

# IR structure of two-loop four-gluon amplitude in $\mathcal{N} = 2$ SQCD

ongoing work with Gregor KÄLIN and Gustav MOGULL

Alexander OCHIROV  
ETH Zürich

Amplitudes in the LHC era,  
GGI, Florence, October 26, 2018

# Invitation

Two-loop amplitudes  $\mathcal{A} = \int \mathcal{I}$  beyond  $2 \rightarrow 2$

- ▶ 5-gluon all-plus

$\mathcal{I}$  : Badger, Frellesvig, Zhang (2013)  
 $\mathcal{I}$  : Badger, Mogull, AO, O'Connell (2015)  
 $\mathcal{A}$  : Gehrmann, Henn, Lo Presti (2015)

- ▶  $n$ -gluon all-plus

$\mathcal{I}$  : Badger, Mogull, Peraro (2016)  
 $\mathcal{A}$  : Dunbar, Jehu, Perkins (2016)

- ▶ 5-gluon all helicities ( $\mathcal{I}$  implicit, numerical  $\int$ )

$\mathcal{A}$  : Badger, Bronnum-Hansen, Hartanto, Peraro (2017)  
 $\mathcal{A}$  : Abreu, Febres Cordero, Ita, Page, Zeng (2017)

- ▶ 5-point general massless QCD

talks by Peraro and Zeng

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Analytic wishlist:

- ▶ control complexity of  $\mathcal{I} \approx$  complexity of  $\mathcal{A}$
- ▶ understand provenance of IR divergences through  $\mathcal{O}(\epsilon^{-2L})$

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**This talk:**

- ▶ inspect 4-pt integrand in  $\mathcal{N} = 2$  SQCD (full color and  $N_f$ )
- ▶ analytic IR+UV structure at  $\mathcal{O}(\epsilon^{-4})$ ,  $\mathcal{O}(\epsilon^{-3})$  and  $\mathcal{O}(\epsilon^{-2})$

# Outline

1. IR factorization review
2.  $\mathcal{N} = 2$  integrand
3. Analytic IR structure
4. Summary & outlook

# IR factorization review

# IR factorization

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Kunszt, Signer, Trocsanyi (1994)  
Catani, Seymour (1996)  
Catani (1998)

$$\widetilde{\mathcal{M}}_n^{(1)} = \mathbf{I}^{(1)}(\epsilon) \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^0)$$

$$\widetilde{\mathcal{M}}_n^{(2)} = \mathbf{I}^{(1)}(\epsilon) \widetilde{\mathcal{M}}_n^{(1)} + \mathbf{I}^{(2)}(\epsilon) \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^0)$$

$$\mathbf{I}^{(1)}(\epsilon) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_{i=1}^n \left( \frac{1}{\epsilon^2} - \frac{\gamma_i^{(1)}}{\mathbf{T}_i^2} \frac{1}{\epsilon} \right) \sum_{j \neq i}^n \left( \frac{-s_{ij}}{\mu^2} \right)^{-\epsilon} \mathbf{T}_i \cdot \mathbf{T}_j$$

$$\mathbf{I}^{(2)}(\epsilon) = -\frac{1}{2} \mathbf{I}^{(1)}(\epsilon) \left( \mathbf{I}^{(1)}(\epsilon) + \frac{2\beta_0}{\epsilon} \right) + \left( \frac{\beta_0}{\epsilon} + 2K_{\text{R.S.}} \right) \mathbf{I}^{(1)}(2\epsilon) + \mathcal{O}(\epsilon^{-1})$$

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\* More modern view — soft exponentiation

$$\widetilde{\mathcal{M}}_n(p_i, \mu, \alpha_s(\mu)) = \mathcal{P} \exp \left\{ - \int_0^\mu \frac{d\lambda}{\lambda} \mathbf{\Gamma} \left( \frac{p_i}{\lambda}, \alpha_s(\lambda) \right) \right\} \mathcal{H}_n(p_i, \mu, \alpha_s(\mu))$$

# IR factorization

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Kunszt, Signer, Trocsanyi (1994)  
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$$\widetilde{\mathcal{M}}_n^{(2)} = \mathbf{I}^{(1)}(\epsilon) \widetilde{\mathcal{M}}_n^{(1)} + \mathbf{I}^{(2)}(\epsilon) \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^0)$$

$$\mathbf{I}^{(1)}(\epsilon) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_{i=1}^n \left( \frac{1}{\epsilon^2} - \frac{\gamma_i^{(1)}}{\mathbf{T}_i^2} \frac{1}{\epsilon} \right) \sum_{j \neq i}^n \left( \frac{-s_{ij}}{\mu^2} \right)^{-\epsilon} \mathbf{T}_i \cdot \mathbf{T}_j$$

$$\mathbf{I}^{(2)}(\epsilon) = -\frac{1}{2} \mathbf{I}^{(1)}(\epsilon) \left( \mathbf{I}^{(1)}(\epsilon) + \frac{2\beta_0}{\epsilon} \right) + \left( \frac{\beta_0}{\epsilon} + 2K_{\text{R.S.}} \right) \mathbf{I}^{(1)}(2\epsilon) + \mathcal{O}(\epsilon^{-1})$$

NB!  $\mathcal{O}(\epsilon^{-2})$  and  $\mathcal{O}(\epsilon^{-1})$  depend on dimreg scheme, e.g.

$$K_{\text{FDH}} = \left( \frac{32-4\epsilon}{9} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} C_f n_f - \frac{6-2\epsilon}{27} C_s n_s$$

Bern, De Freitas, Dixon (2002)

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## Undoing renormalization

$$\widetilde{\mathcal{M}}_n^{(1)} = S_\epsilon^{-1} \mathcal{M}_n^{(1)} - \frac{(n-2)\beta_0}{2\epsilon} \mathcal{M}_n^{(0)}$$

$$\widetilde{\mathcal{M}}_n^{(2)} = S_\epsilon^{-2} \mathcal{M}_n^{(2)} - \frac{n\beta_0}{2\epsilon} S_\epsilon^{-1} \mathcal{M}_n^{(1)} + \frac{(n-2)}{2} \left[ \frac{n\beta_0^2}{4\epsilon^2} - \frac{\beta_1}{\epsilon} \right] \mathcal{M}_n^{(0)}$$

## Undoing renormalization

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Rearrange Catani into

$$\mathcal{M}_n^{(1)} = \mathbf{I}_\alpha^{(1)}(\epsilon) \mathcal{M}_n^{(0)} + \frac{1}{\epsilon} \left[ \frac{n-2}{2} \beta_0 + \sum_{i=1}^n \gamma_i^{(1)} \right] \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^0)$$

$$\mathcal{M}_n^{(2)} = \mathbf{I}_\alpha^{(1)}(\epsilon) \mathcal{M}_n^{(1)} - \frac{1}{2} \mathbf{I}_\alpha^{(1)}(\epsilon) \mathbf{I}_\alpha^{(1)}(\epsilon) \mathcal{M}_n^{(0)} + \left[ \frac{\beta_0}{\epsilon} + 2K_{\text{R.S.}} \right] \mathbf{I}_\alpha^{(1)}(2\epsilon) \mathcal{M}_n^{(0)}$$

$$+ \frac{1}{2\epsilon^2} \left[ \frac{n}{2} \beta_0 + \sum_{i=1}^n \gamma_i^{(1)} \right] \left[ \frac{n-2}{2} \beta_0 + \sum_{i=1}^n \gamma_i^{(1)} \right] \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^{-1})$$

$$\mathbf{I}_\alpha^{(1)}(\epsilon) = \frac{e^{\epsilon\gamma_E}}{\epsilon^2 \Gamma(1-\epsilon)} \sum_{i=1}^n \sum_{j \neq i}^n (-s_{ij})^{-\epsilon} \mathbf{T}_i \cdot \mathbf{T}_j \quad \text{NB! Mixed IR with UV}$$

## Assume IR factorization for $\mathcal{N} = 4$ SYM

- ▶ Specialize to ext. gluons  $\gamma_i^{(1)} = -\beta_0/2$
- ▶ Consider difference w.r.t.  $\mathcal{N} = 4$

$$\mathcal{M}_n^{(1)} - \mathcal{M}_n^{(1)[\mathcal{N}=4]} = -\frac{\beta_0}{\epsilon} \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^0)$$

## Assume IR factorization for $\mathcal{N} = 4$ SYM

- ▶ Specialize to ext. gluons  $\gamma_i^{(1)} = -\beta_0/2$
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$$\begin{aligned}\mathcal{M}_n^{(1)} - \mathcal{M}_n^{(1)[\mathcal{N}=4]} &= -\frac{\beta_0}{\epsilon} \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^0) \\ \mathcal{M}_n^{(2)} - \mathcal{M}_n^{(2)[\mathcal{N}=4]} &= \left( \sum_{i \neq j} s_{ij} \langle \! \! \! \langle \begin{array}{c} i \\ \mathbf{T}_i \cdot \mathbf{T}_j \end{array} \rangle \! \! \! \rangle \right) [\mathcal{M}_n^{(1)} - \mathcal{M}_n^{(1)[\mathcal{N}=4]}] \\ &\quad + \beta_0 \left( \sum_{i \neq j} s_{ij} \langle \! \! \! \langle \begin{array}{c} i \\ \mathbf{T}_i \cdot \mathbf{T}_j \end{array} \rangle \! \! \! \rangle \right) \mathcal{M}_n^{(0)} \\ &\quad + \frac{n N_c}{2 \epsilon^2} (\beta_0 - 2[K - K_{\mathcal{N}=4}]) \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^{-1})\end{aligned}$$

## IR factorization for $\mathcal{N} = 2$ SQCD

Specialize to  $\mathcal{N} = 2$ :

$$\beta_0 = 2N_c - N_f, \quad K_{\mathcal{N}=2} = K_{\mathcal{N}=4} + \frac{\beta_0}{2} + \mathcal{O}(\epsilon)$$

Singularity structure encoded by triangles through  $\mathcal{O}(\epsilon^{-2})$ :

$$\mathcal{M}_n^{(1)} - \mathcal{M}_n^{(1)[\mathcal{N}=4]} = -\frac{\beta_0}{\epsilon} \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^0)$$

$$\begin{aligned} \mathcal{M}_n^{(2)} - \mathcal{M}_n^{(2)[\mathcal{N}=4]} &= \left( \sum_{i \neq j} s_{ij} \triangleleft_j^i \mathbf{T}_i \cdot \mathbf{T}_j \right) [\mathcal{M}_n^{(1)} - \mathcal{M}_n^{(1)[\mathcal{N}=4]}] \\ &+ \beta_0 \left( \sum_{i \neq j} s_{ij} \triangleleft_j^i \mathbf{T}_i \cdot \mathbf{T}_j \right) \mathcal{M}_n^{(0)} + \mathcal{O}(\epsilon^{-1}) \end{aligned}$$

$\mathcal{N} = 2$  integrand

# $\mathcal{N} = 2$ supersymmetric QCD

| hel. \ th. | $\mathcal{N} = 4$ SYM | $\mathcal{N} = 2$ SQCD |   | QCD                                 |
|------------|-----------------------|------------------------|---|-------------------------------------|
| +1         | 1                     | 1                      | $\overbrace{N_f}$                       | 1 $\overbrace{N_f}$                 |
| +1/2       | 4                     | 2                      | $\begin{matrix} 1 & 1 \end{matrix}$     | $\begin{matrix} 1 & 1 \end{matrix}$ |
| 0          | 6                     | 1                      | $\begin{matrix} 1 & 2 & 2 \end{matrix}$ |                                     |
| -1/2       | 4                     |                        | $\begin{matrix} 2 & 1 & 1 \end{matrix}$ | $\begin{matrix} 1 & 1 \end{matrix}$ |
| -1         | 1                     |                        | 1                                       | 1                                   |
| rep.       | $G$                   | $G$                    | $N_c \bar{N}_c$                         | $G \ N_c \bar{N}_c$                 |

$$\alpha_s(\mu) = \frac{\mu_0^{2\epsilon}}{\mu^{2\epsilon}} \frac{\alpha_s(\mu_0)}{1 + \frac{\beta_0 \alpha_s(\mu_0)}{4\pi\epsilon} \left[ 1 - \frac{\mu_0^{2\epsilon}}{\mu^{2\epsilon}} \right]}, \quad \beta_0 = 2N_c - N_f$$

Seiberg (1988)

# One-loop $\mathcal{N} = 2$ integrand

Johansson, AO (2014)

Start with BCJ numerators  $n_i^{[\mathcal{N}=2,\text{pure}]} = n_i^{[\mathcal{N}=4]} - 2n_i^{[\mathcal{N}=2,\text{fund}]}$ ,  
 where  $n_i^{[\mathcal{N}=2,\text{fund}]}$  are

$$n \left( \begin{array}{c} 4 \\ \text{[Diagram: Box with internal lines]} \\ 3 \quad 2 \\ 1 \end{array} \right) = \frac{\kappa_{13}}{u^2} \text{tr}_- + \frac{\kappa_{24}}{u^2} \text{tr}_+ + \mu^2 \left( \frac{\kappa_{12} + \kappa_{34}}{s} + \frac{\kappa_{23} + \kappa_{14}}{t} + \frac{\kappa_{13} + \kappa_{24}}{u} \right)$$

$$n \left( \begin{array}{c} 4 \\ \text{[Diagram: Triangle with internal lines]} \\ 3 \quad 2 \\ 1 \end{array} \right) = \left( \frac{\kappa_{13}}{u^2} + \frac{\kappa_{23}}{t^2} \right) \text{tr}_- + \left( \frac{\kappa_{24}}{u^2} + \frac{\kappa_{14}}{t^2} \right) \text{tr}_+ + \frac{s}{t^2} (\kappa_{23} + \kappa_{14}) (\ell + p_4)^2$$

$$n \left( \begin{array}{c} 4 \\ \text{[Diagram: Bubble with internal lines]} \\ 3 \quad 2 \\ 1 \end{array} \right) = \left( \frac{\kappa_{23} + \kappa_{14}}{t} - \frac{\kappa_{13} + \kappa_{24}}{u} \right) (s - \ell^2 - (\ell - p_{12})^2)$$

$$\text{tr}_\pm = \text{tr}_\pm(1(\ell - p_1)(\ell - p_{12})3)$$



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$$n \left( \begin{array}{c} \text{4} \\ \text{3} \end{array} \begin{array}{c} \text{1} \\ \text{2} \end{array} \right) = \left( \frac{\kappa_{13}}{u^2} + \frac{\kappa_{23}}{t^2} \right) \text{tr}_- + \left( \frac{\kappa_{24}}{u^2} + \frac{\kappa_{14}}{t^2} \right) \text{tr}_+ + \frac{s}{t^2} (\kappa_{23} + \kappa_{14}) (\ell + p_4)^2$$

$$n \left( \begin{array}{c} \text{4} \\ \text{3} \end{array} \begin{array}{c} \text{1} \\ \text{2} \end{array} \right) = \left( \frac{\kappa_{23} + \kappa_{14}}{t} - \frac{\kappa_{13} + \kappa_{24}}{u} \right) (s - \ell^2 - (\ell - p_{12})^2)$$

$$\text{tr}_\pm = \text{tr}_\pm(1(\ell - p_1)(\ell - p_{12})3)$$

Integrand-reduce

$$n \left( \begin{array}{c} \text{4} \\ \text{3} \end{array} \begin{array}{c} \text{1} \\ \text{2} \end{array} \right) = \left( \frac{\kappa_{13}}{u^2} + \frac{\kappa_{23}}{t^2} \right) \text{tr}_+(13\ell 4) + \left( \frac{\kappa_{24}}{u^2} + \frac{\kappa_{14}}{t^2} \right) \text{tr}_-(13\ell 4)$$

$$n \left( \begin{array}{c} \text{4} \\ \text{3} \end{array} \begin{array}{c} \text{1} \\ \text{2} \end{array} \right) = s \left( \frac{\kappa_{13} + \kappa_{24}}{u} - \frac{\kappa_{23} + \kappa_{14}}{t} \right)$$

Pro: triangle integrates to zero; Con: color-kinematics broken

# One-loop $\mathcal{N} = 2$ integrand

$$\begin{aligned}
 \mathcal{M}_4^{(1)} = & \frac{i}{2} \sum_{\text{perms}} I \left[ \frac{1}{8} c \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) + \frac{N_f}{4} c \left( \begin{array}{c} 4 \\ \text{fermion} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{fermion} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{fermion} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{fermion} \\ 2 \end{array} \right) \right. \\
 & + \frac{1}{4} c \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) + \frac{N_f}{2} c \left( \begin{array}{c} 4 \\ \text{fermion} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{fermion} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) \\
 & \left. + \frac{1}{16} c \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) + \frac{N_f}{8} c \left( \begin{array}{c} 4 \\ \text{fermion} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{fermion} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{fermion} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{fermion} \\ 2 \end{array} \right) \right]
 \end{aligned}$$

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 &\quad + \frac{1}{4} c \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) + \frac{N_f}{2} c \left( \begin{array}{c} 4 \\ \text{triangle} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{triangle} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{triangle} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{triangle} \\ 2 \end{array} \right) \\
 &\quad \left. + \frac{1}{16} c \left( \begin{array}{c} 4 \\ \text{star} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{star} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{star} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{star} \\ 2 \end{array} \right) + \frac{N_f}{8} c \left( \begin{array}{c} 4 \\ \text{circle} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{circle} \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ \text{circle} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{circle} \\ 2 \end{array} \right) \right] \\
 &= \frac{i}{2} \sum_{\text{perms}} I \left[ \frac{1}{8} c \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) n^{[\mathcal{N}=4]} \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) \right. \\
 &\quad - \frac{1}{4} \left( c \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ \text{box} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{box} \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ \text{box} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{box} \\ 2 \end{array} \right) \Rightarrow \mathcal{O}(\epsilon^0) \\
 &\quad - \frac{1}{2} \left( c \left( \begin{array}{c} 4 \\ \text{wavy} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{wavy} \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ \text{triangle} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{triangle} \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ \text{triangle} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{triangle} \\ 2 \end{array} \right) \Rightarrow 0 \\
 &\quad \left. - \frac{1}{8} \left( c \left( \begin{array}{c} 4 \\ \text{star} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{star} \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ \text{circle} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{circle} \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ \text{circle} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{circle} \\ 2 \end{array} \right) \right] \\
 &= \mathcal{M}_4^{(1)[\mathcal{N}=4]} - \frac{\beta_0}{\epsilon} \mathcal{M}_4^{(0)} + \mathcal{O}(\epsilon^0)
 \end{aligned}$$

## Finite boxes

$$I\left(\begin{array}{c} 4 \quad \ell \quad 1 \\ \quad \blackrightarrow \quad \\ 3 \quad \quad 2 \end{array}\right) [\text{tr}_+[1(\ell-p_1)(\ell-p_{12})\mathbf{3}]] = -\frac{r_\Gamma}{2} [\log^2(\chi) + \pi^2] + O(\epsilon)$$

e.g. Badger, Mogull, Peraro (2016)

## Finite boxes

$$I\left(\begin{array}{c} 4 \\ \ell \\ \hline 3 \end{array} \begin{array}{c} \ell \\ \hline 1 \\ \hline 2 \end{array}\right) [\text{tr}_+[1(\ell-p_1)(\ell-p_{12})3]] = -\frac{r_\Gamma}{2} [\log^2(\chi) + \pi^2] + O(\epsilon)$$

e.g. Badger, Mogull, Peraro (2016)

$$I\left(\begin{array}{c} 4 \\ \ell_2 \\ \hline 3 \end{array} \begin{array}{c} \ell_1 \\ \hline 1 \\ \hline 2 \end{array}\right) [\text{tr}_+(1\bar{\ell}_1 2 3 \bar{\ell}_2 4)] = -2H_{-1,-1,0,0}(\chi) + \frac{\pi^2}{3} \text{Li}_2(-\chi) \\ - \left( \frac{\pi^2}{2} \log(1+\chi) - \frac{\pi^2}{3} \log \chi + 2\zeta(3) \right) \log(1+\chi) + 6\chi\zeta(3)$$

$$I\left(\begin{array}{c} 4 \\ \ell_2 \\ \hline 3 \end{array} \begin{array}{c} \ell_1 \\ \hline 1 \\ \hline 2 \end{array}\right) [\text{tr}_+(1\bar{\ell}_1 2 4 \bar{\ell}_2 3)] = -2H_{0,-1,0,0}(\chi) + \pi^2 \text{Li}_2(-\chi) \\ + \frac{\pi^2}{6} \log^2 \chi + 4\zeta(3) \log \chi + \frac{\pi^4}{10} + 6(1+\chi)\zeta(3)$$

Caron-Huot, Larsen (2012)

where  $\chi = t/s$

## Two-loop $\mathcal{N} = 2$ integrand

Johansson, Kälin, Mogull (2017)

Start with BCJ numerators

$$n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_1 \xrightarrow{\quad} 1 \\ \downarrow \quad \uparrow \\ 3 \quad \quad 2 \end{array} \right) = -\mu_{12}(\kappa_{12} + \kappa_{34}) + \frac{1}{u^2} (\kappa_{13} \text{tr}_-(1\bar{\ell}_1 2 4 \bar{\ell}_2 3) + \kappa_{24} \text{tr}_+(1\bar{\ell}_1 2 4 \bar{\ell}_2 3)) \\ + \frac{1}{t^2} (\kappa_{14} \text{tr}_-(1\bar{\ell}_1 2 3 \bar{\ell}_2 4) + \kappa_{23} \text{tr}_+(1\bar{\ell}_1 2 3 \bar{\ell}_2 4))$$

$$n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_1 \xrightarrow{\quad} 1 \\ \downarrow \quad \uparrow \\ 3 \quad \quad 2 \end{array} \right) = \mu_{13}(\kappa_{12} + \kappa_{34}) - \frac{1}{u^2} (\kappa_{13} \text{tr}_-(1\bar{\ell}_1 2 4 \bar{\ell}_3 3) + \kappa_{24} \text{tr}_+(1\bar{\ell}_1 2 4 \bar{\ell}_3 3)) \\ - \frac{1}{t^2} (\kappa_{14} \text{tr}_-(1\bar{\ell}_1 2 3 \bar{\ell}_3 4) + \kappa_{23} \text{tr}_+(1\bar{\ell}_1 2 3 \bar{\ell}_3 4)) \quad \text{etc.}$$

Relation to  $\mathcal{N} = 4$  SYM:

$$n^{[\mathcal{N}=4]} \left( \begin{array}{c} 4 \quad \quad 1 \\ \quad \quad \quad \quad \\ 3 \quad \quad 2 \end{array} \right) = n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_1 \xrightarrow{\quad} 1 \\ \downarrow \quad \uparrow \\ 3 \quad \quad 2 \end{array} \right) \\ + 2n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_1 \xrightarrow{\quad} 1 \\ \downarrow \quad \uparrow \\ 3 \quad \quad 2 \end{array} \right) + 2n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_1 \xrightarrow{\quad} 1 \\ \downarrow \quad \uparrow \\ 3 \quad \quad 2 \end{array} \right) + 2n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_1 \xrightarrow{\quad} 1 \\ \downarrow \quad \uparrow \\ 3 \quad \quad 2 \end{array} \right)$$

# Two-loop $\mathcal{N} = 2$ integrand

Integrand assembled with color factors

$$\begin{aligned}
 \mathcal{M}_4^{(2)} = & -\frac{i}{4} \sum_{\text{perms}} I \left[ \frac{1}{4} c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right. \\
 & + N_f c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) + \frac{N_f}{2} c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \\
 & + \frac{1}{4} c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) + N_f c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \\
 & + \frac{N_f}{2} c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) + \frac{1}{2} c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \\
 & + N_f c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) + \frac{1}{4} c \left( \begin{array}{c} 1 \\ 4 \end{array} \begin{array}{c} 2 \\ 3 \end{array} \right) n \left( \begin{array}{c} 1 \\ 4 \end{array} \begin{array}{c} 2 \\ 3 \end{array} \right) \\
 & \left. + \frac{N_f}{2} c \left( \begin{array}{c} 1 \\ 4 \end{array} \begin{array}{c} 2 \\ 3 \end{array} \right) n \left( \begin{array}{c} 1 \\ 4 \end{array} \begin{array}{c} 2 \\ 3 \end{array} \right) \right]
 \end{aligned}$$

# Analytic IR structure



## Nice features of $\mathcal{N} = 2$ integrand

Two-loop numerators related to  $\mathcal{N} = 4$  SYM:

$$\begin{aligned}
 n^{[\mathcal{N}=4]} \left( \begin{array}{c} 4 \\ \text{[Diagram: Two-loop wavy lines]} \\ 3 \end{array} \right) &= n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_2 \ell_1 \xrightarrow{\phantom{\ell_2}} 1 \\ \text{[Diagram: Two-loop wavy lines]} \\ 3 \end{array} \right) \\
 + 2n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_2 \ell_1 \xrightarrow{\phantom{\ell_2}} 1 \\ \text{[Diagram: Two-loop wavy lines with internal arrows]} \\ 3 \end{array} \right) &+ 2n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_2 \ell_1 \xrightarrow{\phantom{\ell_2}} 1 \\ \text{[Diagram: Two-loop wavy lines with internal arrows]} \\ 3 \end{array} \right) + 2n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_2 \ell_1 \xrightarrow{\phantom{\ell_2}} 1 \\ \text{[Diagram: Two-loop wavy lines with internal arrows]} \\ 3 \end{array} \right)
 \end{aligned}$$

Johansson, Kälin, Mogull (2017)

- ▶ obtained from 6d cuts of  $\mathcal{N} = (1, 0)$  SYM with  $N_f$  hypers
- ▶ matter loops IR-regulated by numerators
- ▶  $\mathcal{O}(\epsilon^{-4})$  entirely inside  $\mathcal{M}_4^{(2)[\mathcal{N}=4]}$

$$\begin{aligned}
 I \left[ n \left( \begin{array}{c} 4 \xleftarrow{\ell_2} \ell_2 \ell_1 \xrightarrow{\phantom{\ell_2}} 1 \\ \text{[Diagram: Two-loop wavy lines with internal arrows]} \\ 3 \end{array} \right) \right] &= \mathcal{O}(\epsilon^0) \\
 I \left[ n \left( \begin{array}{c} \ell_1 \xrightarrow{\phantom{\ell_2}} 1 \\ \text{[Diagram: Two-loop wavy lines with internal arrows]} \\ 4 \xleftarrow{\ell_2} \ell_2 \ell_1 \xrightarrow{\phantom{\ell_2}} 3 \end{array} \right) \right] &= \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

## IR factorization of singular double boxes

$$\begin{aligned}
 & \text{Diagram 1} \left[ \text{tr}_{\pm}(1\bar{\ell}_1 23\bar{\ell}_3 4) \right] = \text{Diagram 2} \times \text{Diagram 3} \left[ \text{tr}_{\pm}(1\bar{\ell} 23\bar{\ell} 4) \right] + \mathcal{O}(\epsilon^{-1}) \\
 & = \text{Diagram 2} \times t \left( -s \text{Diagram 3} [\mu^2] - \text{Diagram 4} = \frac{2}{3} + \text{Diagram 5} = \frac{1}{2} \right) + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram 1} \left[ \text{tr}_{\pm}(1\bar{\ell}_1 24\bar{\ell}_3 3) \right] = \text{Diagram 2} \times \text{Diagram 3} \left[ \text{tr}_{\pm}(1\bar{\ell} 24\bar{\ell} 3) \right] + \mathcal{O}(\epsilon^{-1}) \\
 & = \text{Diagram 2} \times \left( -s \text{Diagram 3} [\text{tr}_{\pm}] - su \text{Diagram 3} [\mu^2] + u \text{Diagram 4} = \frac{2}{3} - u \text{Diagram 5} = \frac{1}{2} \right) \\
 & \quad + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

where recall 1-loop  $\text{tr}_{\pm} = \text{tr}_{\pm}(1(\ell-p_1)(\ell-p_{12})3)$

# IR factorization of singular double boxes

$$\begin{aligned}
 & \text{Diagram} \left[ \text{tr}_{\pm}(1\bar{\ell}_1 23\bar{\ell}_3 4) \right] = \text{Diagram} \times \text{Diagram} \left[ \text{tr}_{\pm}(1\bar{\ell} 23\bar{\ell} 4) \right] + \mathcal{O}(\epsilon^{-1}) \\
 & = \text{Diagram} \times t \left( -s \text{Diagram} [\mu^2] - \text{Diagram} = \text{Diagram} \right) + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram} \left[ \text{tr}_{\pm}(1\bar{\ell}_1 24\bar{\ell}_3 3) \right] = \text{Diagram} \times \text{Diagram} \left[ \text{tr}_{\pm}(1\bar{\ell} 24\bar{\ell} 3) \right] + \mathcal{O}(\epsilon^{-1}) \\
 & = \text{Diagram} \times \left( -s \text{Diagram} [\text{tr}_{\pm}] - su \text{Diagram} [\mu^2] + u \text{Diagram} = \text{Diagram} - u \text{Diagram} = \right) \\
 & \quad + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

where recall 1-loop  $\text{tr}_{\pm} = \text{tr}_{\pm}(1(\ell-p_1)(\ell-p_{12})3)$

$$\begin{aligned}
 I \left[ n \left( \text{Diagram} \right) \right] &= s \text{Diagram} \times \left\{ I \left[ n \left( \text{Diagram} \right) \right] \right. \\
 & \quad \left. + \frac{1}{s^2} I \left[ n \left( \text{Diagram} \right) \right] - \frac{1}{s^2} I \left( \text{Diagram} \right) \left[ n \left( \text{Diagram} \right) \right] \right\} + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

# IR factorization of singular cross-boxes

$$\begin{array}{c} \ell_{2 \swarrow} \\ \ell_1 \rightarrow \\ \begin{array}{c} 4 \quad 1 \\ \diagdown \quad / \\ \text{---} 2 \\ \diagup \quad \diagdown \\ 3 \end{array} \end{array} [\text{tr}_{\pm}(2\ell_3\ell_24)] = \begin{array}{c} \ell \\ \begin{array}{c} 4 \quad 1 \\ \diagdown \quad / \\ \text{---} 2 \\ \diagup \quad \diagdown \\ 3 \end{array} \end{array} [\text{tr}_{\mp}(1\ell(\ell+p_4)3)] \times \begin{array}{c} 1 \\ \diagdown \quad / \\ \text{---} 3 \end{array} + \mathcal{O}(\epsilon^{-1})$$

$$\begin{array}{c} \ell_{2 \swarrow} \\ \ell_1 \rightarrow \\ \begin{array}{c} 4 \quad 1 \\ \diagdown \quad / \\ \text{---} 2 \\ \diagup \quad \diagdown \\ 3 \end{array} \end{array} [\text{tr}_{\pm}(1\bar{\ell}_243\bar{\ell}_32)] = \begin{array}{c} \ell \\ \begin{array}{c} 4 \quad 1 \\ \diagdown \quad / \\ \text{---} 2 \\ \diagup \quad \diagdown \\ 3 \end{array} \end{array} [\text{tr}_{\pm}(1\bar{\ell}43(\bar{\ell}-p_1)2)] \times \begin{array}{c} 1 \\ \diagdown \quad / \\ \text{---} 3 \end{array} + \mathcal{O}(\epsilon^{-1})$$

$$\begin{array}{c} \ell_{2 \swarrow} \\ \ell_1 \rightarrow \\ \begin{array}{c} 4 \quad 1 \\ \diagdown \quad / \\ \text{---} 2 \\ \diagup \quad \diagdown \\ 3 \end{array} \end{array} [\text{tr}_{\pm}(1\bar{\ell}_323\bar{\ell}_24)] = \begin{array}{c} \ell \\ \begin{array}{c} 4 \quad 1 \\ \diagdown \quad / \\ \text{---} 2 \\ \diagup \quad \diagdown \\ 3 \end{array} \end{array} [\text{tr}_{\pm}(1\bar{\ell}23\bar{\ell}4)] \times \begin{array}{c} 1 \\ \diagdown \quad / \\ \text{---} 3 \end{array} + \mathcal{O}(\epsilon^{-1})$$

# IR factorization of singular cross-boxes

$$\begin{array}{c} \ell_2 \swarrow \\ \text{---} \text{---} \text{---} \\ \ell_1 \rightarrow \\ \text{---} \text{---} \text{---} \\ \searrow \\ 4 \quad 2 \quad 3 \end{array} \left[ \text{tr}_{\pm}(2\ell_3\ell_24) \right] = \begin{array}{c} 4 \\ \text{---} \text{---} \text{---} \\ \ell \\ \text{---} \text{---} \text{---} \\ 3 \quad 2 \end{array} \left[ \text{tr}_{\mp}(1\ell(\ell+p_4)3) \right] \times \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ \triangleleft \\ 3 \end{array} + \mathcal{O}(\epsilon^{-1})$$

$$\begin{array}{c} \ell_2 \swarrow \\ \text{---} \text{---} \text{---} \\ \ell_1 \rightarrow \\ \text{---} \text{---} \text{---} \\ \searrow \\ 4 \quad 2 \quad 3 \end{array} \left[ \text{tr}_{\pm}(1\bar{\ell}_243\bar{\ell}_32) \right] = \begin{array}{c} 4 \\ \text{---} \text{---} \text{---} \\ \ell \\ \text{---} \text{---} \text{---} \\ 3 \quad 2 \end{array} \left[ \text{tr}_{\pm}(1\bar{\ell}43(\bar{\ell}-p_1)2) \right] \times \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ \triangleleft \\ 3 \end{array} + \mathcal{O}(\epsilon^{-1})$$

$$\begin{array}{c} \ell_2 \swarrow \\ \text{---} \text{---} \text{---} \\ \ell_1 \rightarrow \\ \text{---} \text{---} \text{---} \\ \searrow \\ 4 \quad 2 \quad 3 \end{array} \left[ \text{tr}_{\pm}(1\bar{\ell}_323\bar{\ell}_24) \right] = \begin{array}{c} 4 \\ \text{---} \text{---} \text{---} \\ \ell \\ \text{---} \text{---} \text{---} \\ 3 \quad 2 \end{array} \left[ \text{tr}_{\pm}(1\bar{\ell}23\bar{\ell}4) \right] \times \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ \triangleleft \\ 3 \end{array} + \mathcal{O}(\epsilon^{-1})$$

$$\begin{aligned}
 I \left[ n \left( \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ 4 \quad 2 \quad 3 \end{array} \right) \right] &= u \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ \triangleleft \\ 3 \end{array} \times \left\{ I \left[ n \left( \begin{array}{c} 4 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ 3 \quad 2 \end{array} \right) \right] \right. \\
 &+ \left. \frac{1}{u^2} I \left[ n \left( \begin{array}{c} 4 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ 2 \quad 3 \end{array} \right) \right] - \frac{1}{u^2} I \left( \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ 2 \end{array} \right) \left[ n \left( \begin{array}{c} 4 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ 2 \quad 3 \end{array} \right) \right] \right\} + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

# IR factorization of pentagon-triangles and box-bubbles

$$\begin{aligned}
 & \text{Diagram 1} \left[ \text{tr}_{\pm}(q\ell_3\ell_24) \right] \\
 &= \frac{2p_4 \cdot q}{st} \left( t \text{Diagram 2} + s \text{Diagram 3} + t \text{Diagram 4} + s \text{Diagram 5} \right) + \mathcal{O}(\epsilon^{-1}) \\
 & \text{Diagram 6} = \frac{1}{s} \text{Diagram 7} + \frac{1}{t} \text{Diagram 8} + \frac{1}{s} \text{Diagram 9} + \frac{1}{t} \text{Diagram 10} + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

# IR factorization of pentagon-triangles and box-bubbles

$$\begin{aligned}
 & \text{Diagram: A pentagon with vertices 1, 2, 3, 4. Internal lines connect 1-2, 2-3, 3-4, 4-1. A triangle is formed by lines 1-2, 2-3, and 4-1. External momenta: $l_1 \to$ at vertex 1, $l_2 \swarrow$ at vertex 4. Label: $[\text{tr}_\pm(q l_3 l_2 4)]$} \\
 &= \frac{2p_4 \cdot q}{st} \left( t \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + s \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + t \text{Diagram: Bubble with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + s \text{Diagram: Bubble with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} \right) + \mathcal{O}(\epsilon^{-1}) \\
 & \text{Diagram: A box with vertices 1, 2, 3, 4. Internal lines connect 1-2, 2-3, 3-4, 4-1. A bubble is formed by lines 1-2, 2-3, and 4-1. External momenta: $l_1 \to$ at vertex 1, $l_2 \swarrow$ at vertex 4.} \\
 &= \frac{1}{s} \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + \frac{1}{t} \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + \frac{1}{s} \text{Diagram: Bubble with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + \frac{1}{t} \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

$$\begin{aligned}
 & I \left[ n \left( \text{Diagram: A pentagon with vertices 1, 2, 3, 4. Internal lines connect 1-2, 2-3, 3-4, 4-1. A triangle is formed by lines 1-2, 2-3, and 4-1. External momenta: $l_1 \to$ at vertex 1, $l_2 \swarrow$ at vertex 4.} \right) \right] \\
 &= -i A_{1234}^{(0)} \left( t \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + s \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + t \text{Diagram: Bubble with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + s \text{Diagram: Bubble with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} \right) + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

$$\begin{aligned}
 & I \left[ n \left( \text{Diagram: A box with vertices 1, 2, 3, 4. Internal lines connect 1-2, 2-3, 3-4, 4-1. A bubble is formed by lines 1-2, 2-3, and 4-1. External momenta: $l_1 \to$ at vertex 1, $l_2 \swarrow$ at vertex 4.} \right) \right] \\
 &= i A_{1234}^{(0)} \left( t \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + s \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + t \text{Diagram: Bubble with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} + s \text{Diagram: Triangle with vertices 1, 2, 3, 4. Internal lines 1-2, 2-3, 4-1. External momenta: $l_1 \to$ at 1, $l_2 \swarrow$ at 4.} \right) + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

# Extract difference from $\mathcal{N} = 4$

$$\begin{aligned}
 & \mathcal{M}_4^{(2)} - \mathcal{M}_4^{(2)[\mathcal{N}=4]} \\
 &= \frac{i}{4} \sum_{\text{perms}} I \left[ \left( c \left( \begin{array}{c} 4 \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right) \right. \\
 & \quad + \frac{1}{2} \left( c \left( \begin{array}{c} 4 \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right) \\
 & \quad + \left( c \left( \begin{array}{c} 4 \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \right) \\
 & \quad \left. + \frac{1}{2} \left( c \left( \begin{array}{c} 1 \\ \text{---} \\ 2 \\ \text{---} \\ 3 \\ \text{---} \\ 4 \end{array} \right) - N_f c \left( \begin{array}{c} 1 \\ \text{---} \\ 2 \\ \text{---} \\ 3 \\ \text{---} \\ 4 \end{array} \right) \right) n \left( \begin{array}{c} 1 \\ \text{---} \\ 2 \\ \text{---} \\ 3 \\ \text{---} \\ 4 \end{array} \right) \right] + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$



# Extract difference from $\mathcal{N} = 4$

$$\begin{aligned}
 & \mathcal{M}_4^{(2)} - \mathcal{M}_4^{(2)[\mathcal{N}=4]} \\
 &= \frac{i}{4} \sum_{\text{perms}} I \left[ \left( c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right. \\
 & \quad + \frac{1}{2} \left( c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \\
 & \quad + \left( c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \\
 & \quad \left. + \frac{1}{2} \left( c \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) - N_f c \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) \right) n \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) \right] + \mathcal{O}(\epsilon^{-1}) \\
 &= \frac{i}{4} \sum_{\text{perms}} I \left[ \left( c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right. \\
 & \quad + \frac{1}{2} \left( c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \\
 & \quad \left. + \beta_0 c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \left( n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) + n \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) \right) \right] + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

## Match difference from $\mathcal{N} = 4$

After analytic rearrangements match reorganized Catani

$$\begin{aligned}
 & \mathcal{M}_4^{(2)} - \mathcal{M}_4^{(2)[\mathcal{N}=4]} \\
 &= \frac{i}{4} \sum_{\text{perms}} I \left[ \left( c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \right. \\
 & \quad + \frac{1}{2} \left( c \left( \begin{array}{c} 4 \\ 2 \end{array} \begin{array}{c} 1 \\ 3 \end{array} \right) - N_f c \left( \begin{array}{c} 4 \\ 2 \end{array} \begin{array}{c} 1 \\ 3 \end{array} \right) \right) n \left( \begin{array}{c} 4 \\ 2 \end{array} \begin{array}{c} 1 \\ 3 \end{array} \right) \\
 & \quad \left. + \beta_0 c \left( \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \end{array} \right) \left( n \left( \begin{array}{c} 4 \\ 2 \end{array} \begin{array}{c} 1 \\ 3 \end{array} \right) + n \left( \begin{array}{c} 1 \\ 4 \end{array} \begin{array}{c} 2 \\ 3 \end{array} \right) \right) \right] + \mathcal{O}(\epsilon^{-1}) \\
 &= \left( \sum_{i \neq j} s_{ij} \begin{array}{c} i \\ \triangleleft \\ j \end{array} \mathbf{T}_i \cdot \mathbf{T}_j \right) [\mathcal{M}_4^{(1)} - \mathcal{M}_4^{(1)[\mathcal{N}=4]}] \\
 & \quad + \beta_0 \left( \sum_{i \neq j} s_{ij} \begin{array}{c} i \\ \triangleleft \\ j \end{array} \mathbf{T}_i \cdot \mathbf{T}_j \right) \mathcal{M}_4^{(0)} + \mathcal{O}(\epsilon^{-1})
 \end{aligned}$$

# Summary & outlook

- ▶ Discussed QCD-like IR structure beyond  $\mathcal{N} = 4$
- ▶ Explicit 4-gluon integrand from 6d unitarity cuts  
Johansson, Kälin, Mogull (2017)
- ▶ Matter loops IR regulated  
e.g. box from Badger, Mogull, Peraro (2016)  
double boxes from Caron-Huot, Larsen (2012)
- ▶ Divergences of unregulated loops extracted analytically  
similar to Anastasiou, Sterman (talk 2018)
- ▶ External matter in progress

STAY TUNED!

Thank you!

# Backup slides

## General conventions

$$\widetilde{\mathcal{M}}_n = (4\pi\alpha_s)^{\frac{n-2}{2}} \left[ \widetilde{\mathcal{M}}_n^{(0)} + \frac{\alpha_s}{4\pi} \widetilde{\mathcal{M}}_n^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \widetilde{\mathcal{M}}_n^{(2)} + \dots \right]$$

$$\mathcal{M}_n = (4\pi\alpha_0)^{\frac{n-2}{2}} \left[ \mathcal{M}_n^{(0)} + \frac{\alpha_0}{4\pi} S_\epsilon \mathcal{M}_n^{(1)} + \left(\frac{\alpha_0}{4\pi}\right)^2 S_\epsilon^{-2} \mathcal{M}_n^{(2)} + \dots \right]$$

$$\alpha_0 \mu_0^{2\epsilon} S_\epsilon = \alpha_s \mu_R^{2\epsilon} \left[ 1 - \frac{\alpha_s}{4\pi} \frac{\beta_0}{\epsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left( \frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{\epsilon} \right) + \mathcal{O}(\alpha_s^3) \right]$$

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma_E}$$

$$I \sim \left( e^{\epsilon\gamma_E} \int \frac{d^D \ell}{i\pi^{D/2}} \right)^L$$

$$r_\Gamma = e^{\epsilon\gamma_E} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \frac{1}{2}\zeta_2\epsilon^2 - \frac{7}{3}\zeta_3\epsilon^3 + \mathcal{O}(\epsilon^4)$$

$\mathcal{N} = 0$  conventions

$$\mathcal{M}_4^{(0)} = -\frac{i}{4} \sum_{\text{perms}} c \left( \begin{array}{c} \text{4} \\ \text{3} \end{array} \begin{array}{c} \text{1} \\ \text{2} \end{array} \right) \frac{1}{st} (\kappa_{12} + \kappa_{13} + \kappa_{14} + \kappa_{23} + \kappa_{24} + \kappa_{34})$$
$$\kappa_{ij} = \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \delta^{(4)}(Q) \langle ij \rangle^2 \eta_i^3 \eta_j^3 \eta_i^4 \eta_j^3$$

# All singularities of double box

$$\begin{aligned}
 & \text{Diagram 1} [\text{tr}_\pm(1\bar{\ell}_1 23(\bar{\ell}_1 + \bar{\ell}_2)4)] = \text{Diagram 2} [\text{tr}_\pm(1\bar{\ell}_1 23\bar{\ell}_1 4)] + \mathcal{O}(\epsilon^0) \\
 & = \text{Diagram 3} \times \text{Diagram 4} [\text{tr}_\pm(1\bar{\ell} 23\bar{\ell} 4)] \\
 & + \frac{1}{s} \left[ 4 \text{Diagram 5} [\text{tr}_\pm(1\bar{\ell}_1 23\bar{\ell}_1 4)] - 4 \text{Diagram 6} [\text{tr}_\pm(1\bar{\ell}_1 23\bar{\ell}_1 4)] \right. \\
 & \quad \left. + 3 \text{Diagram 7} [\text{tr}_\pm(1\bar{\ell}_1 23\bar{\ell}_1 4)] - 3 \text{Diagram 8} [\text{tr}_\pm(1\bar{\ell}_1 23\bar{\ell}_1 4)] \right] \\
 & + \mathcal{O}(\epsilon^0)
 \end{aligned}$$