Steps Toward a Two Loop Graphical Coproduct

James Matthew
in collaboration with Samuel Abreu, Ruth Britto, Claude Duhr and Einan Gardi

Amplitudes in the LHC Era
November 2018

Overview

Polylogarithms

- The space of Goncharov polylogarithms $\mathcal{A}$ given by

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t), \qquad G(; z) = 1$$

  possesses a mapping $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$ called a coaction which encodes their analytic structure via the relations $\Delta \circ \mathrm{Disc} = (\mathrm{Disc} \otimes 1) \circ \Delta$ and $\Delta \circ \partial = (1 \otimes \partial) \circ \Delta$, and allows easy derivation of functional relations.
- The coaction takes the form:

$$\Delta G(\underline{a}; z) = \sum_{\emptyset \subseteq \underline{b} \subseteq \underline{a}} G(\underline{b}; z) \otimes G_{\underline{b}}(\underline{a}; z)$$

  - $G(\underline{b}; z)$ has a modified integrand.
  - $G_{\underline{b}}(\underline{a}; z)$ denotes $G(\underline{a}; z)$ with residues taken at poles $\underline{b}$, so the integration contour is modified.

Integrands and Contours in the Coproduct

- This points to a structure of the form

$$\Delta \int_{\gamma} \omega = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$

- What is the relation between the $\{\omega_i\}$, $\{\gamma_i\}$? Let $\Gamma_{\underline{b}}$ be the contour from 0 to z encircling poles in $\underline{b}$ and $\omega_{\underline{b}}$ be the integrand of $G(\underline{b}; z)$, then:

$$\int_{\Gamma_{\underline{b}}} \omega_{\underline{a}} = \begin{cases} z & \underline{b} = \underline{a} = \emptyset \\ (2\pi i)^{|\underline{a}|} & \underline{b} = \underline{a} \neq \emptyset \\ (2\pi i)^{|\underline{b}|} G_{\underline{b}}(\underline{a}, z) & \underline{b} \subsetneq \underline{a} \\ 0 & \underline{b} \not\subseteq \underline{a} \end{cases}$$

- Normalise the contours ($\gamma_\emptyset = \Gamma_\emptyset / z$, $\gamma_{\underline{b}} = \Gamma_{\underline{b}} / (2\pi i)^{|\underline{b}|}$), then

$$\mathcal{P}_{ss} \int_{\gamma_i} \omega_j = \delta_{i,j}$$

  where $\mathcal{P}_{ss}$ projects onto semisimple objects that obey $\Delta x = x \otimes 1$. With this normalisation, the coaction is given by $\Delta \int_\gamma \omega = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$.

One Loop Graphs

- One loop Feynman integrals evaluate to polylogs, so what happens when we take the coaction of such an integral?
- Choose a basis of one loop integrals consisting of

$$\hat{J}_E = e^{\gamma\epsilon} \int \frac{d^D k}{i\pi^{D/2}} \prod_{i=1}^{n} \frac{1}{(k+q_i)^2 - m_i^2} \qquad D = 2\left\lceil \frac{n}{2} \right\rceil - 2\epsilon$$

where E is the set of edges of the graph. Then if we define a new set of graphs J normalised by leading singularity, we can write the coproduct in the form:

One Loop Coproduct

$$\Delta J_E = \sum_{\emptyset \subsetneq x \subseteq E} \left(J_X + a_X \sum_{e \in X} J_{X \setminus e} \right) \otimes \mathcal{C}_X J_E \qquad a_X = \begin{cases} 0 & \text{if } |X| \text{ odd} \\ \frac{1}{2} & \text{if } |X| \text{ even} \end{cases}$$

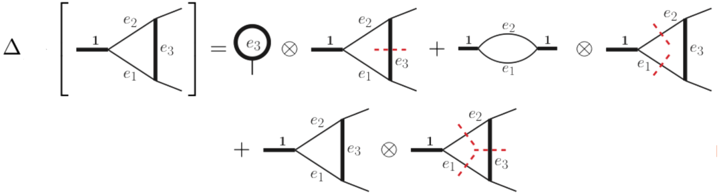
- The cuts are computed as residues in complex kinematics [1702.03163].

Example 1

One Loop Coproduct

$$\Delta J_E = \sum_{\emptyset \subsetneq X \subseteq E} \left(J_X + a_X \sum_{e \in X} J_{X \setminus e} \right) \otimes \mathcal{C}_X J_E \qquad a_X = \begin{cases} 0 & \text{if } |X| \text{ odd} \\ \frac{1}{2} & \text{if } |X| \text{ even} \end{cases}$$

First example: triangle with one external mass and one internal mass

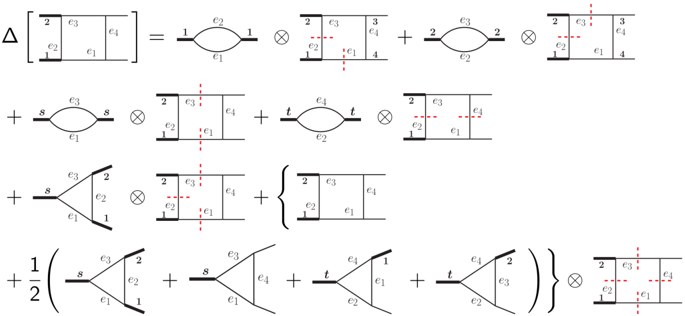


Example 2

One Loop Coproduct

$$\Delta J_E = \sum_{\emptyset \subsetneq x \subseteq E} \left(J_X + a_X \sum_{e \in X} J_{X \setminus e} \right) \otimes \mathcal{C}_X J_E \qquad a_X = \begin{cases} 0 & \text{if } |X| \text{ odd} \\ \frac{1}{2} & \text{if } |X| \text{ even} \end{cases}$$

Second example: box with two adjacent external masses



Structure of the One Loop Coproduct

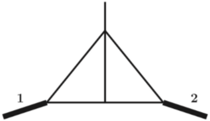
One Loop Coproduct

$$\Delta J_E = \sum_{\emptyset \subsetneq x \subseteq E} \left(J_X + a_X \sum_{e \in X} J_{X \setminus e}\right) \otimes \mathcal{C}_X J_E \qquad a_X = \begin{cases} 0 & \text{if } |X| \text{ odd} \\ \frac{1}{2} & \text{if } |X| \text{ even} \end{cases}$$

- Where does the deformation term $a_X \sum_{e \in X} J_{X \setminus e}$ come from? Can write coaction as $\Delta \int_\gamma \omega = \sum_{i=1}^n \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$ with contours that encircle poles of propagators as well as pole at ∞
- These contours can then be replaced with ordinary cuts, and it can be verified that $\mathcal{P}_{ss} \int_{\gamma_i} \omega_j = \delta_{i,j}$ by using linear relations among the cuts [1703.05064].

Two Loop Coproducts

- The generalisation of the coaction beyond one loop is non-obvious due to:
  - Topologies with multiple master integrals and so multiple cuts for a given collection of propagators.
  - Non-polylogarithmic integrals.
- Take an expression for a Feynman integral to all orders in ϵ, e.g. the graph



which evaluates to

$$e^{2\gamma_E \epsilon} \frac{1}{\epsilon^3(1-2\epsilon)} \frac{\Gamma^2(1+\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(1-2\epsilon)} \frac{(-p_1^2)^{-2\epsilon}}{p_2^2} {}_2F_1\left(1-\epsilon, 1-2\epsilon; 2-2\epsilon; 1-\frac{p_1^2}{p_2^2}\right) + \ldots$$

- We can try to break this into pieces and find the coproduct using linearity of Δ and $\Delta(ab) = \Delta(a)\Delta(b)$.
  - We can show $\Delta z^\epsilon = z^\epsilon \otimes z^\epsilon$
  - $e^{\gamma\epsilon}\Gamma(1+\epsilon) = e^{\sum_{k=2}^{\infty} \frac{(-\epsilon)^k}{k}\zeta_k} \implies \Delta[e^{\gamma\epsilon}\Gamma(1+\epsilon)] = e^{\gamma\epsilon}\Gamma(1+\epsilon) \otimes e^{\gamma\epsilon}\Gamma(1+\epsilon)$
  - What is the coproduct of the hypergeometric function part?

${}_2F_1$ Coproduct 1

- Consider a function ${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n$ where a, b and c take the form $s+t\epsilon$ for $s,t\in\mathbb{Z}$:

$$\int_\gamma \omega$$
$$=\int_0^1 du u^{m+a\epsilon}(1-u)^{n+b\epsilon}(1-uz)^{p+c\epsilon}$$
$$=\frac{\Gamma(1+m+a\epsilon)\Gamma(1+n+b\epsilon)}{\Gamma(2+m+n+(a+b)\epsilon)}{}_2F_1(1+m+a\epsilon,-p-c\epsilon;2+m+n+(a+b)\epsilon;z)$$

- We will deduce the coproduct in the form $\Delta\int_\gamma\omega=\sum_{i=1}^n\int_\gamma\omega_i\otimes\int_{\gamma_i}\omega$ by arranging for $\mathcal{P}_{ss}\int_{\gamma_i}\omega_j=\delta_{i,j}$
- There are two master integrals for the ${}_2F_1$ function (due to contiguous relations), and two independent contours with endpoints at $\{0,1,\frac{1}{z},\infty\}$, so the system is two dimensional. Make the selections

$$\omega_1=u^{a\epsilon}(1-u)^{-1+b\epsilon}(1-uz)^{c\epsilon}du \qquad \Gamma_1=[0,1]$$
$$\omega_2=u^{a\epsilon}(1-u)^{b\epsilon}(1-uz)^{-1+c\epsilon}du \qquad \Gamma_2=[0,1/z]$$

${}_2F_1$ Coproduct 2

- With this choice of $\{\omega_i\}$ and $\{\gamma_i\}$, we normalise the system ($\gamma_1 = b\epsilon\Gamma_1$, $\gamma_2 = c\epsilon z\Gamma_2$), then evaluating the integrals $\int_\gamma \omega_i$ and $\int_{\gamma_i} \omega$ produces the expression

$$\begin{aligned}\Delta {}_2F_1(\alpha,\beta;\gamma;z) =& {}_2F_1(1+a\epsilon,-c\epsilon;1+(a+b)\epsilon;x) \otimes {}_2F_1(\alpha,\beta;\gamma;z) \\ &+ z^{1-\beta}\frac{c\epsilon}{1+(a+b)\epsilon}{}_2F_1(1+a\epsilon,1-c\epsilon;2+(a+b)\epsilon;z) \\ &\otimes \frac{\Gamma(1-\alpha)\Gamma(\gamma)}{\Gamma(1-\alpha+\beta)\Gamma(\gamma-\beta)}{}_2F_1(1+\beta-\gamma,\beta;1-\alpha+\beta;1/z)\end{aligned}$$

- Given a ${}_2F_1$ from a Feynman integral we apply this expression, then use identities on the space of ${}_2F_1$s to re-express the result using Feynman integrals and their cuts.
- Contiguous relations are encoded in the $\sum_n \Delta_{n,0}$ part of the coproduct. The argument of Δ is projected onto the basis of master integrands in the first entry, with coefficients that are determined by $\int_{\gamma_i} \omega$.

F_4 Coproduct 1

- Now consider the function $F_4(a,b;c,d;X,Y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n m!n!} X^m Y^n$ with a, b, c and d written as $s + t\epsilon$ for $s, t \in \mathbb{Z}$. The relevant integrand and contour are:

$$\int_\gamma \omega$$
$$= \int_0^1 du \int_0^1 dv \left[u^{m+a\epsilon} v^{n+b\epsilon} (1-u)^{p+c\epsilon} (1-v)^{q+d\epsilon} (1-ux-vy)^{r+g\epsilon} \right.$$
$$\left. (1-ux)^{s+h\epsilon} (1-vy)^{t+j\epsilon} \right]$$

with $X = x(1-y)$, $Y = y(1-x)$.

- But we cannot replicate the ${}_2F_1$ construction on this integrand $\implies$ expand the integrand by adding extra factor $(1-x-vy)^{w+k\epsilon}$ generated from

$$\frac{1}{(1-u)(1-ux-vy)} = \frac{1}{1-x-vy} \left[\frac{1}{1-u} - \frac{x}{1-ux-vy} \right]$$

F_4 Coproduct 2

General ω is now:

$$u^{m+a\epsilon}v^{n+b\epsilon}(1-u)^{p+c\epsilon}(1-v)^{q+d\epsilon}(1-ux-vy)^{r+g\epsilon}(1-ux)^{s+h\epsilon}(1-vy)^{t+j\epsilon}(1-x-vy)^{w+k\epsilon}$$

Obtain integrands by fixing integer parts of the exponents:

	m	n	p	q	r	s	t	w
ω_1	0	0	−1	−1	0	0	0	0
ω_2	0	0	−1	0	0	0	−1	0
ω_3	0	0	0	−1	0	0	0	−1
ω_4	0	0	0	−1	−1	0	0	0
ω_5	0	0	0	0	−1	0	−1	0
ω_6	0	0	0	0	−1	0	0	−1
ω_7	0	0	0	−1	0	−1	0	0
ω_8	0	0	0	0	0	−1	−1	0
ω_9	0	0	0	0	0	−1	0	−1

Select corresponding contours:

$$\int_{\gamma_1} = \int_0^1 dv \int_0^1 du \qquad \int_{\gamma_2} = \int_0^{1/y} dv \int_0^1 du \qquad \int_{\gamma_3} = \int_0^{\frac{1-x}{y}} dv \int_0^1 du$$
$$\int_{\gamma_4} = \int_0^1 dv \int_0^{\frac{1-yv}{x}} du \qquad \int_{\gamma_5} = \int_0^{1/y} dv \int_0^{\frac{1-yv}{x}} du \qquad \int_{\gamma_6} = \int_0^{\frac{1-x}{y}} dv \int_0^{\frac{1-yv}{x}} du$$
$$\int_{\gamma_7} = \int_0^1 dv \int_0^{1/x} du \qquad \int_{\gamma_8} = \int_0^{1/y} dv \int_0^{1/x} du \qquad \int_{\gamma_9} = \int_0^{\frac{1-x}{y}} dv \int_0^{1/x} du$$

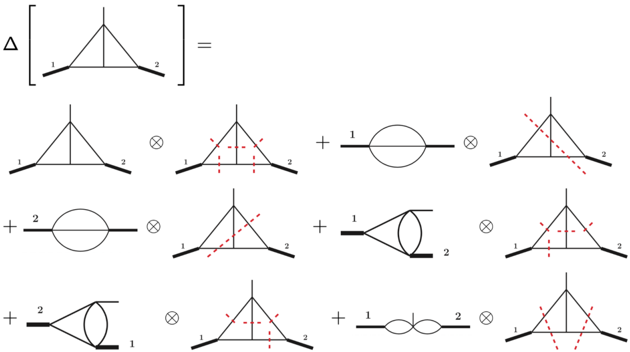
F_4 Coproduct 3

Diagonalise and normalise the system, then need to reduce from the full space of 9 terms to the F_4 case:

- Eliminate extra factor $(1 - x - vy)^{w+k\epsilon}$ by putting $k \to 0$. System develops linear relations that lower number of degrees of freedom.
- Implement constraints among the parameters. The diagonalised system contains terms proportional to vanishing combinations of the parameters.
- Result is a system depending on 4 linear combinations of the integrands, and 4 dual combinations of the contours.
- Coproduct encodes contiguous relations on for F_4 functions in the same way as for the ${}_2F_1$.

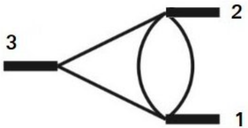
Double Triangle Graph

Consider a double triangle graph with $p_1^2 \neq 0$, $p_2^2 \neq 0$ and $p_3^2 = 0$. Taking coproducts of each term and manipulating the hypergeometric part produces:



where each graph is chosen in a suitable number of dimensions. There are no deformation terms for any of the graphs.

Double Edged triangle



Take $p_1^2 \neq 0$, $p_2^2 \neq 0$ and $p_3^2 \neq 0$ and consider the family of integrals

$$
\begin{aligned}
&P(\nu_1, \nu_2, \nu_3, \nu_4, D_1, D_2) \\
=&e^{2\gamma_E \epsilon} \int \frac{d^{D_1} k}{i\pi^{D_1/2}} \int \frac{d^{D_2} l}{i\pi^{D_2/2}} \frac{1}{(k^2)^{\nu_1} [(k+l+p_2)^2]^{\nu_2} (l^2)^{\nu_3} [(l-p_3)^2]^{\nu_4}} \\
=&(-1)^{D_2/2} (p_3^2)^{-\nu_3} (p_1^2)^{\frac{D_1+D_2}{2} - \nu_1 - \nu_2 - \nu_4} \frac{\Gamma(D_1/2+D_2/2-\nu_1-\nu_2-\nu_3)\Gamma(\nu_1+\nu_2+\nu_4-D_1/2-D_2/2)\Gamma(D_2/2-\nu_4)}{\Gamma(\nu_1+\nu_2-D_1/2)\Gamma(\nu_4)\Gamma(D_2+D_1/2-\nu_1-\nu_2-\nu_3-\nu_4)} \\
&\times F_4 \left(\begin{matrix} \nu_3, D_2/2 - \nu_4 \\ 1 + D_1/2 + D_2/2 - \nu_1 - \nu_2 - \nu_4, 1 + \nu_1 + \nu_2 + \nu_3 - D_1/2 - D_2/2 \end{matrix} ; \frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2} \right) + \dots
\end{aligned}
$$

We will compute the coproduct of $P(1,1,1,1,2-2\epsilon,4-2\epsilon)$, which is proportional to a pure function.

Maximal Cuts

- We compute two maximal cuts of the graph $P(1,1,1,1,2-2\epsilon,4-2\epsilon)$ by considering the object

$$\mathrm{Res}_{l_0=\sqrt{p_3^2}/2}\mathrm{Res}_{\beta=1}\left[e^{2\gamma_E\epsilon}\int\frac{d^{D_2}l}{i\pi^{D_2/2}}\frac{1}{(l^2)^{\nu_3}[(l-p_3)^2]^{\nu_4}}\mathcal{C}_{1,2}B((l+p_2)^2)\right]$$

in the coordinate parametrisation

$$\begin{cases} l = l_0(1,\beta\cos\theta,\beta\sin\theta\underline{1}_{D_2-2}) \\ p_3 = \sqrt{p_3^2}(1,\underline{0}_{D_2-1}) \\ p_2 = \frac{1}{2\sqrt{p_3^2}}(p_1^2-p_2^2-p_3^2,\sqrt{\lambda(p_1^2,p_2^2,p_3^2)},\underline{0}_{D_2-2}) \end{cases}$$

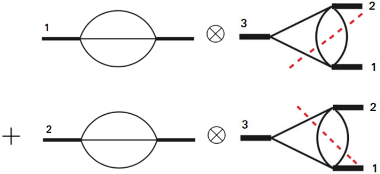
$$\lambda(a,b,c) = a^2+b^2+c^2-2ab-2ac-2bc$$

- There are two independent answers obtained by suitable restrictions of the integration domain. Call them $\mathcal{C}^{(1)}_{1,2,3,4}$ and $\mathcal{C}^{(2)}_{1,2,3,4}$.

Coproduct of the Double Edged Triangle

After manipulating the F4 functions we obtain an expression with two parts:

- Sunset sub-topologies with their corresponding channel cuts



- Two dimensional system with master integrals for the top topology and corresponding maximal cuts:

$$\Big[(1-2\epsilon)(1-3\epsilon)\tfrac{1}{p_3^2}P(1,1,1,1,4-2\epsilon,4-2\epsilon)$$
$$+\tfrac{1}{2}\epsilon xyP(1,1,1,1,2-4\epsilon,4-2\epsilon)\Big]\otimes\mathcal{C}^{(1)}_{1,2,3,4}$$
$$+\Big[-(1-2\epsilon)(1-3\epsilon)\tfrac{1}{p_3^2}P(1,1,1,1,4-2\epsilon,4-2\epsilon)$$
$$+\tfrac{1}{2}\epsilon(1-x-y)P(1,1,1,1,2-2\epsilon,4-2\epsilon)\Big]\otimes\mathcal{C}^{(2)}_{1,2,3,4}$$

with $x(1-y)=p_1^2/p_3^2$ and $y(1-x)=p_2^2/p_3^2$

Conclusions

- The coproducts we have examined take the form $\sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$ with $\mathcal{P}_{ss} \int_{\gamma_i} \omega_j = \delta_{i,j}$.
- For Feynman integrals, this structure features subtopologies of the the graph as well as its cuts.
- Coproducts of hypergeometric functions can be computed from suitable integrands and contours and are useful for deriving graphical coproducts.
- The two loop structure contains the correspondence of graphs and cuts from one loop, but now with topologies that have multiple master integrals / cuts associated with them. Deformation term structure remains to be established.