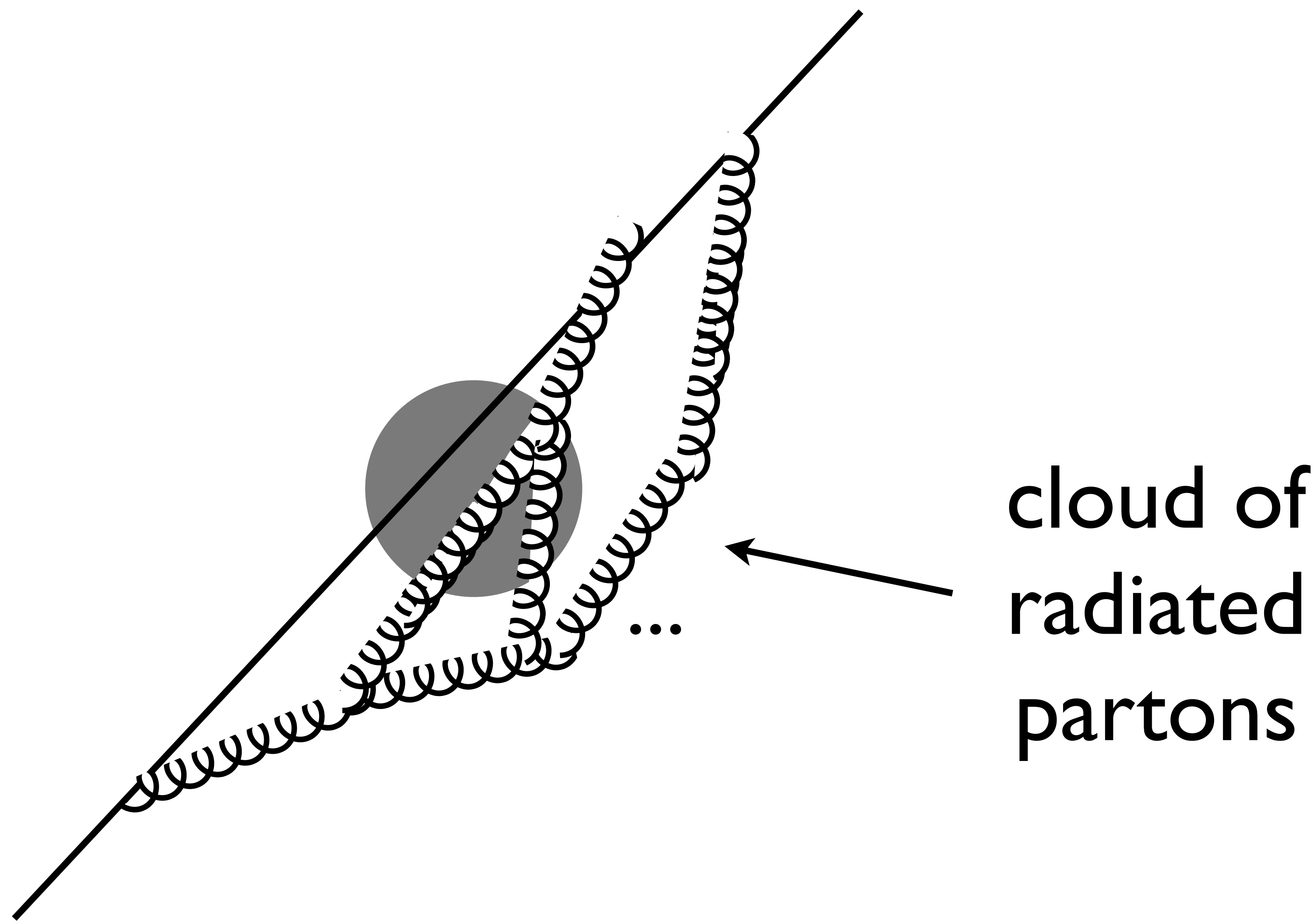
[Balitsky,Fadin,Kuraev,Lipatov '76-78]

A simple, and correct, approach to high-energy scattering: replace each fast parton by a null Wilson line

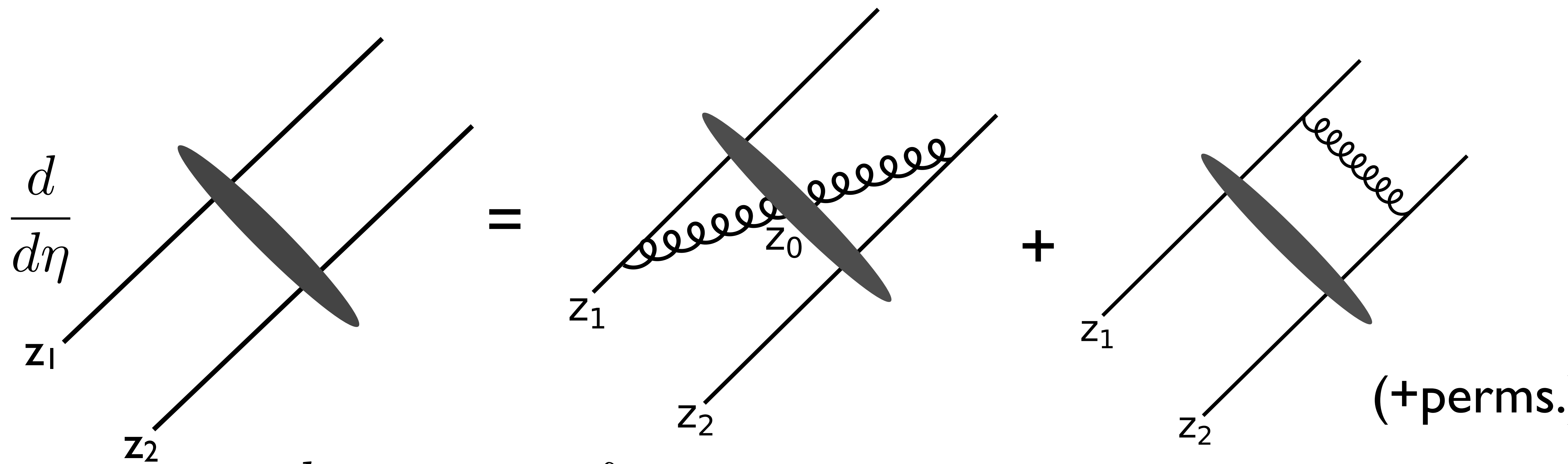
$$U(x_\perp) \equiv \mathcal{P} e^{i \int_{-\infty}^{\infty} dx^+ A_+^a(x^+, 0^-, x_\perp) T^a}$$



The subtlety: projectiles contain more than one parton



Transverse distribution depends on energy resolution



$$\frac{d}{d\eta} UU \sim g^2 \int d^2 z_0 K(z_0, z_1, z_2) \left[U(z_0) UU - UU \right]$$

- ‘shock’ = Lorentz-contracted target
- 45° lines = fast projectile partons
- Each parton crossing the shock gets a Wilson line

The Balitsky-JIMWLK equation

$$\frac{-d}{d\eta} \equiv H = \frac{\alpha_s}{2\pi^2} \int d^2z_i d^2z_j \frac{d^2z_0 \;\; z_{0i}\cdot z_{0j}}{z_{0i}^2 z_{0j}^2} \Big(T^a_{i,L} T^a_{j,L} + T^a_{i,R} T^a_{j,R} - U^{ab}_{\rm ad}(z_0) \big(T^a_{i,L} T^b_{j,R} + T^a_{j,L} T^b_{i,R} \big) \Big)$$

- Well established and tested

[Balitsky ’95, Mueller, Kovchegov, JIMWLK*]

- Now well understood at NLL

[Balitsky&Chirilli ’07&’13; Kovner,Lublinsky&Mulian ’13; SCH ’14]

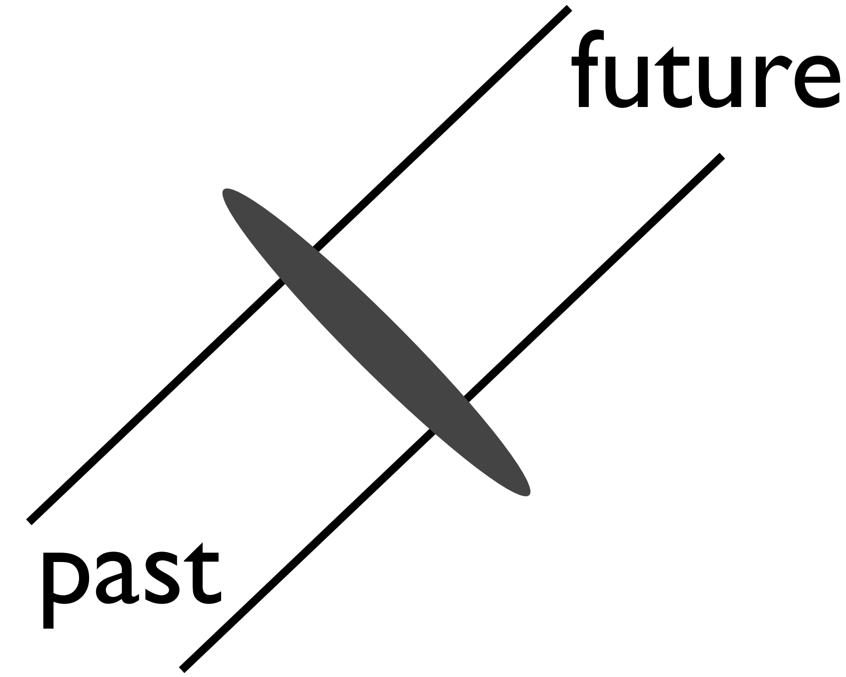
- Partial NNLL results

[SCH&Herranen ’16; Henn& Mistlberger ’17; SCH,Gardi&Vernazza ’17]

*Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov& Kovner

Main feature: index contractions preserve two global symmetries:

$$SU(N)_{past} \times SU(N)_{future}$$



Spontaneously broken to diagonal in vacuum:

$$\langle 0|U(x_\perp)|0\rangle = 1$$

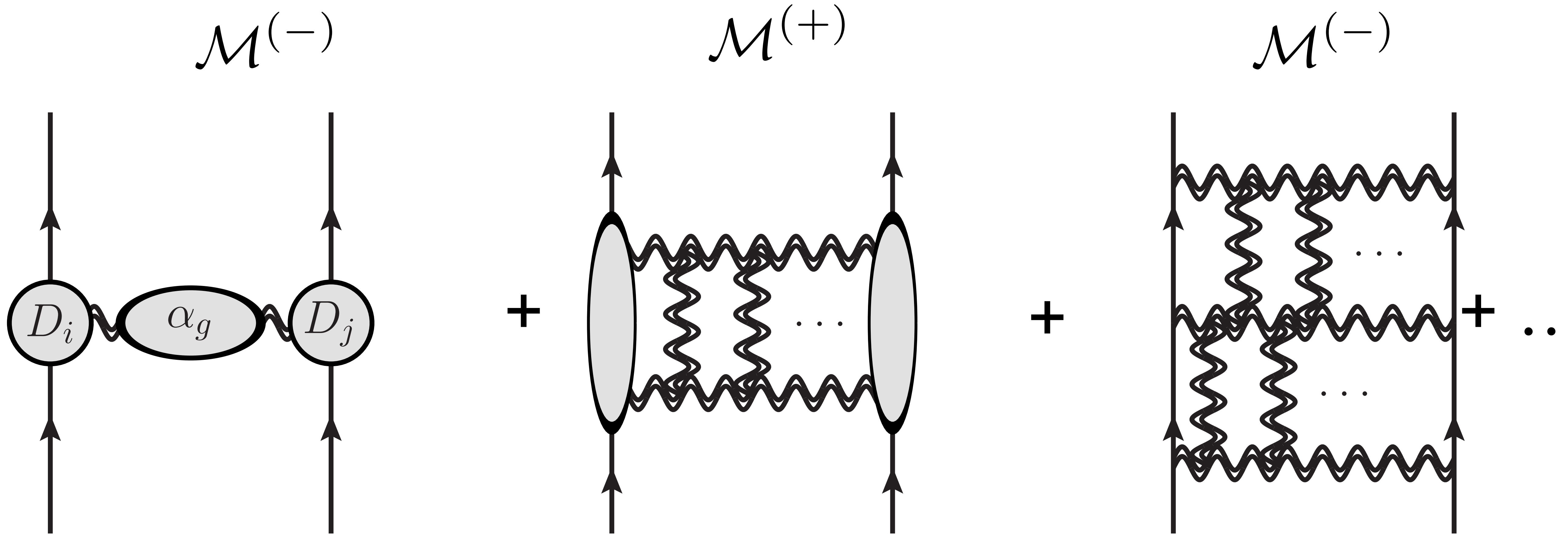
'Goldstone boson' W = Reggeized gluon

[Kovner& Lublinsky '05]
[SCH '13]

$$U(x_\perp) = e^{igT^a W^a(x_\perp)}$$

BFKL : expand in W's and study linearized evolution

Multi-Regge exchanges are suppressed by coupling



LL = one-Reggeon (one-W exchange)

NLL (two W's)

NNLL

$$A_{\rm LL} \propto \frac{s^{\alpha_g(t)}}{t}$$

'Regge pole'

$$A_{\rm NLL} \propto \int d\nu c(\nu) s^{E(\nu)}$$

'Regge cut'

Perturbative structure of the BFKL Hamiltonian

$$e^{igW^aT^a} \sim 1 + igWT + \ldots$$

n→n+k transitions: from LO B-JIMWLK

$$\frac{d}{d\eta}\begin{pmatrix} (W)^1 \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ \cdots \end{pmatrix} = \begin{pmatrix} g^2 & 0 & g^4 & 0 & g^6 & \cdots \\ 0 & g^2 & 0 & g^4 & & \\ g^4 & 0 & g^2 & 0 & \cdots & \\ 0 & g^4 & 0 & g^2 & & \\ & & & \cdots & & \end{pmatrix} \cdot \begin{pmatrix} (W)^1 \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ \cdots \end{pmatrix}$$

Leading BFKL and BKP kernels

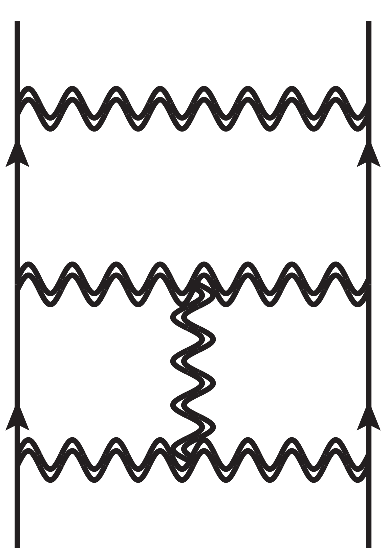
required by symmetry of d/dη

- Matrix is symmetrical: projectile/target symmetry
- Growth/saturation: off-diagonal can't be ignored
- ('Reggeon field theory' which resums all, still elusive)

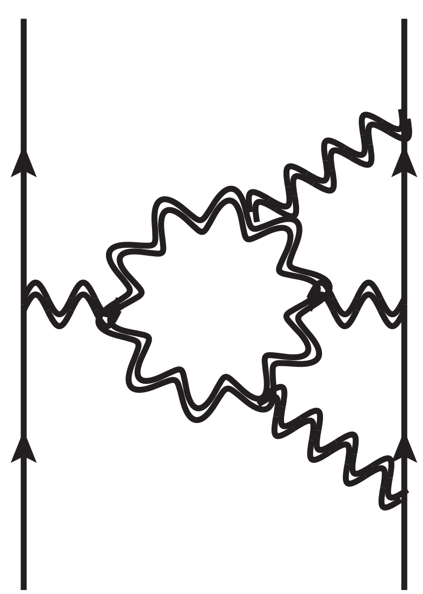
- At NNLL, something new happens: 1 and 3 Reggeon states mix
- @ 2-loops: violation of Regge pole factorization

[Del Duca, Falcioni, Magnea & Vernazza '14]

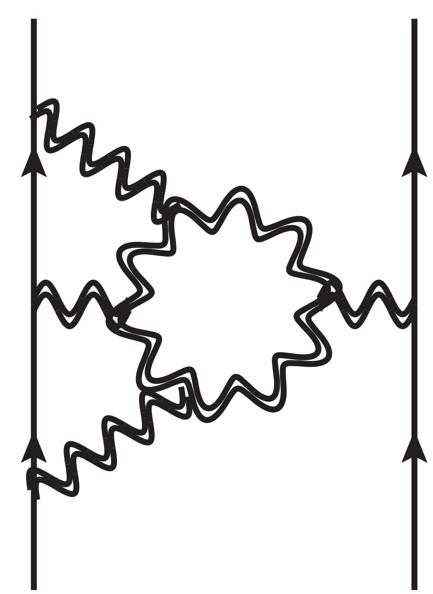
- @ 3-loops: first check of mixing matrix



$H_{3\to 3}$

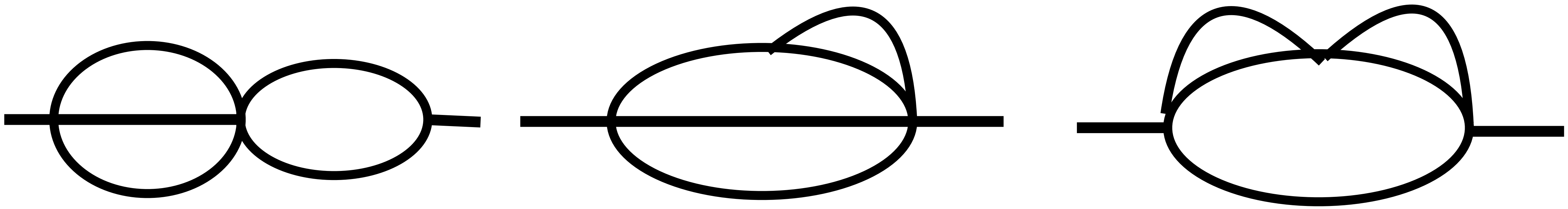


$H_{1\to 3}$



$H_{3\to 1}$

We don't actually compute these diagrams:
the LO B-JIMWLK Hamiltonian gives us simple 2d integrals



all the work is to find the color factors that multiply them, starting from the Hamiltonian.

$$H_{k\to k+2} = \frac{\alpha_s^2}{3\pi}\int [dz_i][dz_0]\, K_{ii;0}\, (W_i - W_0)^x W_0^y (W_i - W_0)^z \,\mathrm{Tr}\big[F^x F^y F^z F^a\big] \frac{\delta}{\delta W_i^a} \qquad (3.11)$$

$$+ \frac{\alpha_s^2}{6\pi}\int [dz_i][dz_j][dz_0]\, K_{ij;0}\, (F^x F^y F^z F^t)^{ab} \Big[(W_i - W_0)^x W_0^y W_0^z (W_j - W_0)^t$$

$$- W_i^x (W_i - W_0)^y W_0^z (W_j - W_0)^t - (W_i - W_0)^x W_0^y (W_j - W_0)^z W_j^t \Big] \frac{\delta^2}{\delta W_i^a \delta W_j^b}.$$

result for 2loops NNLL, in any gauge theory:

$$\hat{\mathcal{M}}^{(-,2)}_{ij\to ij} = \left[D^{(2)}_i + D^{(2)}_j + D^{(1)}_i D^{(1)}_j + \pi^2 R^{(2)}\left((\mathbf{T}^2_{s-u})^2 - \tfrac{1}{12}(C_A)^2\right)\right]\hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

$$R^{(2)} = (r_\Gamma)^2\left(-\frac{1}{8\epsilon^2} + \frac{3}{4}\epsilon\zeta_3 + \frac{9}{8}\epsilon^2\zeta_4 + \ldots\right)$$

Color operator precisely corrects factorization violation! ✔

at 3-loops NNLL, we computed coefficients of 3 color structures:

$$\hat{\mathcal{M}}^{(-,3,1)}_{ij\to ij} = \pi^2\left(R^{(3)}_A\,\mathbf{T}^2_{s-u}[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}] + R^{(3)}_B\,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]\mathbf{T}^2_{s-u} + R^{(3)}_C\,(C_A)^3\right)\hat{\mathcal{M}}^{(0)}_{ij\to ij}$$

$$R^{(3)}_A = \frac{1}{16}(r_\Gamma)^3(\mathcal{I}_a - \mathcal{I}_c) = (r_\Gamma)^3\left(\frac{1}{48\epsilon^3} + \frac{37}{24}\zeta_3 + \ldots\right)$$

Poles are consistent with IR exponentiation! ✔

Removing the 'hat' requires the 3-loop gluon Regge trajectory: $H_{1\to 1}$, which affects only the R_C color structure.

In N=4, we could fix it from [Henn& Mistlberger '16]

$$H^{(3)}_{1\to 1} = N_c^2\left[-\frac{\zeta_2}{144}\frac{1}{\epsilon^3}+\frac{49\zeta_4}{192}\frac{1}{\epsilon}+\frac{107}{144}\zeta_2\zeta_3+\frac{\zeta_5}{4}+\mathcal{O}(\epsilon)\right]+N_c^0\left[0+\mathcal{O}(\epsilon)\right]$$

Upshot:

-new 3loop prediction in QCD, up to one constant

-machine set up to compute NNLL $M^{(-)}$ to 4&higher loops, modulo same constant.

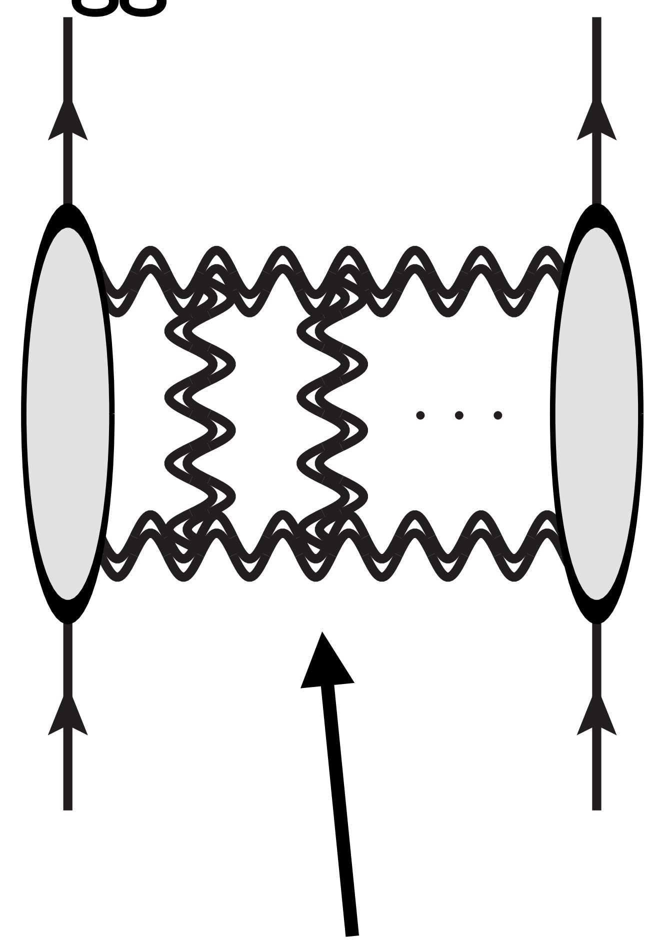
In N=4, that constant is known.

1701.05241

Back to NLL: high-order Solution in the soft limit

two-Reggeon exchange $(\sim \alpha_s^\ell L^{\ell-1})$

$\mathcal{M}^{(+)}$ =



leading order evolution

[1711.04850]

Each rung = the BFKL Hamiltonian $H_{2\to 2}$

$$\hat{H} = (2C_A - \mathbf{T}_t^2)\,\hat{H}_{\rm i} + (C_A - \mathbf{T}_t^2)\,\hat{H}_{\rm m}$$

‘integration’ part:

$$\hat{H}_i \Psi(p,k) = \int \frac{d^{2-2\epsilon}k'}{r_\Gamma (2\pi)^{2-2\epsilon}} f(p,k,k')\left[\Psi(p,k') - \Psi(p,k)\right]$$

‘multiplication’ part:

$$\hat{H}_m \Psi(p,k) = \frac{1}{2\epsilon}\left[2 - \left(\frac{p^2}{k^2}\right)^{\epsilon} - \left(\frac{p^2}{(p-k)^2}\right)^{\epsilon}\right]\Psi(p,k)$$

Both increase transcendental weight by 1

Evolution equation: $\Psi^{(\ell)} = \hat{H}\Psi^{(\ell-1)}, \quad \Psi^{(0)} = 1.$

Exact solution in adjoint channel: $\Psi = 1$

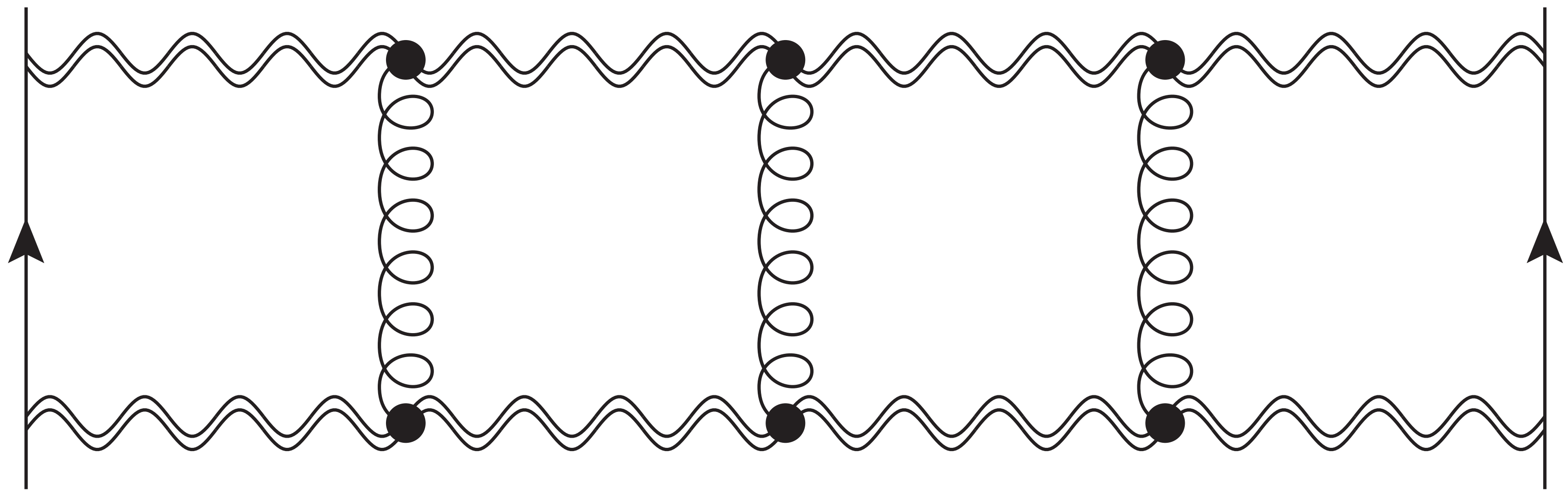
Cases where eigenfunctions are known: [Lipatov]

- Color singlet dipoles (x-space conformal symmetry)
- Color adjoint (p-space 'dual' conformal symmetry)

Unfortunately, for d≠4 / other color reps., eigenfunctions are not known

$\Rightarrow$ iterative solution

Outermost rungs are always easy (multiplication)



4-loop = single nontrivial integral

$$\hat{\mathcal{M}}^{(+,4)}_{\rm NLL} = -i\pi\frac{(B_0)^4}{3!}\int[{\rm D}k]\frac{p^2}{k^2(p-k)^2}\Big\{(C_A-\mathbf{T}_t^2)^3\,\Omega_{\rm mmm}(p,k)$$
$$+(2C_A-\mathbf{T}_t^2)(C_A-\mathbf{T}_t^2)^2\,\Omega_{\rm mim}(p,k)\Big\}\,\mathbf{T}^2_{s-u}\,\mathcal{M}^{(\rm tree)}$$
$$= i\pi\,\frac{(B_0)^4}{4!}\Big\{(C_A-\mathbf{T}_t^2)^3\left(\frac{1}{(2\epsilon)^4}+\frac{175\zeta_5}{2}\epsilon+\mathcal{O}(\epsilon^2)\right) \tag{2.32}$$
$$+C_A(C_A-\mathbf{T}_t^2)^2\left(-\frac{\zeta_3}{8\epsilon}-\frac{3}{16}\zeta_4-\frac{167\zeta_5}{8}\epsilon+\mathcal{O}(\epsilon^2)\right)\Big\}\,\mathbf{T}^2_{s-u}\,\mathcal{M}^{(\rm tree)}.$$

[SCH '13]

Note this has both leading& subleading IR divergences

How to predict the IR divergences at higher-loops?

Facts:

1. Wavefunction $\Psi^{(\ell)}(p,k)$ is finite as $\epsilon \to 0$

   $\Rightarrow$poles can only appear from final integration

$$\int_{k\to 0} \Psi^{(\ell)}(p,k)$$

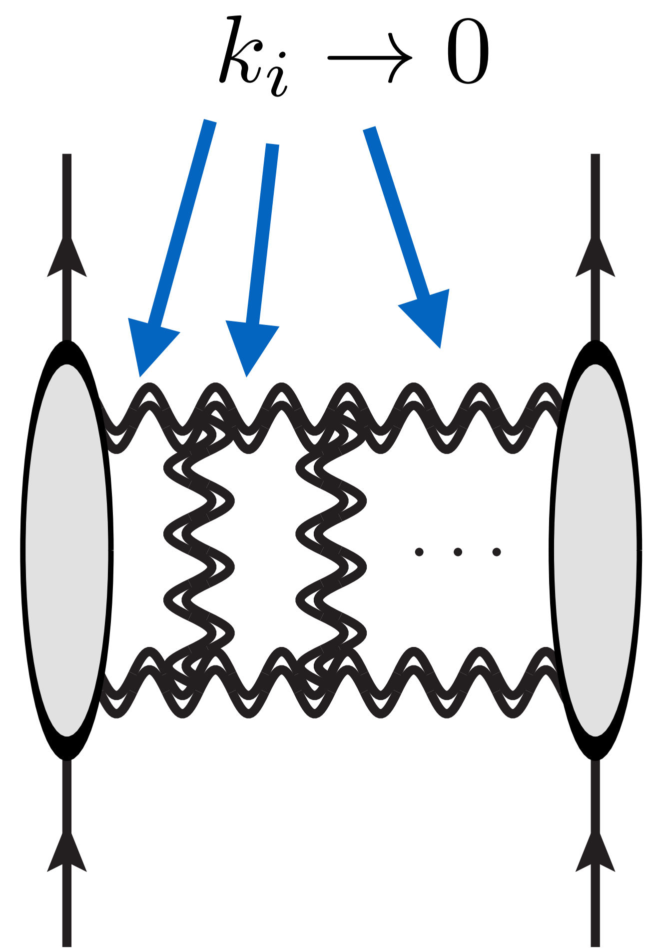
2. Evolution closes in soft limit:

$$\lim_{k\to 0} \psi^{(\ell)}(p,k) \sim \hat{H} \lim_{k\to 0} \psi^{(\ell-1)}(p,k)$$

IR divergences only occur when a full rail goes soft!

$$\hat{H}_i \left(\frac{p^2}{k^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\frac{B_n(\epsilon)}{B_0(\epsilon)}\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}$$

Gamma-functions



$$\hat{H}_m \left(\frac{p^2}{k^2}\right)^{n\epsilon} = \frac{1}{2\epsilon}\left[\left(\frac{p^2}{k^2}\right)^{n\epsilon} - \left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}\right]$$

⇒Soft wave function = polynomial in $\left(\frac{p^2}{k^2}\right)^{\epsilon}$

The soft wavefunction can be easily computed to all orders, and integrated to order $O(\epsilon^0)$

get truckload of Gamma-functions:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\text{NLL}}\Big|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} (1-\hat{B}_{-1}) (C_A - \mathbf{T}_t^2)^{\ell-1} \sum_{n=1}^{\ell} (-1)^{n+1} \binom{\ell}{n} \times \prod_{m=0}^{n-2} \left[1 - \hat{B}_m(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}\right] \mathbf{T}^2_{s-u} \mathcal{M}^{(\text{tree})} + \mathcal{O}(\epsilon^0),$$

However, $\epsilon \to 0$ is not random: WF has to be finite

Whole thing reducible to a geometric series!

$$\hat{\mathcal{M}}^{(+,\ell)}_{\text{NLL}}\Big|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} (1-\hat{B}_{-1}) \left(1 - \hat{B}_{-1}(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}\right)^{-1} \times (C_A - \mathbf{T}_t^2)^{\ell-1} \mathbf{T}^2_{s-u} \mathcal{M}^{(\text{tree})} + \mathcal{O}(\epsilon^0).$$

$$\bar{\mathcal{M}}^{(+,1)}_{\rm NLL} = i\pi \left[\frac{1}{2\epsilon} + O(\epsilon^0) \right] \mathbf{T}^2_{s-u} \mathcal{M}^{(\rm tree)},$$

$$\bar{\mathcal{M}}^{(+,2)}_{\rm NLL} = i\pi \frac{(C_A - \mathbf{T}_t^2)}{2!} \left[\frac{1}{(2\epsilon)^2} + O(\epsilon^0) \right] \mathbf{T}^2_{s-u} \mathcal{M}^{(\rm tree)},$$

$$\bar{\mathcal{M}}^{(+,3)}_{\rm NLL} = i\pi \frac{(C_A - \mathbf{T}_t^2)^2}{3!} \left[\frac{1}{(2\epsilon)^3} + O(\epsilon^0) \right] \mathbf{T}^2_{s-u} \mathcal{M}^{(\rm tree)},$$

$$\bar{\mathcal{M}}^{(+,4)}_{\rm NLL} = i\pi \frac{(C_A - \mathbf{T}_t^2)^3}{4!} \left[\frac{1}{(2\epsilon)^4} - \frac{1}{2\epsilon} \frac{\zeta_3 C_A}{4(C_A - \mathbf{T}_t^2)} + O(\epsilon^0) \right] \mathbf{T}^2_{s-u} \mathcal{M}^{(\rm tree)},$$

$$\bar{\mathcal{M}}^{(+,5)}_{\rm NLL} = i\pi \frac{(C_A - \mathbf{T}_t^2)^4}{5!} \left[\frac{1}{(2\epsilon)^5} - \frac{1}{(2\epsilon)^2} \frac{\zeta_3 C_A}{4(C_A - \mathbf{T}_t^2)} - \frac{1}{2\epsilon} \frac{3\zeta_4 C_A}{16(C_A - \mathbf{T}_t^2)} + O(\epsilon^0) \right] \mathbf{T}^2_{s-u} \mathcal{M}^{(\rm tree)}.$$

iteration of lower loops

single poles = soft anomalous dimension

Recall exponentiation of IR divergences:

$$\mathcal{H} = Z_{\rm IR}\mathcal{M}\,, \qquad Z_{\rm IR} = \mathcal{P}e^{-\int_0^\mu \frac{d\lambda}{\lambda}\Gamma_{\rm s}(\alpha_s(\lambda))}$$

H= IR&UV renormalized scattering = Finite as $\varepsilon \to 0$

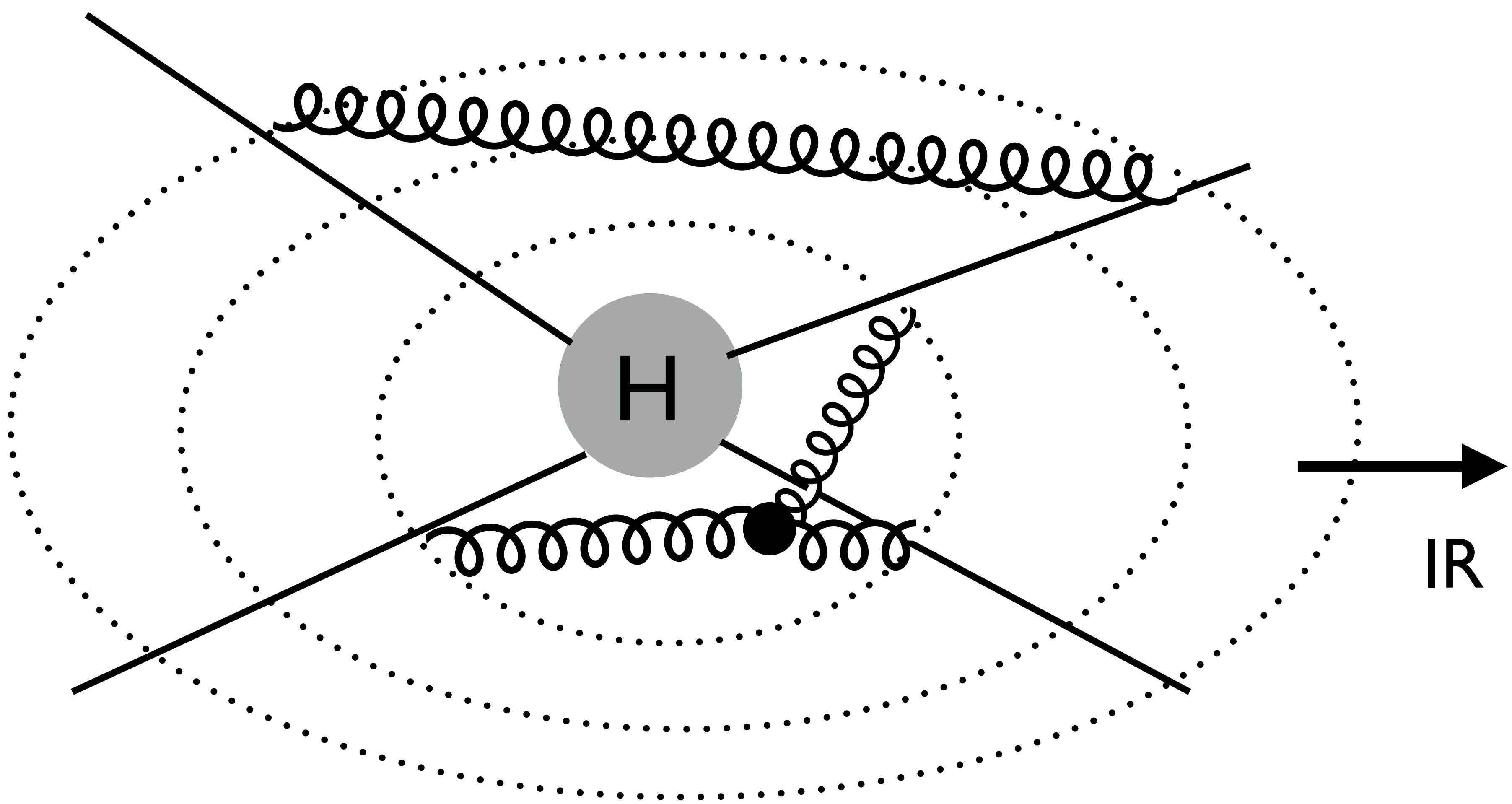
Note that $\varepsilon \to 0$ limit of *H* and $\Gamma_{\rm s}$ contain all physically observable part of S-matrix [Weinzeirl]

(these suffice to compute inclusive cross-sections, when using suitable phase-space subtractions: cf Lorenzo's talk)

Notice similarity when renormalizing UV&IR operators

$$\mathcal{O}_{\rm ren}(x) = Z_{\rm UV}\mathcal{O}_{\rm bare}(x), \qquad Z_{\rm UV} = \mathcal{P}e^{\int_\mu^\infty \frac{d\lambda}{\lambda}\gamma(\alpha_s(\lambda))}$$

$$\mathcal{H} = Z_{\rm IR}\mathcal{M}_{\rm IR-bare} \qquad Z_{\rm IR} = \mathcal{P}e^{-\int_0^\mu \frac{d\lambda}{\lambda}\Gamma_{\rm s}(\alpha_s(\lambda))}$$



Both exponentiate for same reason:
disparate length scales factorize from each other

$$\Gamma_s = \sum_{i\neq j} \frac{\gamma_K(\alpha_s(\lambda))}{4} \log \frac{-s_{ij}}{\lambda^2} T_i^a T_j^a - \sum_i \gamma_{J_i}(\alpha_s(\lambda)) + \Delta$$

dipole ansatz

departure, starts at 3-loops

[Gardi&Magnea; Neubert&Becher '09]
[Almelid,Duhr&Gardi '15]

Can be expanded in Regge limit:

$$\mathbf{\Gamma}\left(\alpha_s(\lambda)\right) = \mathbf{\Gamma}_{\mathrm{LL}}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\mathrm{NLL}}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\mathrm{NNLL}}\left(\alpha_s(\lambda), L\right) + \ldots$$

At LL, gluon Reggeization fixes Γ_s from gluon trajectory:

$$\mathbf{\Gamma}_{\mathrm{LL}}\left(\alpha_s(\lambda)\right) = \frac{\alpha_s(\lambda)}{\pi} \frac{\gamma_K^{(1)}}{2} L\, \mathbf{T}_t^2 = \frac{\alpha_s(\lambda)}{\pi} L\, \mathbf{T}_t^2 \, .$$

[Del Duca, Duhr, Gardi, Magnea& White '11]

The LL Z-factor is a simple exponential:

$$\mathbf{Z}_{\mathrm{LL}}^{(+)}\left(\frac{s}{t},\mu,\alpha_s(\mu)\right)=\exp\left\{\frac{\alpha_s}{\pi}\frac{1}{2\epsilon}L\mathbf{T}_t^2\right\}\simeq s^{\frac{\alpha_s C_A}{2\pi\epsilon}}$$

NLL = perturbation around that

$$\hat{\mathcal{M}}_{\mathrm{NLL}}^{(+)}=\exp\left\{-\frac{\alpha_s(\mu)}{\pi}\frac{B_0(\epsilon)}{2\epsilon}L\mathbf{T}_t^2\right\}\left[\mathbf{Z}_{\mathrm{NLL}}^{(-)}\left(\frac{s}{t},\mu,\alpha_s(\mu)\right)\mathcal{H}_{\mathrm{LL}}^{(-)}\left(\{p_i\},\mu,\alpha_s(\mu)\right)\right.$$
$$\left.+\mathbf{Z}_{\mathrm{LL}}^{(+)}\left(\frac{s}{t},\mu,\alpha_s(\mu)\right)\mathcal{H}_{\mathrm{NLL}}^{(+)}\left(\{p_i\},\mu,\alpha_s(\mu)\right)\right]$$

no poles

$$=-\int_0^p\frac{d\lambda}{\lambda}\exp\left\{\frac{1}{2\epsilon}\frac{\alpha_s(p)}{\pi}L(C_A-\mathbf{T}_t^2)\left[1-\left(\frac{p^2}{\lambda^2}\right)^{\epsilon}\right]\right\}\mathbf{\Gamma}_{\mathrm{NLL}}^{(-)}\left(\alpha_s(\lambda)\right)\mathcal{M}^{(\mathrm{tree})}+\mathcal{O}(\epsilon^0).$$

⇒ single-poles give Γ_{NLL}, higher poles explicitly predicted

All-order result:

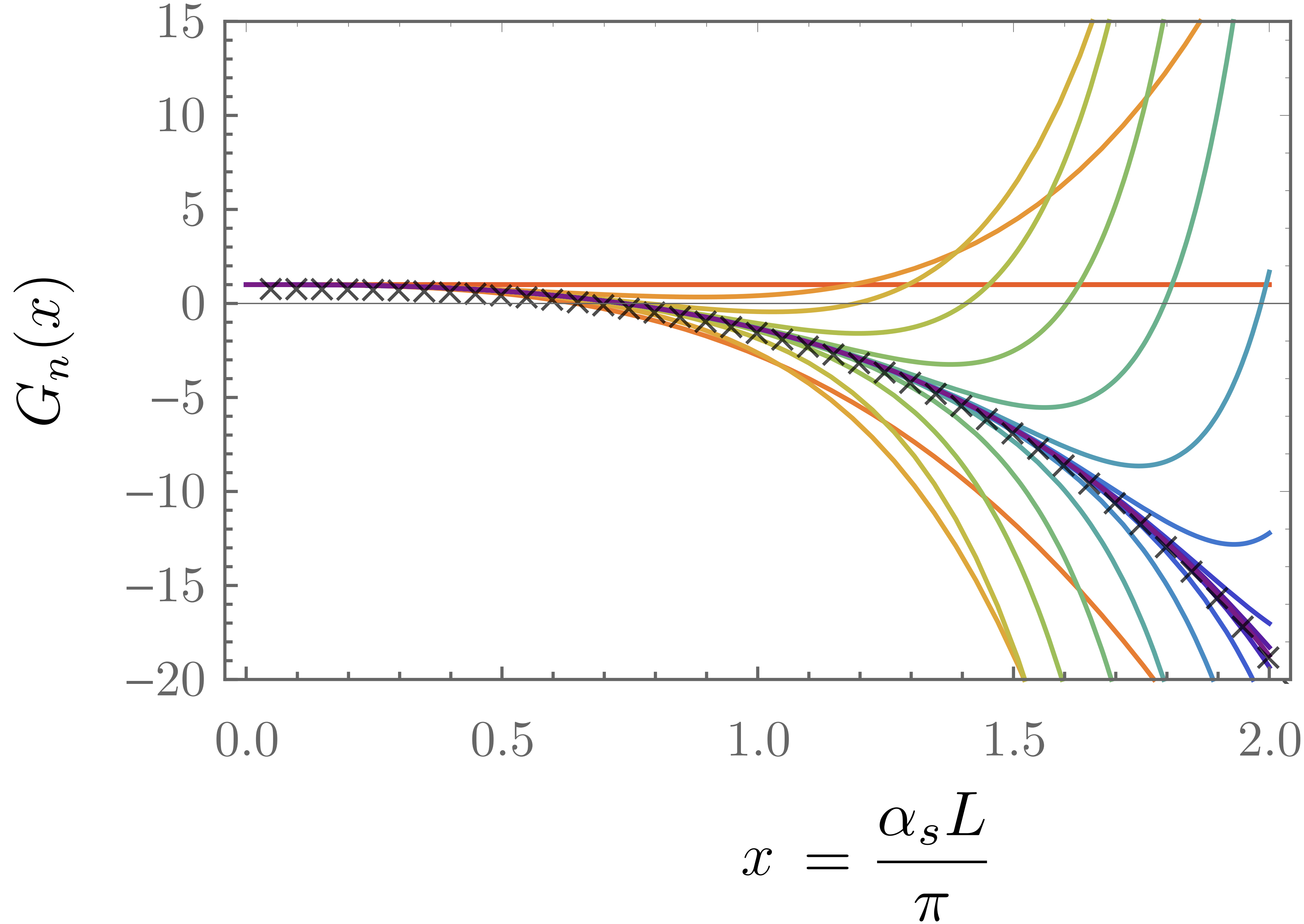
$$\mathbf{\Gamma}_{\rm NLL}^{(-,\ell)} = \frac{i\pi}{(\ell-1)!}\left(1-\frac{C_A}{C_A-\mathbf{T}_t^2}R\left(x(C_A-\mathbf{T}_t^2)/2\right)\right)^{-1}\Bigg|_{x^{\ell-1}}\mathbf{T}_{s-u}^2\,.$$

$$\begin{aligned} R(\epsilon) &= \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 \\ &= -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(10\zeta_6 - 2\zeta_3^2\right)\epsilon^6 + \mathcal{O}(\epsilon^7) \end{aligned}$$

$$
\begin{aligned}
\mathbf{\Gamma}_{\rm NLL}^{(-,1)} &= i\pi\, \mathbf{T}_{s-u}^2 \\
\mathbf{\Gamma}_{\rm NLL}^{(-,2)} &= 0 \\
\mathbf{\Gamma}_{\rm NLL}^{(-,3)} &= 0, \\
\mathbf{\Gamma}_{\rm NLL}^{(-,4)} &= -i\pi\, \frac{\zeta_3}{24}\, C_A (C_A - \mathbf{T}_t^2)^2\, \mathbf{T}_{s-u}^2, \\
\mathbf{\Gamma}_{\rm NLL}^{(-,5)} &= -i\pi\, \frac{\zeta_4}{128}\, C_A (C_A - \mathbf{T}_t^2)^3\, \mathbf{T}_{s-u}^2, \\
\mathbf{\Gamma}_{\rm NLL}^{(-,6)} &= -i\pi\, \frac{\zeta_5}{640}\, C_A (C_A - \mathbf{T}_t^2)^4\, \mathbf{T}_{s-u}^2, \\
\mathbf{\Gamma}_{\rm NLL}^{(-,7)} &= i\pi \frac{1}{720} \left[\frac{\zeta_3^2}{16}\, C_A^2 (C_A - \mathbf{T}_t^2)^4 + \frac{1}{32} \left(\zeta_3^2 - 5\zeta_6 \right) C_A (C_A - \mathbf{T}_t^2)^5 \right] \mathbf{T}_{s-u}^2, \\
\mathbf{\Gamma}_{\rm NLL}^{(-,8)} &= i\pi \frac{1}{5040} \left[\frac{3\zeta_3\zeta_4}{32}\, C_A^2 (C_A - \mathbf{T}_t^2)^5 + \frac{3}{64} \left(\zeta_3\zeta_4 - 3\zeta_7 \right) C_A (C_A - \mathbf{T}_t^2)^6 \right] \mathbf{T}_{s-u}^2. \\
&\ldots
\end{aligned}
$$

1. only classical zeta's, no zeta$_2$.
2. Coefficients decay factorially

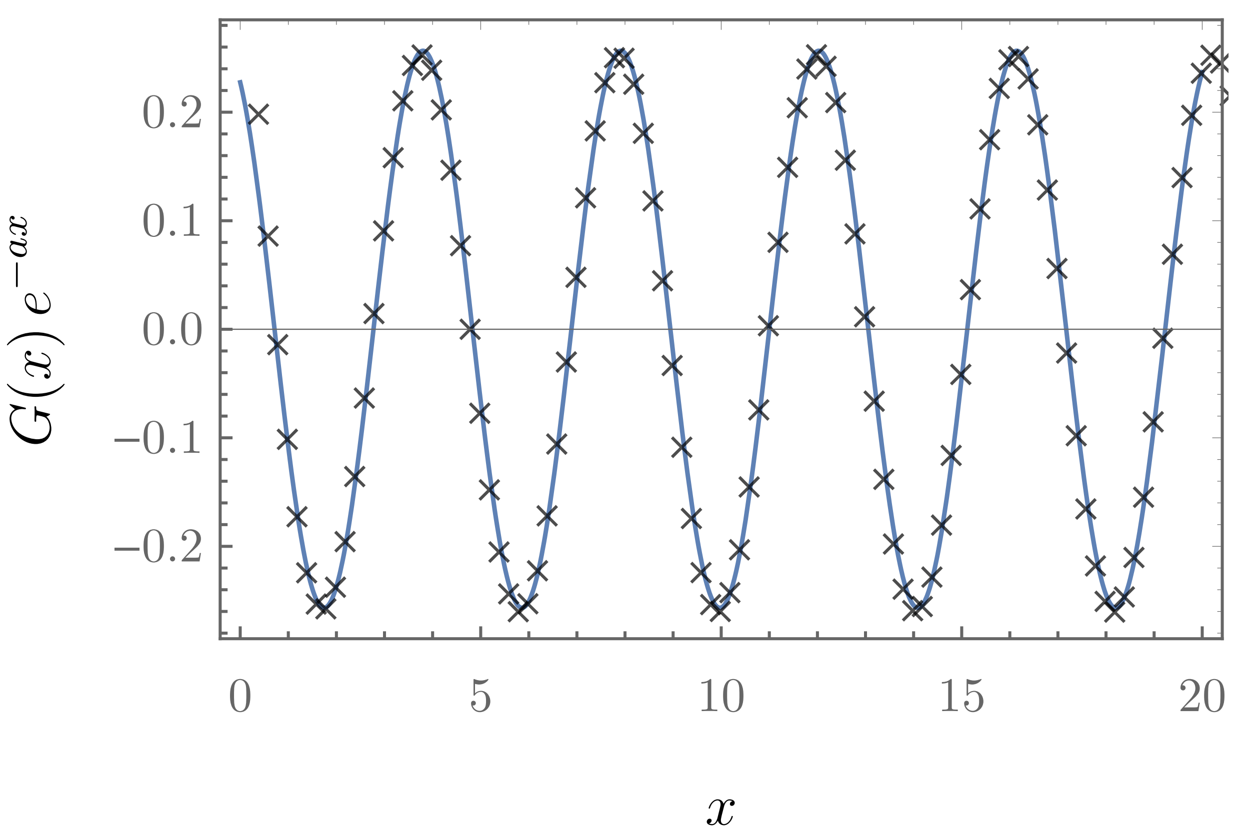
$$\Gamma^{(-)}_{\rm NLL} = i\pi \frac{\alpha_s}{\pi}\, G\left(\frac{\alpha_s}{\pi} L\right) \mathbf{T}^2_{s-u}$$ is entire function

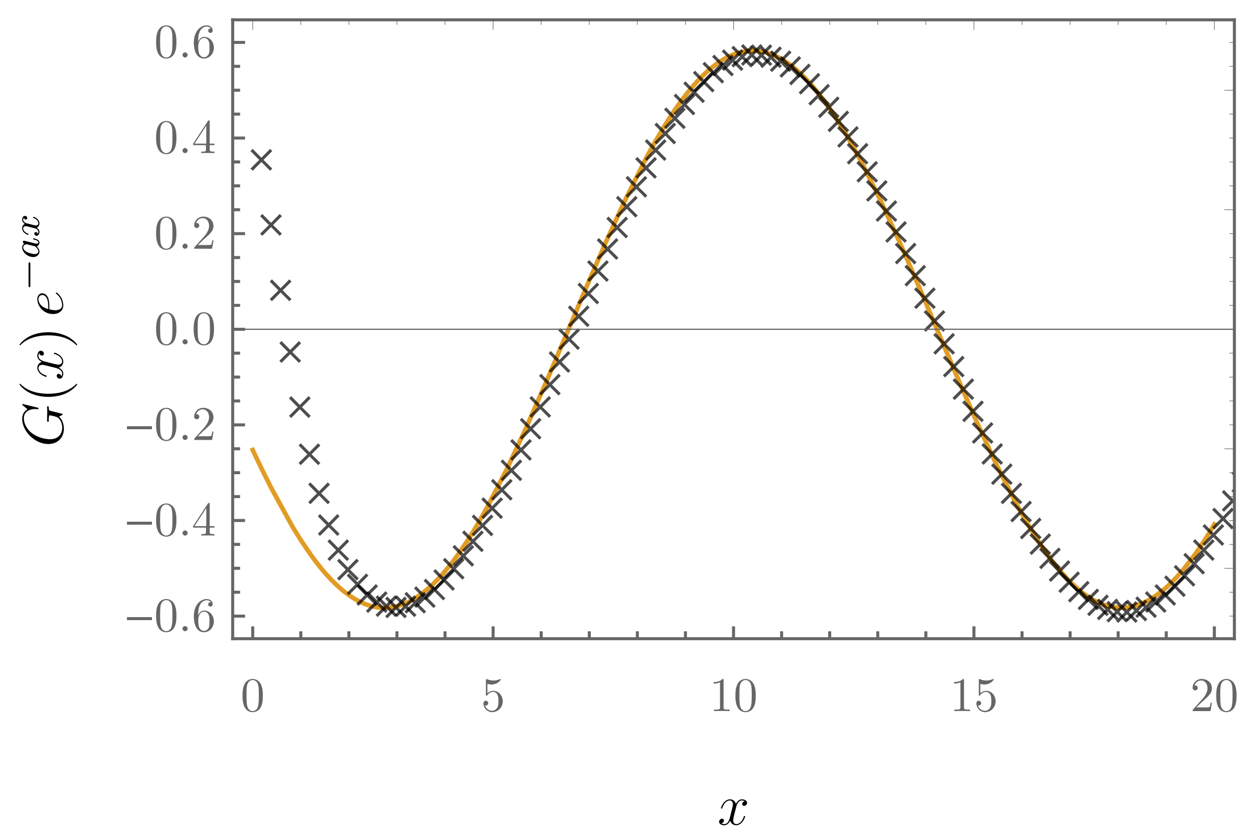


Efficient evaluation via inverse Borel:

$$G(x) = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} d\eta \, g\left(\frac{1}{\eta}\right) \mathrm{e}^{\eta x}$$

Asymptotics: $G(x) \to c\, e^{ax} \cos\left(bx + d\right)$





Note: sign of Γ itself is dominated by Γ_{LL}

Finite part

Recall all physical info is in $\varepsilon \to 0$ limit of H and Γ_s

$$\mathcal{H} = Z_{\mathrm{IR}} \mathcal{M}$$

BFKL ladders

fully understood @ NLL

??

Claim: $\varepsilon\to 0$ limit determined from evolution with $\varepsilon=0$

$$\mathcal{H}^{(+)}_{\mathrm{NLL}} = \int_{k\ \mathrm{soft}} d^{2-2\epsilon}k\Psi(p,k) - (\mathrm{subtractions})$$
$$+ \int_{k\ \mathrm{hard}} d^2 k\Psi(p,k)\Big|_{\epsilon=0}$$

First line computable using soft limit of wavefunction in d dimensions

Second line: wavefunction =sum of SVHPLs

In principle, we would like to diagonalize H:

$$\hat{H} = (2C_A - \mathbf{T}_t^2)\,\hat{H}_\mathrm{i} + (C_A - \mathbf{T}_t^2)\,\hat{H}_\mathrm{m}$$

'integration' & multiplication parts:

$$\hat{H}_\mathrm{i}\psi(z,\bar{z}) = \frac{1}{4\pi}\int \mathrm{d}^2 w K(w,\bar{w},z,\bar{z})\,[\psi(w,\bar{w}) - \psi(z,\bar{z})]$$

$$\hat{H}_\mathrm{m}\psi(z,\bar{z}) = j(z,\bar{z})\psi(z,\bar{z})$$

simple kernels: $j(z,\bar{z}) = \frac{1}{2}\log\left[\frac{z}{(1-z)^2}\frac{\bar{z}}{(1-\bar{z})^2}\right]$

$$K(w,\bar{w},z,\bar{z}) = \frac{1}{\bar{w}(z-w)} + \frac{2}{(z-w)(\bar{z}-\bar{w})} + \frac{1}{w(\bar{z}-\bar{w})}$$

It turns out we can 'integrate-by-parts' derivatives without changing kernel

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left[\hat{H}_{\mathrm{i}}\Psi(z,\bar{z})\right] = \hat{H}_{\mathrm{i}}\left[z\frac{\mathrm{d}}{\mathrm{d}z}\Psi(z,\bar{z})\right]$$

(full algorithm requires (1-z)d/dz, just a bit harder)

That way we easily generate SVHPL expressions

$$\left\{\mathrm{wf}(1) \to \tfrac{1}{2}\mathrm{c2}(L(\{0\}) + 2L(\{1\}))\right\}$$

$$\left\{\mathrm{wf}(2) \to \tfrac{1}{4}\mathrm{c1c2}(-L(\{0,1\}) - L(\{1,0\}) - 2L(\{1,1\})) + \tfrac{1}{2}\mathrm{c2}^2(L(\{0,0\}) + 2L(\{0,1\}) + 2L(\{1,0\}) + 4L(\{1,1\}))\right\}$$

we can get the IR renormalized amplitude to very high order

$$\text{Mfinite}(1) = 0$$

$$\text{Mfinite}(2) = 0$$

$$\text{Mfinite}(3) = \frac{1}{4}(-11)\text{c}2^2\zeta(3)$$

$$\cdots$$

At 11-loops, we do get $SVZ_{5,3,3}$

Coefficient grow exponentially:
finite radius of convergence in $\alpha_s L$

series seems alternating, for unitary representations $Tt^2>0$

Conclusions

- Modern approach to high-energy scattering via Wilson lines: new theoretical control @NNLL
- Systematic and now well-tested theory, simplifies and exponentiate many diagrams in the forward limit
- Possible applications to
  -Mueller Navelet jets, small-x physics
  -Predictions and new techniques for fixed-order multi-loop QCD computations