Super Geometry and Supermoduli

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This is a very mild case of non commutative geometry.

Much of what can be done in commutative algebra and geometry carries over. But the main interest is in new phenomena that do not have straightforward 'bosonic' analogues.

Outline

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(1) Background:

Background: particles

Background: string perturbation theory

Background: super string perturbation theory

(2) Supermanifolds

(3) Super symmetric manifolds

(4) Non splitness of supermoduli

(5) Atiyah classes vs obstructions

(6) Ramond boundary

(7) Further topics

Background: particles

bosons
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In its simplest form, "**super**" refers to: a $\mathbf{Z}$-graded ring A, which is graded-commutative:

$$b \cdot a = (-1)^{deg(a)deg(b)} a \cdot b,$$

or to its geometric spectrum. In fact, this needs only $\mathbf{Z}/2$-grading.

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Lots of symmetry =>
integrand depends only on the complex structure.
So: **amplitudes = integrals over** $M_{g,n}$.
($M_{g,n}$ = moduli space of complex structures.)

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More precisely, $\pi : \mathcal{C}_g := M_{g,n+1} \to M_{g,n}$ is the universal curve;
$\omega := \omega_{\mathcal{C}_g/M_{g,n}}$ is the bundle of holomorphic 1-forms on the moving curve;
$\omega^{\otimes i}$ is the bundle of holomorphic i-uple differentials;
$V_i := \pi_*(omega^{\otimes i})$ the vector bundle on $M_{g,n}$ of all global holomorphic i-uple differentials;
L_i is the detereminant of V_i, aka determinant of cohomology of $\omega^{\otimes i}$.

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d is the dimension of space-time.
Mumford's theorem: $L_2 = (L_1)^{13}$, independent of g.
=> Bosonic string is consistent in $d = 26$ dimensions.
Also need to compactify M_g: Deligne-Mumford $\overline{M_g}$.

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There are actually two different kinds of punctures: Neveu-Schwarz and Ramond.

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It involves a ratio $(L_{\frac{3}{2}})/(L_{\frac{1}{2}})^{d/2}$.
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Also need to compactify $\mathcal{M}_g$: super Deligne-Mumford $\overline{\mathcal{M}_g}$.

Calculations

The actual calculation?

Elementary at "tree level" ($g = 0$) and elliptic ($g = 1$).

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Van Geemen et al proposed: similar calculations for $g = 3$:

- Push the integrand forward from $\mathcal{M}_g$ to M_g
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trying to push this up to higher and higher genus. (up to 6? ∞?)
[DW]: this must fail at $g = 5$, maybe sooner.
Reason: there is no projection $\mathcal{M}_g \to M_g$.

Supermanifolds

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$M =$ body, $V =$ soul.
$dim(S) = (m|n)$ if $m = dim(M), n = rank(V)$.
Can define T_S, a (super) vector bundle on S of rank $(m|n)$.
Its restriction to M splits into even and odd parts:
$T_{S,+} = T_M$, $T_{S,-} = V$.

Supermanifolds

The **split** supermanifold $S = S(M, V)$ is $S = (M, \mathcal{O}_S)$ where

$$\mathcal{O}_S := \mathcal{O}_M \otimes \wedge^\bullet(V^*).$$

A supermanifold $S = (M, \mathcal{O}_S)$ is **split** if $S = S(M, V)$ for some vector bundle V on M. It is **projected** if there is a projection $S \to M$.

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Is every supermanifold split and/or projected?
There is an obstruction class in cohomology:

$$\omega \in H^1(M, T_M \otimes \wedge^2 V^*)$$

Every C^∞ manifold is split. The obstruction class vanishes because the sheaf is fine. (There is a partition of unity.)
It can be non-0 in the holomorphic world. Which is where physics needs it.

Super symmetric manifolds

A supersymmetric manifold is a supermanifold $S = (M, \mathcal{O}_S)$ whose underlying $V = T_{S,-}$ is a direct sum $V \cong \mathcal{S}^{\mathcal{N}}$ of $\mathcal{N}$ copies of a spinor bundle of T_M.
It is a much tighter structure than a supermanifold.
In particular, $dim(S) = (m|n)$ with $n = \mathcal{N}2^{m'}$, where $m' \cong [(m-1)/2]$.
First example: $m = \mathcal{N} = n = 1$ in the holomorphic world: a Super Riemann Surface.
Spinors = square root of T_M = spin structure, theta characteristic.

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$v : f(x) + \theta g(x) \mapsto g(x) + \theta f'(x)$
$v^2 : f(x) + \theta g(x) \mapsto f'(x) + \theta g'(x)$
$v^2 = \partial_x$.

Super Riemann Surfaces

Key point: can do algebraic geometry with SRSs.

There are moduli spaces $\mathcal{M}_g$: super, not susy

Riemann's (super) count: $dim(\mathcal{M}_g) = (3g-3|2g-2)$.

$T_+\mathcal{M}_{g\,|[S]} = H^1(T_M), \quad T_-\mathcal{M}_{g\,|[S]} = H^1((T_M)^{\frac{1}{2}})$

$T_+^*\mathcal{M}_{g\,|[S]} = H^0(K_M)^2), \quad T_-^*\mathcal{M}_{g\,|[S]} = H^0((K_M)^{\frac{3}{2}})$

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$T_+^*\mathcal{M}_{g|[S]} = H^0(K_M)^2), \quad T_-^*\mathcal{M}_{g|[S]} = H^0((K_M)^{\frac{3}{2}})$
Underlying manifold: SM_g is the modui space of Riemann surfaces with a spin structure.
Two types of puncture: N and R.
Neveu-Schwarz puncture lives at a point (a submanifold of dimension $(0|0)$), can be forgotten.
Ramond puncture lives on a divisor (a submanifold of dimension $(0|1)$), cannot be forgotten.
Local picture: $v := \partial_\theta + x\theta\partial_x$.
$v^2 = x\partial_x$: v is maximally non integrable except where $x = 0$.
DM compactification: two types of nodes
Gluing rules

Non splitness of supermoduli

[DW1]: $\mathcal{M}_g$ (and others) are non split and non projected, for $g \geq 5$.
(Note: the analogous question for the DM compactification is easier.)
Idea: find compact curve C in M_g,
described as a family of branched covers.
Lift it to $\mathcal{M}_g$
Calculate: the obstruction, restricted to a neighborhood of C, is $\neq 0$.
Lift requires: all branching **odd**.

Atiyah classes vs obstructions

Atiyah class := obstruction to existence of a connection
On a manifold: in $H^1(X, \wedge^2 T^*X \otimes TX)$
On a vector bundle V : $H^1(X, T^*X \otimes V^* \otimes V)$
On a principal G-bundle P : $H^1(X, T^*X \otimes ad(P))$
(Case of a manifold: $V = T^*X$ but one component vanishes due to torsion freeness)
Bundles on supermanifolds have super Atiyah classes
[DW2]: The super Atiyah class of a supermanifold $S = (M, \mathcal{O}_S)$ has 3 components:

- the Atiyah class of M, i.e. of T_+S
- the Atiyah class of $V = T_-S$
- the obstruction class to splitting S.

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In: *More on superstring perturbation theory*, arXiv:1304.2832,
Witten explained several subtleties of superstring perturbation theory in the RNS framework.
One of these is the spontaneous breakdown of supersymmetry at one loop in certain superstring models through the appearance of a Goldstone fermion in supersymmetric Ward identities.
Witten's discussion depended on a rather subtle mathematical result about the geometry of super moduli spaces in genus 1 near the Ramond boundary. We explain and prove that mathematical result.

Ramond boundary

There is a large class of heterotic string compactifications to four dimensions that are supersymmetric at tree level but have an anomalous U(1) gauge field. A concrete example treated by Witten is the SO(32) heterotic string compactified on a Calabi-Yau manifold with the spin connection embedded in the gauge group in the standard way. (The anomalous U(1) arises in this case as the first factor in the commutant, $U(1) \times SO(26)$, in the gauge group SO(32), of the SU(3) spin connection when this is embedded in SO(32).)

Ramond boundary

The breaking of supersymmetry at one loop in this case can be reduced to a particular result about the boundary of the Deligne-Mumford compactification of the moduli space $\mathfrak{M}_{1,1,2}$ of super Riemann surfaces of genus 1 with one Neveu-Schwarz puncture and two Ramond punctures.

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In general, this boundary is a divisor with normal crossings. Its generic points parametrize stable super Riemann surfaces with a single node. This node can be of NS or R type.

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In general, this boundary is a divisor with normal crossings. Its generic points parametrize stable super Riemann surfaces with a single node. This node can be of NS or R type.

We show that in the case of $\mathfrak{M}_{1,1,2}$, the boundary consists of two divisors parametrizing stable super Riemann surfaces with an NS node, and two others parametrizing stable super Riemann surfaces with an R node. Let $\mathcal{R} \subset \mathfrak{M}_{1,1,2}$ be one of the two Ramond boundary components, $\mathcal{R}_{\mathrm{red}}$ its reduced space, $\mathcal{L} := N_{\mathcal{R} \setminus \mathfrak{M}_{1,1,2}}$ the normal bundle to $\mathcal{R}$ in $\mathfrak{M}_{1,1,2}$, and $\mathcal{L}_0 = \mathcal{L}_{|\mathcal{R}_{\mathrm{red}}}$ its restriction to $\mathcal{R}_{\mathrm{red}}$, which is also the normal bundle to $\mathcal{R}_{\mathrm{red}}$ in $\mathcal{SM}_{1,1,2} = \mathfrak{M}_{1,1,2_{\mathrm{red}}}$.

Ramond boundary

Turns out $\mathcal{R}$ is split, hence projected: $p : \mathcal{R} \to \mathcal{R}_{\mathrm{red}}$. The needed result is that the normal bundle $\mathcal{L}_1 := \mathcal{L} \otimes p^*(\mathcal{L}_0)^{-1}$ is non trivial. Equivalently, our main mathematical result is that $\mathcal{L}$ is not the pull back by p of any bundle on $\mathcal{R}_{\mathrm{red}}$.

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In case $S = \mathfrak{M}_{1,1,2}$, we have the forgetful map $\pi : \mathfrak{M}_{1,1,2} \to \mathfrak{M}_{1,0,2}$. This identifies the moduli space $\mathfrak{M}_{1,1,2}$ with the universal SRS over $\mathfrak{M}_{1,0,2}$. In terms of π, the odd tangent bundle can be described as an extension involving two simpler bundles:

$$0 \to L \to T_{-}\mathfrak{M}_{1,1,2} \to \pi^* T_{-}\mathfrak{M}_{1,0,2} \to 0. \tag{1}$$

Here $L := T_{\pi,-}$ is the bundle of odd tangent vectors along the fibers of π. These fibers are super Riemann surfaces (with two R punctures), so L can be interpreted as the family of generalized spin bundles along the fibers. The map $T_{-}\mathfrak{M}_{1,1,2} \to \pi^* T_{-}\mathfrak{M}_{1,0,2}$ is the differential of π. We prove the non-splitness of this extension, on all of $\mathfrak{M}_{1,1,2}$ as well as when restricted to $\mathcal{R}$, and we reduce our main result to this non-splitness.

Ramond boundary

In more detail:

The restriction $\pi_{\mathcal{R}}$ of the forgetful map π to the Ramond divisor $\mathcal{R}$ exhibits $\mathcal{R}$ as an affine $\mathbb{C}^{0|1}$-bundle over $\mathfrak{M}_{1,0,2}$. One may ask whether this bundle has a section, i.e. whether it is actually a line bundle.

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We show that this happens if and only if the normal bundle $\mathcal{L} := N_{\mathcal{R}\backslash\mathfrak{M}_{1,1,2}}$ is pulled back by p from some bundle on $\mathcal{R}_{\mathrm{red}}$.

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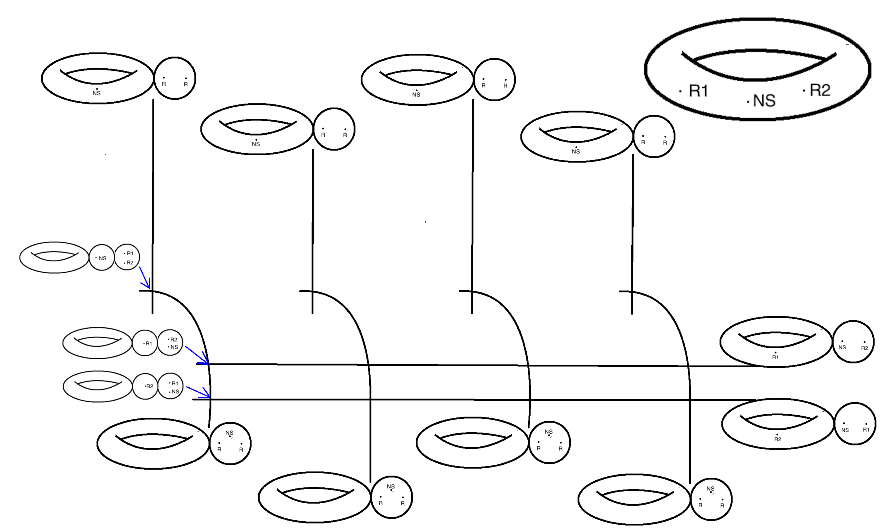
This restricted sequence exhibits $T_{-}\mathfrak{M}_{1,1,2|\mathcal{R}_{red}}$, the restriction to $\mathcal{R}_{red}$ of the odd tangent budle $T_{-}\mathfrak{M}_{1,1,2}$, as an extension involving two simpler bundles. The middle term can be identified with $\mathcal{R}$ itself, and the map $T_{-}\mathfrak{M}_{1,1,2|Rred} \to \pi^{*}T_{-}\mathfrak{M}_{1,0,2|Rred}$ is then identified with $\pi_{\mathcal{R}} : \mathcal{R} \to \mathfrak{M}_{1,0,2}$.
It is straightforward to show that the sequence (1) on $\mathcal{SM}_{1,1,2}$ does not split. It takes more detailed calculations to show that the sequence still does not split after restriction to R_{red}.


R1
NS
R2

Ongoing:

- Justify van Geemen's original proposal: meromorphic projection $\mathcal{M}_3 \to M_3$.
- Heterotic MS, via SCY? (cf. Melnikov-Plesser)
- Super Hilbert schemes (Jang)
- super Teichmueller theory, cluster coordinates (Penner-Zeitlin)
- Super DM via super log strs? (With Ionita, Morissey)
- Super toric geometry? Fans, projectivity, apply toroidal embeddings to DM. (With Jang)

Morals:

- Origins in perturbative super string theory
- Supermanifolds vs supersymmetric manifolds
- Split vs non-split supermanifolds, e.g. M_g
- Obstruction theory, analogous to Atiyah classes
- Rich theory of supermoduli, DM compactifications
- R vs NS punctures and nodes

Thank you!!!