Scattering Equations in Multi-Regge Kinematics

Zhengwen Liu

Center for Cosmology, Particle Physics and Phenomenology
Institut de Recherche en Mathématique et Physique
UCLouvain

Base on `1811.xxxxx` (with C. Duhr) and `1811.yyyyy`

Galileo Galilei Institute, Firenze

November 15, 2018

Outline

Introduction to scattering equations

Multi-Regge kinematics (MRK)

- Scattering equations in MRK
- Gauge theory amplitudes in MRK
- Gravity amplitudes in MRK

Quasi Multi-Regge kinematics

- Scattering equations in QMRK
- Generalized Impact factors and Lipatov vertices

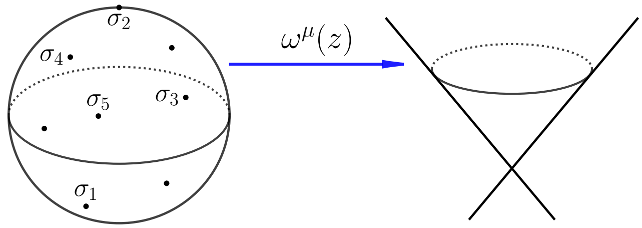
Summary & Outlook

Scattering equations

Let us start with a rational map from the moduli space $\mathfrak{M}_{0,n}$ to the space of momenta for n massless particles scattering:

$$k_a^\mu = \frac{1}{2\pi i} \oint_{|z-\sigma_a|=\epsilon} dz\, \omega^\mu(z)$$

$$\omega^\mu(z) = \sum_{a=1}^{n} \frac{k_a^\mu}{z-\sigma_a} = \frac{P^\mu(z)}{\prod_{a=1}^{n}(z-\sigma_a)}$$

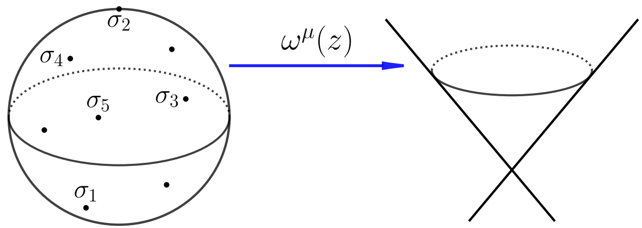


Scattering equations

Let us start with a rational map from the moduli space $\mathfrak{M}_{0,n}$ to the space of momenta for n massless particles scattering:

$$k_a^\mu = \frac{1}{2\pi i} \oint_{|z-\sigma_a|=\epsilon} dz\, \omega^\mu(z)$$

$$\omega^\mu(z) = \sum_{a=1}^{n} \frac{k_a^\mu}{z-\sigma_a} = \frac{P^\mu(z)}{\prod_{a=1}^{n}(z-\sigma_a)}$$



$$\omega_\mu(z)\omega^\mu(z) = 0$$

$\omega^\mu(z)$ maps the $\mathfrak{M}_{0,n}$ to the null cone of momenta

$$0 = \frac{1}{2\pi i} \oint_{|z-\sigma_a|=\epsilon} dz\, \omega(z)^2 = \sum_{b\neq a} \frac{2k_a \cdot k_b}{\sigma_a - \sigma_b}, \qquad a = 1, 2, \ldots, n$$

which are named as the **scattering equations**.

[Cachazo, He & Yuan, 1306.2962, 1306.6575]

Scattering equations

The scattering equations: $\mathfrak{M}_{0,n} \to \mathcal{K}_n$

$$f_a = \sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \quad a = 1, 2, \ldots, n$$

- This system has an $\mathrm{SL}(2, \mathbb{C})$ redundancy, only $(n-3)$ out of n equations are independent
- Equivalent to a system of homogeneous polynomial equations [Dolan & Goddard, 1402.7374]
- The total number of independent solutions is $(n-3)!$

Scattering equations

The scattering equations: $\mathfrak{M}_{0,n} \to \mathcal{K}_n$

$$f_a = \sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \quad a = 1, 2, \ldots, n$$

- This system has an $\mathrm{SL}(2, \mathbb{C})$ redundancy, only $(n-3)$ out of n equations are independent
- Equivalent to a system of homogeneous polynomial equations [Dolan & Goddard, 1402.7374]
- The total number of independent solutions is $(n-3)!$
- The scattering equations have appeared before in different contexts, e.g.,
  - D. Fairlie and D. Roberts (1972): amplitudes in dual models
  - D. Gross and P. Mende (1988): the high energy behavior of string scattering
  - E. Witten (2004): twistor string
- Cachazo, He and Yuan rediscovered them in the context of field theory amplitudes

[CHY, 1306.2962, 1306.6575, 1307.2199, 1309.0885]

Scattering equations in 4d

In 4 dimensions, the null map vector $P^{\mu}(z)$ can be rewritten in spinor variables as follows:

$$P^{\alpha\dot{\alpha}}(z) \equiv \left(\prod_{a=1}^{n}(z-\sigma_a)\right)\sum_{b=1}^{n}\frac{\lambda_b^{\alpha}\tilde{\lambda}_b^{\dot{\alpha}}}{z-\sigma_b} = \lambda^{\alpha}(z)\tilde{\lambda}^{\dot{\alpha}}(z)$$

$\deg \lambda(z) = d \in \{1,\ldots,n-3\}$, $\deg \tilde{\lambda}(z) = \tilde{d}$, $d+\tilde{d} = n-2$. A simple construction is

$$\lambda^{\alpha}(z) = \prod_{a\in\mathfrak{N}}(z-\sigma_a)\sum_{I\in\mathfrak{N}}\frac{t_I\lambda_I^{\alpha}}{z-\sigma_I}, \qquad \lambda^{\dot{\alpha}}(z) = \prod_{a\in\mathfrak{P}}(z-\sigma_a)\sum_{i\in\mathfrak{P}}\frac{t_i\tilde{\lambda}_i^{\dot{\alpha}}}{z-\sigma_i}$$

We divide $\{1,\ldots,n\}$ into two subsets $\mathfrak{N}$ and $\mathfrak{P}$, $|\mathfrak{N}| = k = d+1$, $|\mathfrak{P}| = n-k = \tilde{d}+1$.

Scattering equations in 4d

In 4 dimensions, the null map vector $P^\mu(z)$ can be rewritten in spinor variables as follows:

$$P^{\alpha\dot\alpha}(z) \equiv \left(\prod_{a=1}^{n}(z-\sigma_a)\right)\sum_{b=1}^{n}\frac{\lambda_b^\alpha\tilde\lambda_b^{\dot\alpha}}{z-\sigma_b} = \lambda^\alpha(z)\tilde\lambda^{\dot\alpha}(z)$$

$\deg\lambda(z) = d \in \{1,\ldots,n-3\}$, $\deg\tilde\lambda(z) = \tilde d$, $d+\tilde d = n-2$. A simple construction is

$$\lambda^\alpha(z) = \prod_{a\in\mathfrak{N}}(z-\sigma_a)\sum_{I\in\mathfrak{N}}\frac{t_I\lambda_I^\alpha}{z-\sigma_I}, \qquad \lambda^{\dot\alpha}(z) = \prod_{a\in\mathfrak{P}}(z-\sigma_a)\sum_{i\in\mathfrak{P}}\frac{t_i\tilde\lambda_i^{\dot\alpha}}{z-\sigma_i}$$

We divide $\{1,\ldots,n\}$ into two subsets $\mathfrak{N}$ and $\mathfrak{P}$, $|\mathfrak{N}| = k = d+1$, $|\mathfrak{P}| = n-k = \tilde d+1$.

Then the two spinor maps leads to

$$\bar{\mathcal{E}}_I^{\dot\alpha} = \tilde\lambda_I^{\dot\alpha} - \sum_{i\in\mathfrak{P}}\frac{t_I t_i}{\sigma_I-\sigma_i}\tilde\lambda_i^{\dot\alpha} = 0,\ \ I\in\mathfrak{N}; \quad \mathcal{E}_i^\alpha = \lambda_i^\alpha - \sum_{I\in\mathfrak{N}}\frac{t_i t_I}{\sigma_i-\sigma_I}\lambda_I^\alpha = 0,\ \ i\in\mathfrak{P}$$

4D scattering equations

Geyer-Lipstein-Mason (GLM) scattering equations:

$$\bar{\mathcal{E}}_I^{\dot\alpha} = \tilde\lambda_I^{\dot\alpha} - \sum_{i\in\mathfrak{P}} \frac{t_I t_i}{\sigma_I - \sigma_i} \tilde\lambda_i^{\dot\alpha} = 0,\ I \in \mathfrak{N}; \quad \mathcal{E}_i^{\alpha} = \lambda_i^{\alpha} - \sum_{I\in\mathfrak{N}} \frac{t_i t_I}{\sigma_i - \sigma_I} \lambda_I^{\alpha} = 0,\ i \in \mathfrak{P}$$

• These equations are originally derived from the four-dimensional ambitwistor string model, based on them tree superamplitudes in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity are obtained.

[Geyer, Lipstein & Mason, 1404.6219]

Scattering equations in 4d

Geyer-Lipstein-Mason (GLM) scattering equations:

$$\bar{\mathcal{E}}_I^{\dot\alpha} = \tilde\lambda_I^{\dot\alpha} - \sum_{i\in\mathfrak{P}} \frac{t_I t_i}{\sigma_I - \sigma_i} \tilde\lambda_i^{\dot\alpha} = 0, \ I \in \mathfrak{N}; \quad \mathcal{E}_i^{\alpha} = \lambda_i^{\alpha} - \sum_{I\in\mathfrak{N}} \frac{t_i t_I}{\sigma_i - \sigma_I} \lambda_I^{\alpha} = 0, \ i \in \mathfrak{P}$$

• These equations are originally derived from the four-dimensional ambitwistor string model, based on them tree superamplitudes in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity are obtained.

[Geyer, Lipstein & Mason, 1404.6219]

• Equivalent polynomial versions [Roiban, Spradlin & Volovich, hep-th/0403190; He, ZL & Wu, 1604.02834]

$$\sum_{a=1}^{n} t_a \sigma_a^m \tilde\lambda_a^{\dot\alpha} = 0, \quad m = 0, 1, \ldots, d; \quad \lambda_a^{\alpha} - t_a \sum_{m=0}^{d=k-1} \rho_m^{\alpha} \sigma_a^m = 0$$

• In 4d, the scattering eqs fall into "helicity sector" are characterized by $k \in \{2, \ldots, n-2\}$

• In sector k, the number of independent solutions is $\left\langle {n-3 \atop k-2} \right\rangle$

$$\sum_{k=2}^{n-2} \left\langle {n-3 \atop k-2} \right\rangle = (n-3)!$$

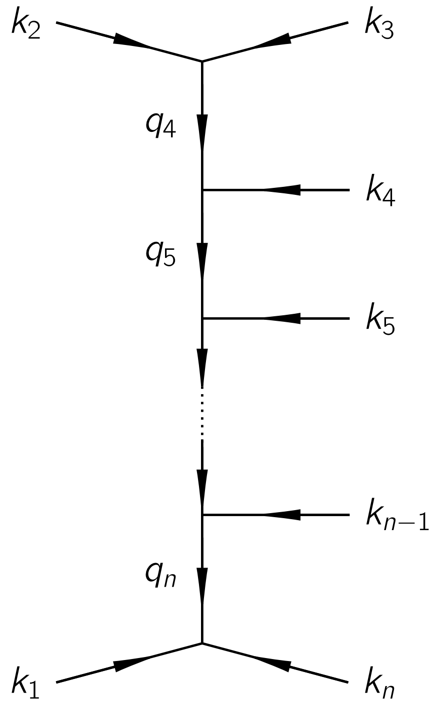
Multi-Regge Kinematics (MRK)

Multi-Regge kinematics is defined as a $2 \to n{-}2$ scattering where the final state particles are strongly ordered in rapidity while having comparable transverse momenta,

$$y_3 \gg y_4 \gg \cdots \gg y_n \quad \text{and} \quad |\mathbf{k_3}| \simeq |\mathbf{k_4}| \simeq \ldots \simeq |\mathbf{k_n}|$$

- In lightcone coordinates $k_a = (k_a^+, k_a^-; k_a^\perp)$ with $k_a^\pm = k_a^0 \pm k_a^z$ and $k_a^\perp = k_a^x + i k_a^y$

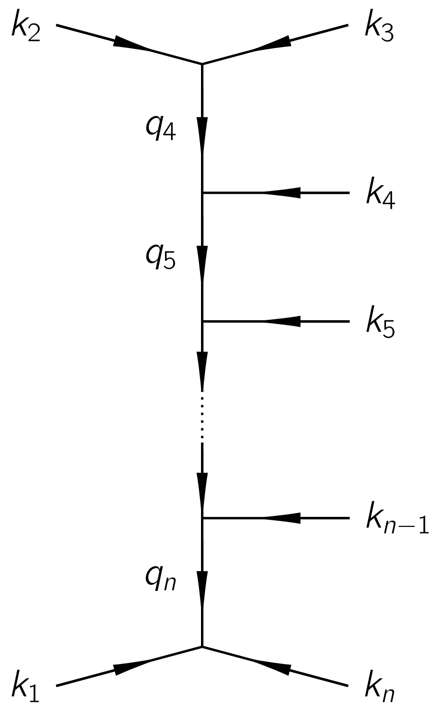
$$k_3^+ \gg k_4^+ \gg \cdots \gg k_n^+$$



Multi-Regge Kinematics (MRK)

Multi-Regge kinematics is defined as a $2 \to n{-}2$ scattering where the final state particles are strongly ordered in rapidity while having comparable transverse momenta,

$$y_3 \gg y_4 \gg \cdots \gg y_n \quad \text{and} \quad |\mathbf{k_3}| \simeq |\mathbf{k_4}| \simeq \ldots \simeq |\mathbf{k_n}|$$



- In lightcone coordinates $k_a = (k_a^+, k_a^-; k_a^\perp)$ with $k_a^\pm = k_a^0 \pm k_a^z$ and $k_a^\perp = k_a^x + i k_a^y$

$$k_3^+ \gg k_4^+ \gg \cdots \gg k_n^+$$

- We work in center-of-momentum frame:

$$k_1 = (0, -\kappa; 0), \quad k_2 = (-\kappa, 0; 0), \quad \kappa \equiv \sqrt{s}$$

- In this region, tree amplitudes in gauge and gravity factorize

$$\mathcal{A}_n \sim s^{\text{spin}}\, C_{2;3} \frac{1}{t_4} V_4 \cdots \frac{1}{t_{n-1}} V_{n-1} \frac{1}{t_n} C_{1;n}$$

[Kuraev, Lipatov & Fadin, 1976; Del Duca, 1995; Lipatov, 1982]

When scattering equations meet MRK

Scattering equations in MRK

- The simplest example: four points

$$\sigma_1 = 0, \quad \sigma_2 \to \infty, \qquad f_1 = -\frac{s_{13}}{\sigma_3} - \frac{s_{14}}{\sigma_4} = 0 \quad \Longrightarrow \quad \frac{\sigma_3}{\sigma_4} = \frac{s+t}{t}$$

In the Regge limit, $s \gg -t$, we have

$$\left|\frac{\sigma_3}{\sigma_4}\right| \simeq \left|\frac{s}{t}\right| \gg 1 \quad \Longrightarrow \quad |\sigma_3| \gg |\sigma_4|$$

Scattering equations in MRK

- The simplest example: four points

$$\sigma_1 = 0, \quad \sigma_2 \to \infty, \qquad f_1 = -\frac{s_{13}}{\sigma_3} - \frac{s_{14}}{\sigma_4} = 0 \quad \Longrightarrow \quad \frac{\sigma_3}{\sigma_4} = \frac{s+t}{t}$$

In the Regge limit, $s \gg -t$, we have $|\sigma_3/\sigma_4| \simeq |s/t| \gg 1 \quad \Longrightarrow \quad |\sigma_3| \gg |\sigma_4|$

- The next-to-simplest: five points

$$\sigma_1 = 0, \quad \sigma_2 \to \infty, \quad \sigma_a^{(1)} = \frac{k_a^+}{k_a^\perp}, \quad \sigma_a^{(2)} = \frac{k_a^+}{k_a^{\perp *}} \quad a = 3, 4, 5$$

In MRK, $k_3^+ \gg k_4^+ \gg k_5^+$, we have again

$$|\sigma_3| \gg |\sigma_4| \gg |\sigma_5|$$

Scattering equations in MRK

- The simplest example: four points

$$\sigma_1 = 0, \quad \sigma_2 \to \infty, \qquad f_1 = -\frac{s_{13}}{\sigma_3} - \frac{s_{14}}{\sigma_4} = 0 \quad \Longrightarrow \quad \frac{\sigma_3}{\sigma_4} = \frac{s+t}{t}$$

In the Regge limit, $s \gg -t$, we have $|\sigma_3/\sigma_4| \simeq |s/t| \gg 1 \quad \Longrightarrow \quad |\sigma_3| \gg |\sigma_4|$

- The next-to-simplest: five points [Fairlie & Roberts, 1972]

$$\sigma_1 = 0, \quad \sigma_2 \to \infty, \quad \sigma_a^{(1)} = \frac{k_a^+}{k_a^\perp}, \quad \sigma_a^{(2)} = \frac{k_a^+}{k_a^{\perp *}} \quad a = 3, 4, 5$$

In MRK, $k_3^+ \gg k_4^+ \gg k_5^+$, we have again $|\sigma_3| \gg |\sigma_4| \gg |\sigma_5|$

- Any n-point scattering eqs have a MHV ($\overline{\text{MHV}}$) solution [Fairlie, 2008]

$$\sigma_1 = 0, \quad \sigma_2 \to \infty, \quad \sigma_a^{(\text{MHV})} = \frac{k_a^+}{k_a^\perp}, \quad \sigma_a^{(\overline{\text{MHV}})} = \frac{k_a^+}{k_a^{\perp *}} \quad a = 3, \ldots, n$$

In MRK, $k_3^+ \gg \cdots \gg k_n^+$, we have

$$|\sigma_3| \gg | \cdots \gg |\sigma_n|$$

Scattering equations in MRK

- The simplest example: four points

$$\sigma_1 = 0, \quad \sigma_2 \to \infty, \qquad f_1 = -\frac{s_{13}}{\sigma_3} - \frac{s_{14}}{\sigma_4} = 0 \quad \Longrightarrow \quad \frac{\sigma_3}{\sigma_4} = \frac{s+t}{t}$$

In the Regge limit, $s \gg -t$, we have $|\sigma_3/\sigma_4| \simeq |s/t| \gg 1 \quad \Longrightarrow \quad |\sigma_3| \gg |\sigma_4|$

- The next-to-simplest: five points [Fairlie & Roberts, 1972]

$$\sigma_1 = 0, \quad \sigma_2 \to \infty, \quad \sigma_a^{(1)} = \frac{k_a^+}{k_a^\perp}, \quad \sigma_a^{(2)} = \frac{k_a^+}{k_a^{\perp *}} \quad a = 3, 4, 5$$

In MRK, $k_3^+ \gg k_4^+ \gg k_5^+$, we have again $|\sigma_3| \gg |\sigma_4| \gg |\sigma_5|$

- Any n-point scattering eqs have a MHV (and $\overline{\text{MHV}}$) solution [Fairlie, 2008]

$$\sigma_1 = 0, \quad \sigma_2 \to \infty, \quad \sigma_a^{(\text{MHV})} = \frac{k_a^+}{k_a^\perp}, \quad \sigma_a^{(\overline{\text{MHV}})} = \frac{k_a^+}{k_a^{\perp *}} \quad a = 3, \ldots, n$$

In MRK, $k_3^+ \gg \cdots \gg k_n^+$, we have again $|\sigma_3| \gg |\cdots \gg |\sigma_n|$

- In MRK, we conjecture for arbitrary multiplicity n

$$|\Re(\sigma_3)| \gg \cdots \gg |\Re(\sigma_n)| \quad \& \quad |\Im(\sigma_3)| \gg \cdots \gg |\Im(\sigma_n)| \quad \text{with } (\sigma_1, \sigma_2) \to (0, \infty)$$

Scattering equations in MRK

Conjecture: In MRK, the solutions of the scattering eqs behave as

$$|\Re(\sigma_3)| \gg \cdots \gg |\Re(\sigma_n)| \;\;\&\;\; |\Im(\sigma_3)| \gg \cdots \gg |\Im(\sigma_n)| \quad \text{fixing } (\sigma_1, \sigma_2) \to (0, \infty)$$

Similarly, for t-solutions in the 4d scattering equations, we conjecture

$$|t_{i_1}| \gg |t_{i_2}| \gg \cdots, \quad i_a < i_{a+1} \in \mathfrak{P}; \qquad |t_{I_1}| \gg |t_{I_2}| \gg \cdots, \quad I_a < I_{a+1} \in \mathfrak{N}_{\neq 1,2}$$

where we fix $\{1, 2\} \subseteq \mathfrak{N}$, and gauge fix $\sigma_1 = 0, \sigma_2 = t_2 \to \infty, t_1 = -1$.

Scattering equations in MRK

Conjecture: In MRK, the solutions of the scattering eqs behave as

$$|\Re(\sigma_3)| \gg \cdots \gg |\Re(\sigma_n)| \;\;\&\;\; |\Im(\sigma_3)| \gg \cdots \gg |\Im(\sigma_n)| \quad \text{fixing } (\sigma_1, \sigma_2) \to (0, \infty)$$

Similarly, for t-variables in the 4d scattering equations, we conjecture

$$|t_{i_1}| \gg |t_{i_2}| \gg \cdots, \quad i_a < i_{a+1} \in \mathfrak{P}; \qquad |t_{I_1}| \gg |t_{I_2}| \gg \cdots, \quad I_a < I_{a+1} \in \mathfrak{N}_{\neq 1,2}$$

where we fix $\{1, 2\} \subseteq \mathfrak{N}$, and gauge fix $\sigma_1 = 0, \sigma_2 = t_2 \to \infty, t_1 = -1$.

- We numerically checked the scattering eqs up to 8 points. Furthermore, we conjecture that

$$\Re(\sigma_a) = \mathcal{O}\left(k_a^+\right), \quad \Im(\sigma_a) = \mathcal{O}\left(k_a^+\right), \quad t_a = \mathcal{O}\left(\sqrt{k_a^+\,\kappa^{-h_a}}\right), \quad a = 3, \ldots, n$$

$h_a = 1$ when $a \in \mathfrak{P}$, otherwise $h_a = -1$

- Here $\{3, n\} \subseteq \mathfrak{P}$, $\{1, 2\} \subset \mathfrak{N}$; for other cases, the solutions have the similar behavior

Solving scattering equations in MRK

Scattering equations in lightcone

- We choose the 4d (Geyer-Lipstein-Mason) scattering equations:
  - They have simpler structure compared with the CHY scattering equations;
  - The 4d formalism is more suitable to study helicity amplitudes;
  - 4d equations are written in spinors, MRK is naturally defined in lightcone coordinates.

Scattering equations in lightcone

- We choose the 4d (Geyer-Lipstein-Mason) scattering equations:
  - They have simpler structure compared with the CHY scattering equations;
  - The 4d formalism is more suitable to study helicity amplitudes;
  - 4d equations are written in spinors, MRK is naturally defined in lightcone coordinates.
- Perform rescalings for variables, $t_i = \tau_i \sqrt{k_i^+/\kappa}$ and $t_I = \tau_I \sqrt{\kappa\, k_I^+}/k_I^\perp$, and for equations

$$\mathcal{S}_i^1 \equiv \frac{1}{\lambda_i^1}\mathcal{E}_i^1 = 1 + \tau_i - \sum_{I\in\overline{\mathfrak{N}}} \frac{\tau_i \tau_I}{\sigma_i - \sigma_I}\frac{k_I^+}{k_I^\perp} = 0, \quad \overline{\mathfrak{N}} \equiv \mathfrak{N}\backslash\{1,2\}$$

$$\mathcal{S}_i^2 \equiv \frac{\lambda_i^1}{k_i^\perp}\mathcal{E}_i^2 = 1 + \frac{k_i^+}{k_i^\perp}\frac{\tau_i}{\sigma_i} - \frac{k_i^+}{k_i^\perp}\sum_{I\in\overline{\mathfrak{N}}} \frac{\tau_i \tau_I}{\sigma_i - \sigma_I} = 0,$$

$$\bar{\mathcal{S}}_I^{\dot{1}} \equiv \lambda_I^2\, \bar{\mathcal{E}}_I^{\dot{1}} = k_I^\perp - \sum_{i\in\mathfrak{P}} \frac{\tau_i \tau_I}{\sigma_I - \sigma_i} k_i^+ = 0,$$

$$\bar{\mathcal{S}}_I^{\dot{2}} \equiv \lambda_I^1\, \bar{\mathcal{E}}_I^{\dot{2}} = (k_I^\perp)^* - \frac{k_I^+}{k_I^\perp}\sum_{i\in\mathfrak{P}} \frac{\tau_i \tau_I}{\sigma_I - \sigma_i}(k_i^\perp)^* = 0$$

- Perfectly suitable for the study of Multi-Regge kinematics.

Scattering equations in MRK

In MRK, according to our conjecture

$$\frac{1}{\sigma_a - \sigma_b} \simeq \frac{1}{\sigma_a}, \qquad a < b$$

The 4d scattering equations get greatly simplified at leading order:

$$\mathcal{S}_i^1 = 1 + \tau_i \left(1 + \sum_{l \in \overline{\mathfrak{N}}_{<i}} \zeta_l \right) = 0\,, \qquad \bar{\mathcal{S}}_l^{\dot{1}} = k_l^{\perp} + \tau_l \sum_{i \in \mathfrak{P}_{<l}} \zeta_i\, k_i^{\perp} = 0\,,$$

$$\mathcal{S}_i^2 = 1 + \zeta_i \left(1 - \sum_{l \in \overline{\mathfrak{N}}_{>i}} \tau_l \right) = 0\,, \qquad \bar{\mathcal{S}}_l^{\dot{2}} = (k_l^{\perp})^* - \zeta_l \sum_{i \in \mathfrak{P}_{>l}} \tau_i (k_i^{\perp})^* = 0\,,$$

where $A_{>i} := \{a \in A |\, a > i\}$, and we define

$$\zeta_a \equiv \frac{k_a^+}{k_a^\perp} \frac{\tau_a}{\sigma_a}, \quad 3 \leq a \leq n$$

- 4d scattering equations become 'almost linear' in MRK.
- Indeed, as I will show later, they exactly have a unique solution.

Solving scattering eqs in MRK

Let us rewrite the equations as:

$$\mathcal{S}_i^1 = 1 + a_i\,\tau_i = 0\,, \qquad \bar{\mathcal{S}}_l^{\dot{2}} = (k_l^\perp)^* + b_l\,\zeta_l = 0$$
$$\mathcal{S}_i^2 = 1 + c_i\,\zeta_i = 0\,, \qquad \bar{\mathcal{S}}_l^{\dot{1}} = k_l^\perp + d_l\,\tau_l = 0$$

with

$$a_i \equiv 1 + \sum_{l\in\overline{\mathfrak{N}}_{<i}} \zeta_l\,, \quad b_l \equiv -\sum_{i\in\mathfrak{P}_{>l}} \tau_i\,k_i^{\perp *}\,, \quad c_i \equiv 1 - \sum_{l\in\overline{\mathfrak{N}}_{>i}} \tau_l\,, \quad d_l \equiv \sum_{i\in\mathfrak{P}_{<l}} \zeta_i\,k_i^\perp$$

- At the first step, we can use the equations $\mathcal{S}_i^\alpha = 0$ to obtain

$$\tau_i = -\frac{1}{a_i}\,, \qquad \zeta_i = -\frac{1}{c_i}$$

Solving scattering eqs in MRK

Let us rewrite the equations as:

$$\mathcal{S}_i^1 = 1 + a_i\,\tau_i = 0\,, \qquad \bar{\mathcal{S}}_I^{\dot{2}} = (k_I^{\perp})^* + b_I\,\zeta_I = 0$$
$$\mathcal{S}_i^2 = 1 + c_i\,\zeta_i = 0\,, \qquad \bar{\mathcal{S}}_I^{\dot{1}} = k_I^{\perp} + d_I\,\tau_I = 0$$

with

$$a_i \equiv 1 + \sum_{I\in\overline{\mathfrak{N}}_{<i}} \zeta_I\,, \quad b_I \equiv -\sum_{i\in\mathfrak{P}_{>I}} \tau_i\,{k_i^{\perp}}^*, \quad c_i \equiv 1 - \sum_{I\in\overline{\mathfrak{N}}_{>i}} \tau_I, \quad d_I \equiv \sum_{i\in\mathfrak{P}_{<I}} \zeta_i\,k_i^{\perp}$$

- At the first step, we can use the equations $\mathcal{S}_i^\alpha = 0$ to obtain

$$\tau_i = -\frac{1}{a_i}, \qquad \zeta_i = -\frac{1}{c_i}$$

- Then the equations $\bar{\mathcal{S}}_I^{\dot{1}} = k_I^{\perp} + d_I\,\tau_I = 1$ are independent with $\bar{\mathcal{S}}_I^{\dot{2}} = k_I^{\perp} + d_I\,\tau_I = 0$, and two sets of equations have the same structure.

$$d_I = -\sum_{i\in\mathfrak{P}_{<I}} k_i^{\perp}\left(1 - \sum_{J\in\overline{\mathfrak{N}}_{>i}} \tau_J\right)^{-1}, \quad b_I = \sum_{i\in\mathfrak{P}_{>I}} (k_i^{\perp})^*\left(1 + \sum_{J\in\overline{\mathfrak{N}}_{<i}} \zeta_J\right)^{-1}$$

Solving scattering eqs in MRK

Let us try to solve

$$\bar{\mathcal{S}}_I^{\mathrm{i}} = k_I^{\perp} + d_I\,\tau_I, \qquad d_I = -\sum_{i\in\mathfrak{P}_{<I}} k_i^{\perp}\Bigg(1-\sum_{J\in\overline{\mathfrak{N}}_{>i}}\tau_J\Bigg)^{-1}$$

First, we reorder labels: $I_1 < \cdots < I_{m=k-2}$. The coefficients d_I satisfy the following recursion

$$d_{I_r} = -\Bigg(\sum_{i\in\mathfrak{P}_{<I_{r-1}}} k_i^{\perp} + \sum_{I_{r-1}<i<I_r} k_i^{\perp}\Bigg)\Bigg(1-\sum_{J\in\overline{\mathfrak{N}}_{>i}}\tau_J\Bigg)^{-1} = d_{I_{r-1}} - \Bigg(1-\sum_{l=r}^{m}\tau_{I_l}\Bigg)^{-1}\sum_{I_{r-1}<a<I_r} k_a^{\perp}$$

which starts with $d_{I_0} = 0$.

Solving scattering eqs in MRK

Let us try to solve

$$\bar{\mathcal{S}}_I^{\mathfrak{i}} = k_I^{\perp} + d_I\,\tau_I, \qquad d_I = -\sum_{i\in\mathfrak{P}_{<I}} k_i^{\perp}\left(1-\sum_{J\in\overline{\mathfrak{N}}_{>i}}\tau_J\right)^{-1}$$

First, we reorder labels: $I_1 < \cdots < I_{m=k-2}$. The coefficients d_I satisfy the following recursion

$$d_{I_r} = -\left(\sum_{i\in\mathfrak{P}_{<I_{r-1}}} k_i^{\perp} + \sum_{I_{r-1}<i<I_r} k_i^{\perp}\right)\left(1-\sum_{J\in\overline{\mathfrak{N}}_{>i}}\tau_J\right)^{-1} = d_{I_{r-1}} - \left(1-\sum_{l=r}^{m}\tau_{I_l}\right)^{-1}\sum_{I_{r-1}<a<I_r} k_a^{\perp}$$

which starts with $d_{I_0}=0$. Using it, we can get

$$0 = \bar{\mathcal{S}}_{I_r}^{\mathfrak{i}} = \left(1-\sum_{l=r}^{m}\tau_{I_l}\right)^{-1}\left[k_{I_r}^{\perp}\left(1-\sum_{l=r+1}^{m}\tau_{I_l}\right) - \tau_{I_r} q_{I_r+1}^{\perp}\right]$$

It naturally leads to the recursion of the solution of the 4d scattering equations

$$\tau_{I_m} = \frac{k_{I_m}^{\perp}}{q_{I_m+1}^{\perp}}, \qquad \tau_{I_r} = \frac{k_{I_r}^{\perp}}{q_{I_r+1}^{\perp}}\left(1-\sum_{l=r+1}^{m}\tau_{I_l}\right)$$

Solving scattering eqs in MRK

Let us try to solve

$$\bar{\mathcal{S}}_I^{\mathfrak{i}} = k_I^{\perp} + d_I\,\tau_I, \qquad d_I = -\sum_{i\in\mathfrak{P}_{<I}} k_i^{\perp}\left(1-\sum_{J\in\overline{\mathfrak{N}}_{>i}}\tau_J\right)^{-1}$$

First, we reorder labels: $I_1 < \cdots < I_{m=k-2}$. The coefficients d_I satisfy the following recursion

$$d_{I_r} = -\left(\sum_{i\in\mathfrak{P}_{<I_{r-1}}} k_i^{\perp} + \sum_{I_{r-1}<i<I_r} k_i^{\perp}\right)\left(1-\sum_{J\in\overline{\mathfrak{N}}_{>i}}\tau_J\right)^{-1} = d_{I_{r-1}} - \left(1-\sum_{l=r}^{m}\tau_{I_l}\right)^{-1}\sum_{I_{r-1}<a<I_r} k_a^{\perp}$$

which starts with $d_{I_0}=0$. Using it, we can get

$$0 = \bar{\mathcal{S}}_{I_r}^{\mathfrak{i}} = \left(1-\sum_{l=r}^{m}\tau_{I_l}\right)^{-1}\left[k_{I_r}^{\perp}\left(1-\sum_{l=r+1}^{m}\tau_{I_l}\right) - \tau_{I_r}\,q_{I_r+1}^{\perp}\right]$$

It naturally leads to the recursion of the solution of the 4d scattering equations

$$\tau_{I_m} = \frac{k_{I_m}^{\perp}}{q_{I_m+1}^{\perp}}, \qquad \tau_{I_r} = \frac{k_{I_r}^{\perp}}{q_{I_r+1}^{\perp}}\left(1-\sum_{l=r+1}^{m}\tau_{I_l}\right) = \frac{k_{I_r}^{\perp}}{q_{I_r+1}^{\perp}}\prod_{l=r+1}^{m}\frac{q_{I_l}^{\perp}}{q_{I_l+1}^{\perp}}$$

Solving scattering eqs in MRK

Solving $\bar{\mathcal{S}}_l^{\dot{1}} = 0$ gives

$$\tau_{l_r} = \frac{k_{l_r}^{\perp}}{q_{l_r+1}^{\perp}} \left(1 - \sum_{l=r+1}^{m} \tau_{l_l}\right) = \frac{k_{l_r}^{\perp}}{q_{l_r+1}^{\perp}} \prod_{l=r+1}^{m} \frac{q_{l_l}^{\perp}}{q_{l_l+1}^{\perp}}$$

Similarly, we can solve $\bar{\mathcal{S}}_l^{\dot{2}} = 0$ and obtain

$$\zeta_{l_r} = \left(\frac{k_{l_r}^{\perp}}{q_{l_r}^{\perp}}\right)^* \left(1 + \prod_{l=1}^{r-1} \zeta_{l_l}\right) = \left(\frac{k_{l_r}^{\perp}}{q_{l_r}^{\perp}}\right)^* \left(\prod_{l=1}^{r-1} \frac{q_{l_l+1}^{\perp}}{q_{l_l}^{\perp}}\right)^*$$

For τ_i and ζ_i, we have

$$\tau_i = -\frac{1}{a_i} = \left(-1 + \sum_{l \in \overline{\mathfrak{N}}_{<i}} \zeta_l\right)^{-1} = -\left(\prod_{l \in \overline{\mathfrak{N}}_{<i}} \frac{q_l^{\perp}}{q_{l+1}^{\perp}}\right)^*$$

$$\zeta_i = -\frac{1}{c_i} = \left(1 - \sum_{l \in \overline{\mathfrak{N}}_{>i}} \tau_l\right)^{-1} = -\prod_{l \in \overline{\mathfrak{N}}_{>i}} \frac{q_{l+1}^{\perp}}{q_l^{\perp}} \,.$$

Finally, in MRK we exactly solve the 4d scattering eqs of any sector k and any multiplicity!

MRK solutions

• For each "helicity configuration" of any sector k and any multiplicity n, we exactly solved the 4d scattering equations

$$\tau_I = \frac{k_I^\perp}{q_{I+1}^\perp} \prod_{J \in \mathfrak{N}_{>I}} \frac{q_J^\perp}{q_{J+1}^\perp}, \qquad \zeta_I = \left(\frac{k_I^\perp}{q_I^\perp}\right)^* \left(\prod_{J \in \mathfrak{N}_{<I}} \frac{q_{J+1}^\perp}{q_J^\perp}\right)^*, \quad I \in \mathfrak{N}$$

$$\tau_i = -\left(\prod_{I \in \overline{\mathfrak{N}}_{<i}} \frac{q_I^\perp}{q_{I+1}^\perp}\right)^*, \qquad \zeta_i = -\prod_{I \in \overline{\mathfrak{N}}_{>i}} \frac{q_{I+1}^\perp}{q_I^\perp}, \quad i \in \mathfrak{P}$$

• It is very rare that one can analytically solve the scattering eqs for arbitrary multiplicities.

- MHV (and $\overline{\text{MHV}}$) [Fairlie & Roberts, 1972]
- A very special two parameter family of kinematics [Kalousios, 1312.7743]

• Very natural to ask: how to evaluate amplitudes using this MRK solution?

Gauge theory amplitudes in MRK

YM amplitudes

N^{k-2}MHV gluon amplitudes: [Geyer, Lipstein & Mason, 1404.6219]

$$\mathcal{A}_n(1^-, 2^-, \ldots, n)$$
$$= -s \int \prod_{a=3}^{n} \frac{d\sigma_a d\tau_a}{\tau_a} \frac{1}{\sigma_{34} \cdots \sigma_{n-1,n}\sigma_n} \left(\prod_{i \in \mathfrak{P}} \frac{1}{k_i^\perp} \delta^2(S_i^\alpha) \right) \left(\prod_{I \in \overline{\mathfrak{N}}} k_I^\perp \delta^2(\bar{S}_I^{\dot{\alpha}}) \right)$$

- $\mathfrak{P}$ ($\mathfrak{N}$) collects the labels of negative (positive) gluons, $|\mathfrak{N}| = k$ and $\overline{\mathfrak{N}} = \mathfrak{N} \backslash \{1, 2\}$
- $\mathcal{A}_n(1^\pm, 2^\mp, \ldots)$ can be evaluated using the almost same formula via "SUSY Ward identity"
- Similarly, we can obtain the formula for amplitudes with a few massless quark pairs

[He & Zhang, 1607.0284; Dixon, Henn, Plefka & Schuster, 1010.3991]

YM amplitudes in MRK

In MRK, gluon amplitudes become

$$\mathcal{A}_n(1^-,2^-,\ldots,n) \simeq -s\left(\int\prod_{a=3}^{n}\frac{d\tau_a d\zeta_a}{\zeta_a\tau_a}\right)\left(\prod_{i\in\mathfrak{P}}\frac{1}{k_i^\perp}\,\delta^2(S_i^\alpha)\right)\left(\prod_{I\in\overline{\mathfrak{N}}}k_I^\perp\,\delta^2(\bar{S}_I^{\dot\alpha})\right)$$

- Using the procedure similar to solving the equations, we can localize these integrals

$$\mathcal{A}_n(1,\ldots,n) \simeq s\,C(2;3)\frac{-1}{|q_4^\perp|^2}V(q_4;4;q_5)\cdots\frac{-1}{|q_{n-1}^\perp|^2}V(q_{n-1};n{-}1;q_n)\frac{-1}{|q_n^\perp|^2}C(1;n)$$

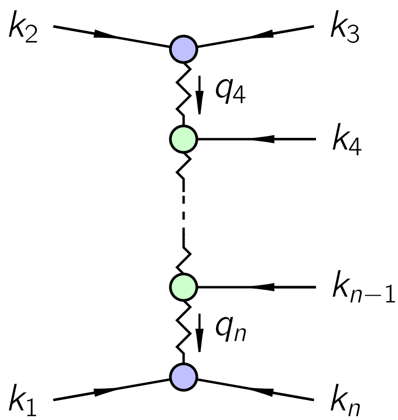
Buildling blocks:

$$C(2^\pm;3^\pm)=C(1^\pm;n^\pm)=0,\quad C(2^\pm;3^\mp)=1$$

$$C(1^-;n^+)=C(1^+;n^-)^*=\frac{(k_n^\perp)^*}{k_n^\perp}$$

$$V\big(q_a;a^+;q_{a+1}\big)=V\big(q_a;a^-;q_{a+1}\big)^*=\frac{(q_a^\perp)^*\,q_{a+1}^\perp}{k_a^\perp}$$

[Kuraev, Lipatov & Fadin, 1976; Lipatov, 1976; Lipatov, 1991; Del Duca, 1995]



How about gravity?

Graviton amplitudes in MRK

- Similarly, the formula for tree superamplitudes in $\mathcal{N}=8$ SUGRA is constructed from 4d ambitwistor strings [Geyer, Lipstein & Mason, 1404.6219].

- In MRK, the Geyer-Lipstein-Mason formula of graviton amplitudes takes

$$\mathcal{M}_n = s^2 \left(\int \prod_{a=3}^{n} \frac{d\zeta_a d\tau_a}{\zeta_a^2 \tau_a^2} \right) \left(\prod_{I \in \overline{\mathfrak{N}}} (k_I^{\perp})^2 \delta^2(\bar{\mathcal{S}}_I^{\dot\alpha}) \right) \left(\prod_{i \in \mathfrak{P}} \frac{\delta^2(\mathcal{S}_i^{\alpha})}{(k_i^{\perp})^2} \right) \det{}' \overline{\mathrm{H}} \det{}' \mathrm{H}$$

where

$$\overline{\mathrm{H}}_{ij} = (k_j^{\perp}\zeta_j)(k_i^{\perp *}\tau_i), \quad i > j \in \mathfrak{P}; \quad \overline{\mathrm{H}}_{ii} = -\sum_{j \in \mathfrak{P}, j \neq i} \overline{\mathrm{H}}_{ij};$$

$$\mathrm{H}_{12} = -1, \quad \mathrm{H}_{1I} = -\zeta_I, \quad \mathrm{H}_{2I} = -\tau_I, \quad \mathrm{H}_{IJ} = \tau_I \zeta_J, \quad I > J \in \overline{\mathfrak{P}}$$

$$\mathrm{H}_{11} = -\mathrm{H}_{12} - \sum_{I \in \overline{\mathfrak{N}}} \mathrm{H}_{1I}, \quad \mathrm{H}_{22} = -\mathrm{H}_{12} - \sum_{I \in \overline{\mathfrak{N}}} \mathrm{H}_{2I}, \quad \mathrm{H}_{II} = -\mathrm{H}_{1I} - \mathrm{H}_{2I} - \sum_{b \in \mathfrak{N}, b \neq a} \mathrm{H}_{ab}$$

MHV graviton amplitudes

In MHV sector, the GLM formula is simply reduced to Hodges formula [Hodges, 1204.1930]

$$\mathcal{M}_n(1^-, 2^-, 3^+, \ldots, n^+) \simeq \frac{s^2}{(k_3^\perp)^2} \det \phi,$$

In MRK

$$\phi = \begin{pmatrix} x_4 + v_4 & x_5 & x_6 & \cdots & x_7 & x_n \\ x_5 & x_5 + v_5 & x_6 & \cdots & x_7 & x_n \\ x_6 & x_6 & x_6 + v_6 & \cdots & x_7 & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & x_{n-1} & x_{n-1} & \cdots & x_{n-1} + v_{n-1} & x_n \\ x_n & x_n & x_n & \cdots & x_n & x_n \end{pmatrix}.$$

with

$$\phi_{ab} = \frac{k_a^{\perp *}}{k_a^\perp} = x_a, \qquad a > b \geq 3,$$

$$\phi_{aa} = v_a + x_a, \quad v_a = \frac{k_a^\perp q_a^{\perp *} - q_a^\perp k_a^{\perp *}}{(k_a^\perp)^2}, \qquad 3 \leq a \leq n,$$

MHV graviton amplitudes

In MHV sector, the GLM formula is simply reduced to Hodges formula [Hodges, 1204.1930]

$$\mathcal{M}_n(1^-, 2^-, 3^+, \ldots, n^+) \simeq \frac{s^2}{(k_3^\perp)^2} \det \phi,$$

In MRK

$$\det\phi = \left\| \begin{matrix} x_4+v_4 & x_5 & x_6 & \cdots & x_7 & x_n \\ x_5 & x_5+v_5 & x_6 & \cdots & x_7 & x_n \\ x_6 & x_6 & x_6+v_6 & \cdots & x_7 & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & x_{n-1} & x_{n-1} & \cdots & x_{n-1}+v_{n-1} & x_n \\ x_n & x_n & x_n & \cdots & x_n & x_n \end{matrix} \right\|$$

triangularization: $\text{column}_i - \text{column}_1$

MHV graviton amplitudes

In MHV sector, the GLM formula is simply reduced to Hodges formula [Hodges, 1204.1930]

$$\mathcal{M}_n(1^-, 2^-, 3^+, \ldots, n^+) \simeq \frac{s^2}{(k_3^\perp)^2} \det \phi,$$

Almost triangular!

$$\det \phi = \left\| \begin{matrix} x_4 + v_4 & x_5 - x_4 - v_4 & x_6 - x_4 - v_4 & \cdots & x_{n-1} - x_4 - v_4 & x_n - x_4 - v_4 \\ 0 & v_5 & x_6 - x_5 & \cdots & x_{n-1} - x_5 & x_n - x_5 \\ 0 & 0 & v_6 & \cdots & x_{n-1} - x_6 & x_n - x_6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & v_{n-1} & x_n - x_{n-1} \\ x_n & 0 & 0 & \cdots & 0 & x_n \end{matrix} \right\|$$

$$\text{row}_1 - \frac{v_4}{x_n} \times \text{row}_{n-3}$$

MHV graviton amplitudes

In MHV sector, the GLM formula is simply reduced to Hodges formula [Hodges, 1204.1930]

$$\mathcal{M}_n(1^-,2^-,3^+,\ldots,n^+) \simeq \frac{s^2}{(k_3^\perp)^2}\det\phi,$$

Almost triangular!

$$\det\phi = \left\|\begin{matrix} x_4 & x_5-x_4-v_4 & x_6-x_4-v_4 & \cdots & x_{n-1}-x_4-v_4 & x_n-x_4 \\ 0 & v_5 & x_6-x_5 & \cdots & x_{n-1}-x_5 & x_n-x_5 \\ 0 & 0 & v_6 & \cdots & x_{n-1}-x_6 & x_n-x_6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & v_{n-1} & x_n-x_{n-1} \\ x_n & 0 & 0 & \cdots & 0 & x_n \end{matrix}\right\|$$

MHV graviton amplitudes

In MHV sector, the GLM formula is simply reduced to Hodges formula [Hodges, 1204.1930]

$$\mathcal{M}_n(1^-, 2^-, 3^+, \ldots, n^+) \simeq \frac{s^2}{(k_3^\perp)^2} \det \phi,$$

where

$$\det \phi = \left\| \begin{matrix} x_4 & x_5 - x_4 - v_4 & \cdots & x_{n-1} - x_4 - v_4 & x_n - x_4 \\ 0 & v_5 & \cdots & x_{n-1} - x_5 & x_n - x_5 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & v_{n-1} & x_n - x_{n-1} \\ {\color{red} x_n} & 0 & \cdots & 0 & x_n \end{matrix} \right\|$$

$$= \left(v_4 v_5 \ldots v_{n-1} x_n\right) \left(1 + x_n \left(\psi^{-1}\right)_{1,n-3}\right) = \frac{k_3^\perp}{k_n^\perp} v_4\, v_5 \cdots v_{n-1}\, x_n$$

Matrix determinant lemma [Harville, 1997; Ding & Zhou, 2007]

$$\det\left(\psi + u v^\mathsf{T}\right) = \left(1 + v^\mathsf{T} \psi^{-1} u\right) \det \psi$$

Here we can take $u = (x_n, 0, \ldots, 0)^\mathsf{T}$ and $v = (0, 0, \ldots, 0, 1)^\mathsf{T}$

MHV graviton amplitudes

In MRK, the MHV amplitude of gravitons factorizes

$$\mathcal{M}_n(1^-,2^-,\dots) = s^2\,\mathcal{C}(2^-;3^+)\frac{-1}{|q_4^\perp|^2}\,\mathcal{V}(q_4;4^+;q_5)\,\cdots\,\frac{-1}{|q_{n-1}^\perp|^2}\,\mathcal{V}(q_{n-1},(n-1)^+,q_n)\,\frac{-1}{|q_n^\perp|^2}\mathcal{C}(1^-;n^+)$$

Building blocks:

$$\mathcal{C}(2^-;3^+) = 1,$$

$$\mathcal{C}(1^-;n^+) = x_n^2 = \left(\frac{k_n^{\perp *}}{k_n^\perp}\right)^2$$

$$\mathcal{V}(q_i,i^+,q_{i+1}) = q_i^{\perp *}v_i\,q_{i+1}^\perp = \frac{q_i^{\perp *}\left(k_i^\perp q_i^{\perp *} - k_i^{\perp *}q_i^\perp\right)q_{i+1}^\perp}{(k_i^\perp)^2}$$

All graviton amplitudes

Beyond MHV, the formula become complicated; but fortunately the similar trick works and wen can obtain

$$\mathcal{M}_n = s^2 \mathcal{C}(2;3)\frac{-1}{|q_4^\perp|^2}\mathcal{V}(q_4;4;q_5)\cdots\frac{-1}{|q_{n-1}^\perp|^2}\mathcal{V}(q_{n-1},n-1,q_n)\frac{-1}{|q_n^\perp|^2}\mathcal{C}(1;n)$$

Building blocks:

$$\mathcal{C}(2^\pm;3^\mp)=1, \quad \mathcal{C}(1^-;n^+)=\mathcal{C}(1^+;n^-)^*=\left(\frac{k_n^{\perp *}}{k_n^\perp}\right)^2, \quad \mathcal{C}(a^\pm;b^\pm)=0$$

$$\mathcal{V}(q_i,i^+,q_{i+1})=q_i^{\perp *}v_i\,q_{i+1}^\perp=\frac{q_i^{\perp *}\left(k_i^\perp q_i^{\perp *}-k_i^{\perp *}q_i^\perp\right)q_{i+1}^\perp}{(k_i^\perp)^2}$$

$$\mathcal{V}(q_l,l^-,q_{l+1})=q_l^\perp v_l^*\,q_{l+1}^{\perp *}=\frac{q_l^\perp\left(k_l^{\perp *}q_l^\perp-k_l^\perp q_l^{\perp *}\right)q_{l+1}^{\perp *}}{(k_l^{\perp *})^2}$$

- Complicated amplitudes of gravitons simply factorizes into a t-channel ladder in MRK!
- The result agrees with the one from dispersion relations [Lipatov 1982]

Quasi Multi-Regge Kinematics

Scattering equations in QMRK

When relaxing the strong rapidity ordering in MRK, e.g.

$$y_3 \simeq \cdots \simeq y_m \gg y_{m+1} \simeq \cdots \simeq y_r \gg y_{r+1} \cdots \quad \text{and} \quad |k_3^\perp| \simeq \cdots \simeq |k_n^\perp|$$

• Very similar to MRK, in QMRK we conjecture that all solutions of the scattering equations satisfy the same hierarchy as the ordering of the rapidities. More precisely,

$$\Re(\sigma_a) = \mathcal{O}\left(k_a^+\right), \quad \Im(\sigma_a) = \mathcal{O}\left(k_a^+\right), \quad t_a = \mathcal{O}\left(\sqrt{k_a^+ \,\kappa^{-h_a}}\right), \quad a = 3, \ldots, n$$

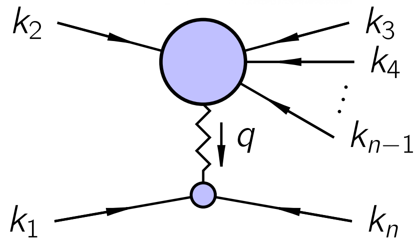
• Fix $(\sigma_1, \sigma_2, \sigma_3) \to (0, \infty, k_3^+)$ or $(\sigma_1, \sigma_2 = t_2, t_1) \to (0, \infty, -1)$

• $\{3, n\} \subseteq \mathfrak{P}$, $\{1, 2\} \subset \mathfrak{N}$, the solutions have similar behaviors for other cases

• We numerically checked the scattering eqs up to 8 points

• Using the conjecture, we can obtain the correct factorization of amplitudes

Gluon amplitudes in QMRK (I)



- Let us study $y_3 \simeq \cdots \simeq y_{n-1} \gg y_n$. Our conjecture gives

$$\mathcal{S}_n^1 = 1 + \tau_n \Big(1 + \sum_{I \in \overline{\mathfrak{N}}} \zeta_I \Big) = 0, \quad \mathcal{S}_n^2 = 1 + \zeta_n = 0$$

- Localize the integrals over ζ_n and τ_n by $\mathcal{S}_n^\alpha = 0$

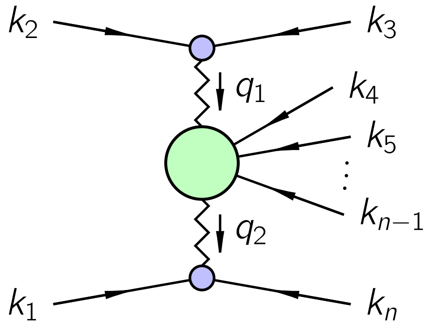
$$\mathcal{A}_n(1^-, 2^-, \ldots, n^+) \simeq s\, C(2^-; 3, \ldots, n{-}1) \frac{-1}{|q_n^\perp|^2} C(1^-; n^+)\,,$$

- The generalized impact factor is given by a CHY-type formula

$$C\big(2^-; 3, \ldots, n{-}1\big) = q_n^\perp \int \prod_{a=3}^{n-1} \frac{d\sigma_a d\tau_a}{\tau_a} \frac{1}{\sigma_{34} \cdots \sigma_{n-2,n-1} \sigma_{n-1}} \Bigg(\prod_{i \in \mathfrak{P}, I \in \overline{\mathfrak{N}}} \frac{k_I^\perp}{k_i^\perp} \Bigg)$$
$$\times \prod_{I \in \overline{\mathfrak{N}}} \delta\Bigg(k_I^\perp - \sum_{i \in \mathfrak{P}} \frac{\tau_I \tau_i}{\sigma_I - \sigma_i} k_i^+ \Bigg) \delta\Bigg(k_I^{\perp *} - \frac{k_I^+}{k_I^\perp} \sum_{i \in \mathfrak{P}} \frac{\tau_I \tau_i}{\sigma_I - \sigma_i} k_i^{\perp *} - \zeta_I \frac{q_n^{\perp *}}{1 + \sum_{J \in \overline{\mathfrak{N}}} \zeta_J} \Bigg)$$
$$\times \prod_{i \in \mathfrak{P}} \delta\Bigg(1 + \tau_i - \sum_{I \in \overline{\mathfrak{N}}} \frac{\tau_i \tau_I}{\sigma_i - \sigma_I} \frac{k_I^+}{k_I^\perp} \Bigg) \delta\Bigg(1 + \zeta_i - \frac{k_i^+}{k_i^\perp} \sum_{I \in \overline{\mathfrak{N}}} \frac{\tau_i \tau_I}{\sigma_i - \sigma_I} \Bigg),$$

Gluon amplitudes in QMRK (II)

- Similarly, in the limit



$$y_3 \gg y_4 \simeq \cdots \simeq y_{n-1} \gg y_n$$

using our conjecture, we can fix the integrals corresponding to legs 3 and n and obtain

$$\mathcal{A}_n(1^-, 2^-, 3, \ldots, n) \simeq s\, C(2^-; 3) \frac{-1}{|q_4^\perp|^2} V(q_4; 4, \ldots, n{-}1; q_n) \frac{-1}{|q_n^\perp|^2} C(1^-; n)$$

- Generalised Lipatov vertices admit the following CHY-type representation

$$V\big(q_4; 4, \ldots, n{-}1; q_n\big) = (q_4^{\perp *} q_n^\perp) \int \prod_{a=4}^{n-1} \frac{d\sigma_a dt_a}{t_a} \frac{1}{\sigma_{45} \cdots \sigma_{n-2,n-1} \sigma_{n-1}} \left(\prod_{i \in \mathfrak{P}, I \in \mathfrak{N}} \frac{k_I^\perp}{k_i^\perp} \right)$$

$$\times \prod_{I \in \mathfrak{N}} \delta\left(k_I^\perp - \sum_{i \in \mathfrak{P}} \frac{t_i t_I}{\sigma_I - \sigma_i} k_i^+ + \frac{t_I}{1 - \sum_{J \in \mathfrak{N}} t_J} q_4^\perp \right) \delta\left(k_I^{\perp *} - \frac{k_I^+}{k_I^\perp} \sum_{i \in \mathfrak{P}} \frac{t_i t_I}{\sigma_I - \sigma_i} k_i^{\perp *} - \frac{\zeta_I}{1 + \sum_{J \in \mathfrak{N}} \zeta_J} q_n^{\perp *} \right)$$

$$\times \prod_{i \in \mathfrak{P}} \delta\left(1 - \sum_{I \in \mathfrak{N}} \frac{t_i t_I}{\sigma_i - \sigma_I} \frac{k_I^+}{k_I^\perp} + t_i \right) \delta\left(1 - \frac{k_i^+}{k_i^\perp} \sum_{I \in \mathfrak{N}} \frac{t_i t_I}{\sigma_i - \sigma_I} + \zeta_i \right).$$

Impact factors and Lipatov vertices

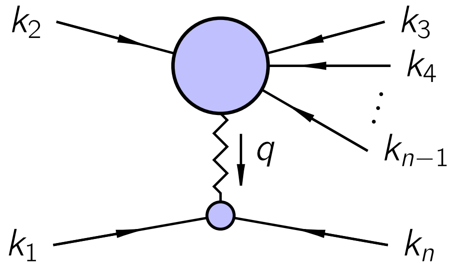
- Byproducts: the CHY-type formulas for generalized impact factors and Lipatov vertices

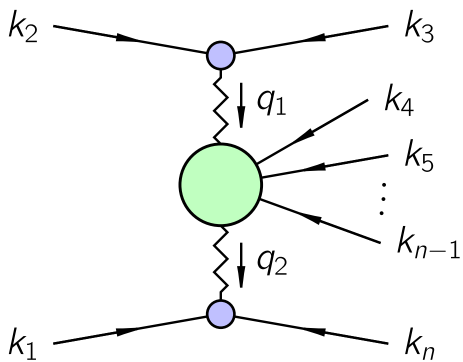
- We numerically checked these two formulas up to $n = 8$

- In particular, we can reproduce correct results for all Lipatov vertices $V(q_1; a, b; q_2)$ $(g^* g^* \to gg)$ and impact factors $C(2; 3, 4, 5)$ $(gg^* \to gg)$ analytically

- We checked these formulas have correct factorization in soft, collinear limits

- We checked they have correct factorization in the Regge limit $y_3 \gg \cdots \gg y_a \simeq \cdots \simeq y_b \gg y_{b+1} \gg \cdots$





[Lipatov, hep-ph/9502308; Del Duca, hep-ph/9503340, hep-ph/9601211, hep-ph/9909464...]

Summary & Outlook

• We have initiated the study of Regge kinematics through the lens of the scattering equations.

• We found the asymptotic behaviour of the solutions in (quasi) Multi-Regge regime.

• While have no a proof of our conjecture, our conjecture implies the expected factorization of the amplitudes in YM and gravity. This gives strong support to our conjecture!

• In particular, an application of our conjecture leads to solving the 4d scattering equations exactly in MRK.

• Byproduct: we obtain the CHY-type formulas for impact factors and Lipatov vertices.

Summary & Outlook

- We have initiated the study of Regge kinematics through the lens of the scattering equations.
- We found the asymptotic behaviour of the solutions in (quasi) Multi-Regge regime.
- While have no a proof of our conjecture, our conjecture implies the expected factorization of the amplitudes in YM and gravity. This gives strong support to our conjecture!
- In particular, an application of our conjecture leads to solving the 4d scattering equations exactly in MRK.
- Byproduct: we obtain the CHY-type representations for impact factors and Lipatov vertices.

- It would be interesting to
  - find a rigorous mathematical proof of our conjecture
  - apply this framework for other theories
  - extend to loop level

Grazie!