Conformal Supergravity, 4D Scattering Equations (and Monte Carlo Methods)

Joe Farrow

Based on Farrow, "A Monte Carlo Approach to the 4D Scattering Equations", 1806.02732

Farrow & Lipstein, "New Worldsheet Formulae for Conformal Supergravity Amplitudes", 1805.04504

Geyer, Lipstein & Mason, "Ambitwistor Strings in 4 Dimensions", 1404.6219

Introduction

- Review of 4D ambitwistor string theory
- $\mathcal{N} = 4$ conformal supergravity
- Solving the 4D scattering equations
- Current work and future directions

4D Ambitwistor Review

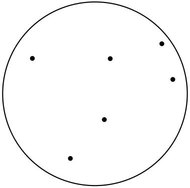
Witten 2003 considers a string theory where the target space is twistor space

4D Ambitwistor Review

Witten 2003 considers a string theory where the target space is twistor space

Cachazo, He and Yuan 2013 introduce the scattering equations

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{k_i \cdot k_j}{s_i - s_j} = 0$$



4D Ambitwistor Review

Geyer, Lipstein and Mason 2014 consider worldsheet action

$$S = \int d^2\sigma \left(Z \cdot \overline{\partial} W + cZ \cdot W \right)$$
$$= \int d^2\sigma \left(\langle\mu|\, \overline{\partial}\, |\lambda\rangle + [\lambda|\overline{\partial}|\mu] + c\left(\langle\mu\lambda\rangle + [\lambda\mu]\right) \right)$$

Amplitudes in field theory are correlation function of worldsheet vertex operators

$$Z = \begin{pmatrix} |\lambda\rangle \\ |\mu] \end{pmatrix} \qquad W = \begin{pmatrix} \langle\mu| \\ [\lambda| \end{pmatrix}$$

Penrose transform from twistor theory motivates plane-wave vertex opeartors

$$\tilde{\mathcal{V}}^{S-}_{|l]\langle l|}(\sigma) = \langle l\lambda(\sigma)\rangle^{S-1} \int \frac{dt}{t^{2S-1}} \quad \delta^2\Big(|l] - t|\lambda(\sigma)]\Big) e^{it\langle\mu(\sigma) l\rangle}$$

Penrose transform from twistor theory motivates plane-wave vertex opeartors

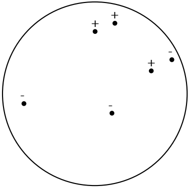
$$\tilde{\mathcal{V}}^{S-}_{|l]\langle l|}(\sigma) = \langle l \lambda(\sigma) \rangle^{S-1} \int \frac{dt}{t^{2S-1}} \quad \delta^2\Big(|l] - t|\lambda(\sigma)] \Big) e^{it\langle \mu(\sigma) l \rangle}$$

$$\mathcal{V}^{S+}_{|r]\langle r|}(\sigma) = [r \lambda(\sigma)]^{S-1} \int \frac{dt}{t^{2S-1}} \quad \delta^2\Big(\langle r| - t\, \langle \lambda(\sigma)| \Big) e^{it[\mu(\sigma) r]}$$

4D Ambitwistor Review

Amplitudes are supported on 4D scattering equations refined by MHV degree

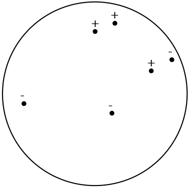
$$|l] = \sum_{r \in R} \frac{|r]}{(lr)} \qquad \langle r| = \sum_{l \in L} \frac{\langle l|}{(rl)}$$



4D Ambitwistor Review

Amplitudes are supported on 4D scattering equations refined by MHV degree

$$|l] = \sum_{r \in R} \frac{|r]}{(lr)} \qquad \langle r| = \sum_{l \in L} \frac{\langle l|}{(rl)}$$



$$\sigma_i = \frac{1}{t_i}\begin{pmatrix} 1 \\ s_i \end{pmatrix} \qquad (ij) = \det(\sigma_i \sigma_j)$$

$$\sigma = (\sigma_1 \sigma_2 ... \sigma_n) \in Gr(2, n)$$

4D Ambitwistor Review

Geyer, Lipstein and Mason 2014 write tree-level S matrices as integrals over these equations

$$\mathcal{A}_{n,L}^{(0)} = \int \frac{d^{2\times n}\sigma}{GL(2)} \frac{1}{\prod_i (i\, i{+}1)} \prod_l \delta^{2|\mathcal{N}}\left(|l] - \sum_r \frac{|r]}{(lr)} \right) \prod_r \delta^2\left(\langle r| - \sum_l \frac{\langle l|}{(rl)} \right)$$

Geyer, Lipstein and Mason 2014 write tree-level S matrices as integrals over these equations

$$\mathcal{A}^{(0)}_{n,L} = \int \frac{d^{2\times n}\sigma}{GL(2)} \frac{1}{\prod_i (i\, i{+}1)} \prod_l \delta^{2|\mathcal{N}}\left(|l] - \sum_r \frac{|r]}{(lr)}\right) \prod_r \delta^2\left(\langle r| - \sum_l \frac{\langle l|}{(rl)}\right)$$

$$\mathcal{M}^{(0)}_{n,L} = \int \frac{d^{2\times n}\sigma}{GL(2)} \det{}' H \det{}' \tilde{H} \prod_l \delta^{2|\mathcal{N}}\left(|l] - \sum_r \frac{|r]}{(lr)}\right) \prod_r \delta^2\left(\langle r| - \sum_l \frac{\langle l|}{(rl)}\right)$$

Conformal Supergravity

Berkovits and Witten 2004 consider $\mathcal{N} = 4$ conformal supergravity amplitudes in twistor string framework.

Action is schematically

$$S = \int d^4x\sqrt{-g}\; f(\phi)\, W^2$$

Conformal Supergravity

Berkovits and Witten 2004 consider $\mathcal{N} = 4$ conformal supergravity amplitudes in twistor string framework.

Action is schematically

$$S = \int d^4x\sqrt{-g}\; f(\phi)\; W^2$$

So equations of motion are now fourth order, ie.

$$\Box^2\phi(x) = 0$$

solved by $\phi(x) = (A + B \cdot x)e^{ik\cdot x}$

Graviton supermultiplet is

$$\Phi^- = h^-\eta_1\eta_2\eta_3\eta_4 + \eta_I\eta_J\eta_K\psi^{IJK} + \eta_I\eta_J A^{IJ} + \eta_I\psi^I + \phi^-$$

Conformal Supergravity

Graviton supermultiplet is

$$\Phi^- = h^-\eta_1\eta_2\eta_3\eta_4 + \ \eta_I\eta_J\eta_K\psi^{IJK} + \eta_I\eta_J A^{IJ} + \eta_I\psi^I + \phi^-$$

Out-of-MHV amplitudes can now be non-zero

$$\mathcal{M}^{(0)}(- - -) = \delta^8(Q)$$

$$\mathcal{M}^{(0)}(h^-h^-\phi^-) = \langle 12\rangle^4$$

$$\mathcal{M}^{(0)}(h^-h^-h^-) = 0, \quad \mathcal{M}^{(0)}(h^-h^-h^+) = 0$$

So we grade amplitude by both MHV degree and a separate Grassmann degree

4 types of plane wave vertex operator

$$\tilde{\mathcal{V}}^{-}_{|l]\langle l|}(\sigma) = \langle l\lambda(\sigma)\rangle \int \frac{dt}{t^2} \quad \delta^{2|4}\Big(|l] - t|\lambda(\sigma)]\Big) e^{it\langle\mu(\sigma)l\rangle}$$

$$\tilde{\mathcal{V}}^{+}_{|l]\langle l|}(\sigma) = [\lambda\partial\lambda(\sigma)] \int dt\, t \quad \delta^{2|4}\Big(|l] - t|\lambda(\sigma)]\Big) e^{it\langle\mu(\sigma)l\rangle}$$

4 types of plane wave vertex operator

$$\tilde{\mathcal{V}}^{-}_{|l]\langle l|}(\sigma) = \langle l\lambda(\sigma)\rangle \int \frac{dt}{t^2} \quad \delta^{2|4}\Big(|l] - t|\lambda(\sigma)]\Big) e^{it\langle\mu(\sigma)l\rangle}$$

$$\tilde{\mathcal{V}}^{+}_{|l]\langle l|}(\sigma) = [\lambda\partial\lambda(\sigma)] \int dt\, t \quad \delta^{2|4}\Big(|l] - t|\lambda(\sigma)]\Big) e^{it\langle\mu(\sigma)l\rangle}$$

$$\mathcal{V}^{+}_{|r]\langle r|}(\sigma) = [r\lambda(\sigma)] \int \frac{dt}{t^2} \quad \delta^{2}\Big(\langle r| - t\langle\lambda(\sigma)|\Big) e^{it([\mu(\sigma)r] + \chi(\sigma)\cdot\eta_i)}$$

$$\tilde{\mathcal{V}}^{-}_{|r]\langle r|}(\sigma) = \langle\lambda\partial\lambda(\sigma)\rangle \int dt\, t \quad \delta^{2}\Big(\langle r| - t\langle\lambda(\sigma)|\Big) e^{it([\mu(\sigma)r] + \chi(\sigma)\cdot\eta_i)}$$

Plane wave graviton multiplet S-matrix

$$\mathcal{M}^{(0)}_{n,L,\Phi^-} = \int \frac{d^{2\times n}\sigma}{GL(2)} \prod_l \delta^{2|4}\left(|l] - \sum_r \frac{|r]}{(lr)}\right) \prod_r \delta^2\left(\langle r| - \sum_l \frac{\langle l|}{(rl)}\right) \prod_{l^-\in L\cap\Phi^-} \mathcal{H}_{l^-} \prod_{l^+\in L\cap\Phi^+} \tilde{\mathcal{F}}_{l^+} \prod_{r^-\in R\cap\Phi^-} \mathcal{F}_{r^-} \prod_{r^+\in R\cap\Phi^+} \tilde{\mathcal{H}}_{r^+}$$

Non-plane wave states

$$\phi(x) = B \cdot x e^{ik\cdot x} = -iB \cdot \frac{\partial}{\partial k} e^{ik\cdot x}$$

Conformal Supergravity

Non-plane wave states

$$\phi(x) = B\cdot x e^{ik\cdot x} = -iB\cdot\frac{\partial}{\partial k}e^{ik\cdot x}$$

Vertex operators

$$\tilde{\mathcal{V}}^{-}_{|l]\langle l|}(\sigma) = B\cdot\int\frac{dt}{t^2}\left(|l\rangle\,[\mu(\sigma)| - |\lambda(\sigma)\rangle\,\frac{\partial}{\partial|l]}\right)\;\delta^{2|4}\Big(|l]-t|\lambda(\sigma)]\Big)e^{it\langle\mu(\sigma)l\rangle}$$

$$\tilde{\mathcal{V}}^{+}_{|l]\langle l|}(\sigma) = B\cdot\int t dt\Big(\,|\partial\mu(\sigma)\rangle\,[\lambda(\sigma)| - |\mu(\sigma)\rangle[\partial\lambda(\sigma)|\Big)\;\delta^{2|4}\Big(|l]-t|\lambda(\sigma)]\Big)e^{it\langle\mu(\sigma)l\rangle}$$

Conformal Supergravity

$\mathcal{M}(h_x^- h^- h^+ ... h^+)$

Conformal Supergravity

$$\mathcal{M}(h_x^- h^- h^+ ... h^+)$$

$$= B_1 \cdot \int \frac{d^{2\times n}\sigma}{GL(2)} \frac{\langle 12\rangle}{(12)} \left(\left(\frac{|1\rangle \frac{\partial}{\partial|2]} - |2\rangle \frac{\partial}{\partial|1]}}{(12)} \right) \prod_{r\in R} \sum_{r'\in R} \frac{[rr']}{(rr')} + \sum_{r\in R} \frac{|1\rangle\, [r|}{(1r)} \prod_{r'\neq r\in R} \sum_{r''\in R} \frac{[r'r'']}{(r'r'')} \right) \delta(SE_L^n)$$

Conformal Supergravity

$\mathcal{M}(h_x^- h^- h^+ ... h^+)$

$$= B_1 \cdot \int \frac{d^{2\times n}\sigma}{GL(2)} \frac{\langle 12\rangle}{(12)} \left(\left(\frac{|1\rangle \frac{\partial}{\partial |2]} - |2\rangle \frac{\partial}{\partial |1]}}{(12)} \right) \prod_{r\in R} \sum_{r'\in R} \frac{[rr']}{(rr')} + \sum_{r\in R} \frac{|1\rangle [r|}{(1r)} \prod_{r'\neq r\in R} \sum_{r''\in R} \frac{[r'r'']}{(r'r'')} \right) \delta(SE_L^n)$$

$$= \langle 12\rangle^4 B_1 \cdot \left(\prod_{r\in R} \psi_{r,n}^{|1\rangle|2\rangle} \frac{\partial}{\partial P_1} + \sum_{r\in R} \frac{\langle 12\rangle |1\rangle [r|}{\langle 1r\rangle^2 \langle 2r\rangle} \prod_{r'\in R, r'\neq r} \psi_{r',n} \right) \delta^4(P)$$

Conformal Supergravity

$\mathcal{M}(h_x^- h^- h^+ \dots h^+)$

$$= B_1 \cdot \int \frac{d^{2\times n}\sigma}{GL(2)} \frac{\langle 12\rangle}{(12)} \left(\left(\frac{|1\rangle \frac{\partial}{\partial |2]} - |2\rangle \frac{\partial}{\partial |1]}}{(12)} \right) \prod_{r\in R} \sum_{r'\in R} \frac{[rr']}{(rr')} + \sum_{r\in R} \frac{|1\rangle [r|}{(1r)} \prod_{r'\neq r\in R} \sum_{r''\in R} \frac{[r'r'']}{(r'r'')} \right) \delta(SE_L^n)$$

$$= \langle 12\rangle^4 B_1 \cdot \left(\prod_{r\in R} \psi_{r,n}^{|1\rangle|2\rangle} \frac{\partial}{\partial P_1} + \sum_{r\in R} \frac{\langle 12\rangle |1\rangle [r|}{\langle 1r\rangle^2 \langle 2r\rangle} \prod_{r'\in R, r'\neq r} \psi_{r',n} \right) \delta^4(P)$$

$$= \langle 12\rangle^4 B_1 \cdot \frac{\partial}{\partial P_1} \left(\prod_{r\in R} \psi_{r,n}^{|1\rangle|2\rangle} \delta^4(P) \right)$$

Solving the Equations

How do we extract amplitudes from worldsheet integrals?

$$\mathcal{A}^{(0)}_{n,L} = \int \frac{d^{2\times n}\sigma}{GL(2)} \delta^{2\times n}\left(SE^n_L\right) f(\sigma)$$

$$= \delta^4(P) \sum_{\sigma_{\text{sol}}\in\text{solutions}} \frac{f(\sigma_{\text{sol}})}{\langle ll'\rangle^{-2} \det(J^{n\,ll'}_L(\sigma_{\text{sol}}))}$$

Solving the Equations

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0$$

Solving the Equations

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0$$

$$|l] = \sum_{r\in R} \frac{|r]}{(lr)} \qquad \langle r| = \sum_{l\in L} \frac{\langle l|}{(rl)}$$

Solving the Equations

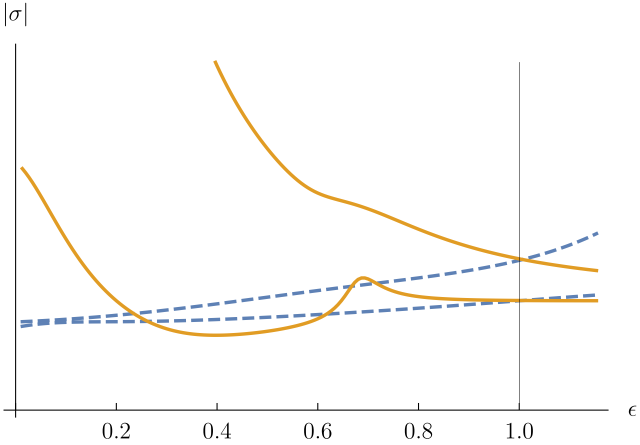
n	# solutions
4	1
5	1 1
6	1 4 1
7	1 11 11 1
8	1 26 66 26 1

Solving the Equations

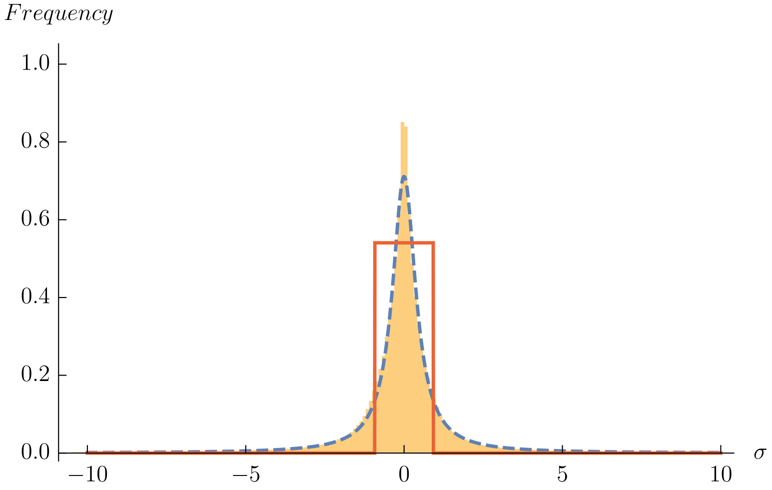
n	# solutions
4	1
5	1 1
6	1 4 1
7	1 11 11 1
8	1 26 66 26 1

$$\sigma_{\mathsf{MHV}} = \begin{pmatrix} 1 & 0 & \frac{\langle 12\rangle}{\langle 31\rangle} & \dots & \frac{\langle 12\rangle}{\langle n1\rangle} \\ 0 & 1 & \frac{\langle 12\rangle}{\langle 32\rangle} & \dots & \frac{\langle 12\rangle}{\langle n2\rangle} \end{pmatrix}$$

CHY's inverse soft algorithm



Solution point histogram



Solving the Equations

Additional points to address:

- Grassmann integrals
- Solving can be time-consuming so tabulate solutions
- Changing left set

Solving the Equations

Additional points to address:

- Grassmann integrals
- Solving can be time-consuming so tabulate solutions
- Changing left set

Go to `Mathematica`

Current work

Cachazo, Mizera and Zhang 2016 work with subset of Mandelstam invariants where solutions all fit into the interval $[0, 1]$

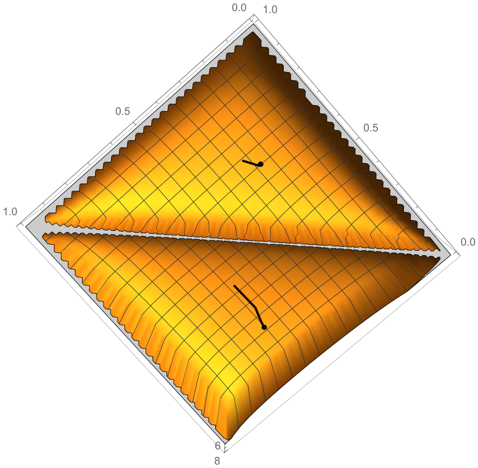
Current work

Cachazo, Mizera and Zhang 2016 work with subset of Mandelstam invariants where solutions all fit into the interval $[0,1]$

$$
G(s_{ij}) = \begin{array}{c}
\begin{array}{ccccc|c|c|c}
1 & 2 & 3 & \cdots & n\text{-}3 & A & B & C
\end{array} \\
\left[\begin{array}{ccccc|c|c|c}
0 & & & & & & & \\
 & 0 & & \mathbf{+} & & \mathbf{+} & \mathbf{+} & - \\
 & & \ddots & & & & & \\
 & + & & 0 & & & & \\
 & & & & 0 & & & \\
\hline
 & + & & & & 0 & - & + \\
\hline
 & + & & & & - & 0 & + \\
\hline
 & - & & & & + & + & 0
\end{array}\right]
\end{array}
\begin{array}{l}
1 \\ 2 \\ 3 \\ \vdots \\ n\text{-}3 \\ A \\ B \\ C
\end{array}
$$

Current work

Solutions are now labelled by $(n-3)!$ orderings



End

- Review of 4D ambitwistor string theory
- $\mathcal{N} = 4$ conformal supergravity
- Solving the 4D scattering equations
- Current work on general d equations

Thank you for listening