Massive Automorphic Green's functions

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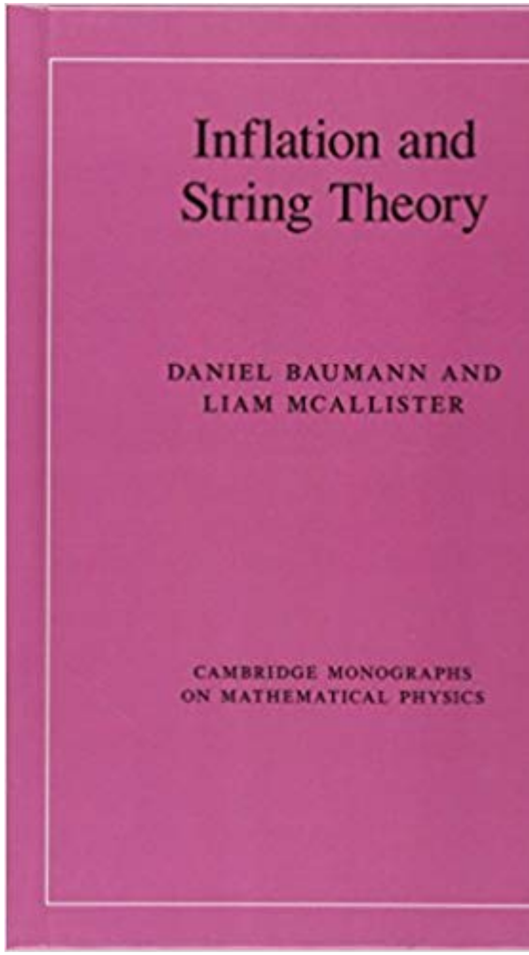
work in progress with W. Linch, C.Mafra, O.Schlotterer
work in progress with K.Bringmann, T.Gannon

Why string loops?



- moduli stabilization

  e.g. M.B, Haack, Körs ’05, Michael’s talk

- models of inflationary cosmology

  e.g. Kim, McAllister ‘18

- models of supersymmetry phenomenology

  e.g M.B., Marsh, McAllister, Pajer ’12

- 1/N corrections in AdS/CFT duality

  e.g Alday ’18
  Cho, Collier, Yin ‘18

Plane wave in pure spinor/hybrid

Berkovits, "Conformal Field Theory for the Superstring in a Ramond-Ramond Plane Wave Background", hep-th/0203248

$$S_{(0)} = \frac{1}{2\pi\alpha'} \int d^2z \left(\frac{1}{2}\partial X_m \bar{\partial} X^m + \frac{1}{2} m^2 \partial X^+ \bar{\partial} X^+ (X^i)^2 + \text{fermions} \right)$$

"It has therefore been proven that the action for the superstring in an R-R plane wave background is an exact conformal field theory"

Dreams

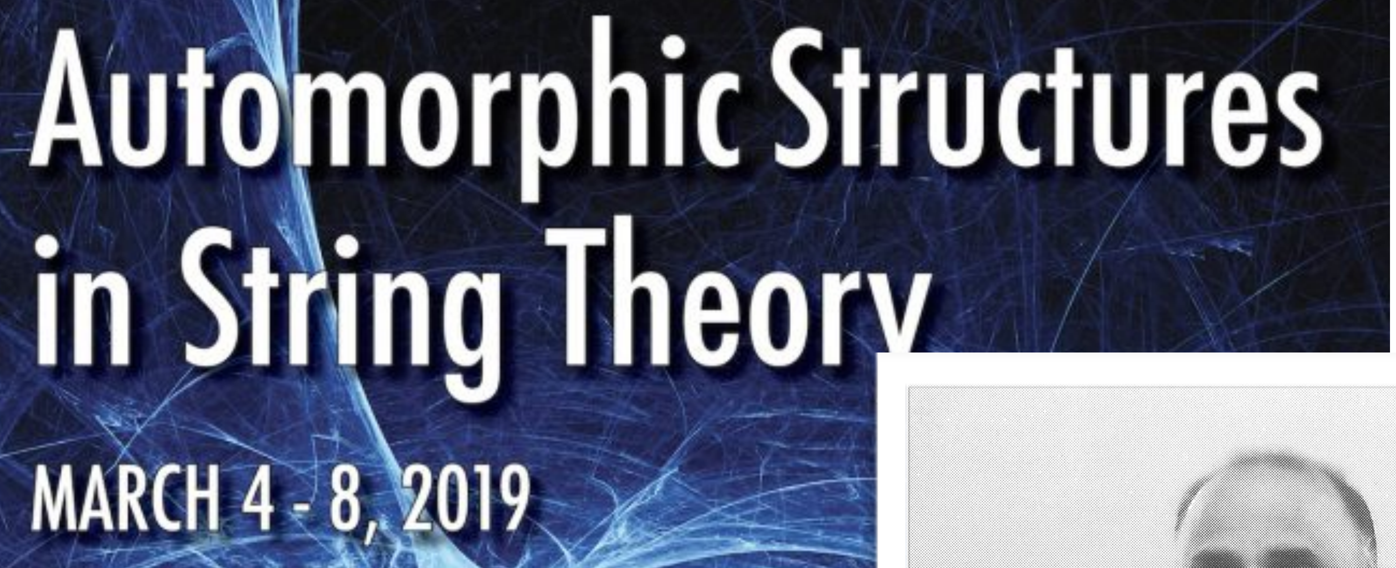
covariant pure spinor/hybrid AdS or plane wave worldsheet equation of motion + spacetime supersymmetry

↓

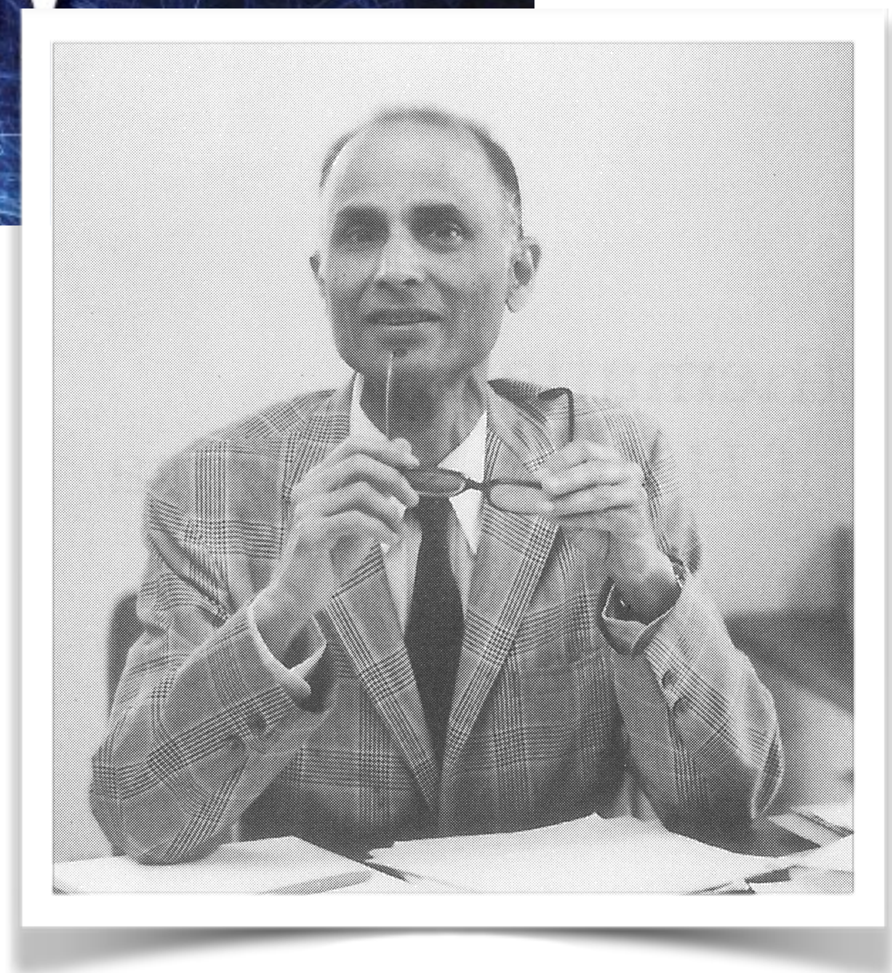
partial differential equations

↑

[some math argument]

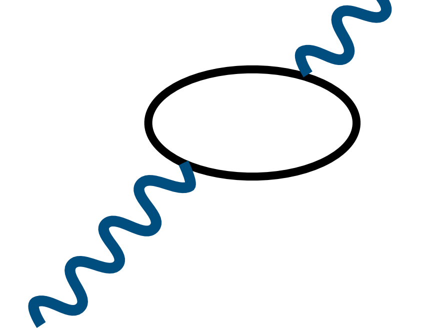


see also Angelantonj, Florakis, Pioline



Harish-Chandra defined *automorphic*:

1. invariant under G
2. differential equation
3. falloff condition

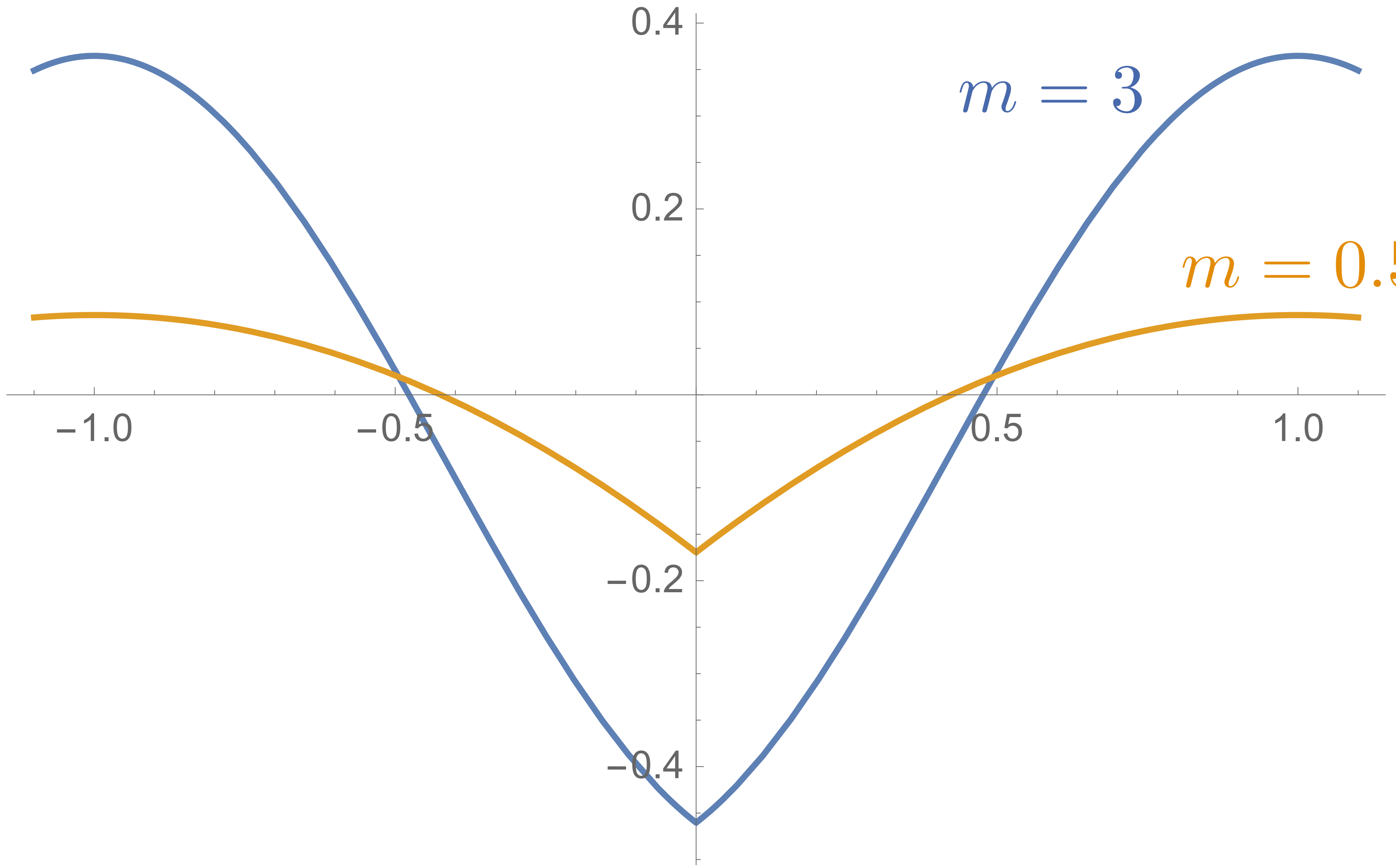


```
Helmholtz1D = {y''[x] + m^2 y[x], y'[-t] == 0, y'[t] == 0}

FullSimplify[GreenFunction[Helmholtz1D, y[x], {x, -t, t}, 0], x > 0]
```

$$\frac{\cos(m(t-|x|))}{2m\sin(mt)} - \frac{1}{2m^2 t}$$

$t = 1$

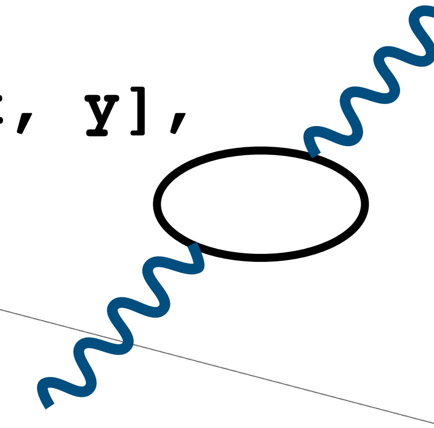


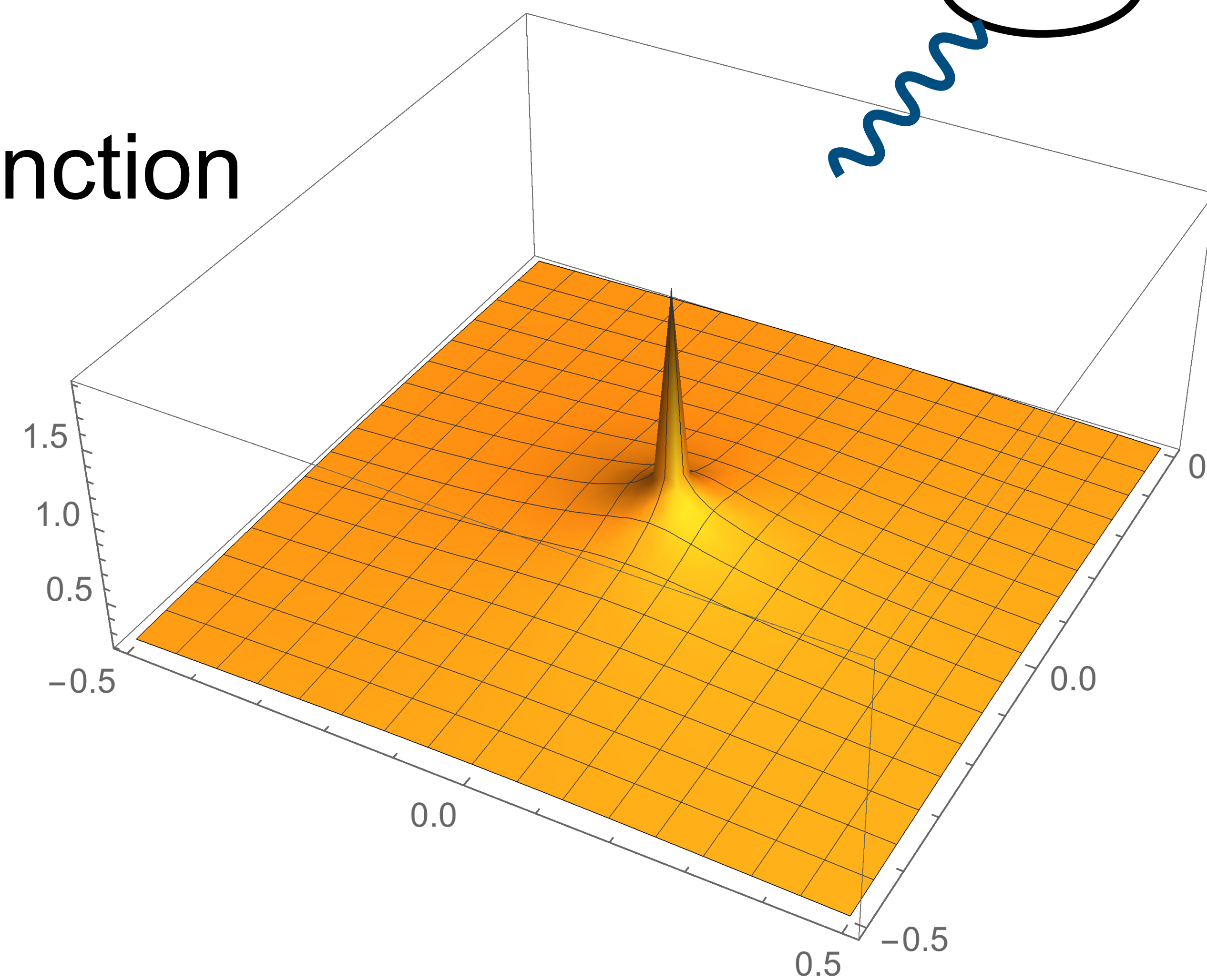
```
Simplify[GreenFunction[Laplacian[G[x, y],{x, y}] - m^2 G[x, y],
        G[x, y], {x, y} ∈ FullRegion[2], {k, l}], m > 0]
                              ℝ²
```

$$-\frac{1}{2\pi}K_0(m|\mathbf{k}-\mathbf{x}|)$$

Bessel function

$\sim \log m$ for small mass





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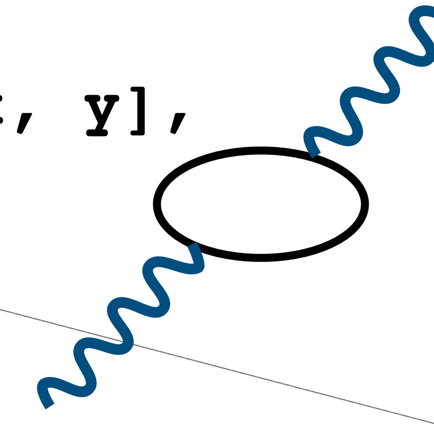
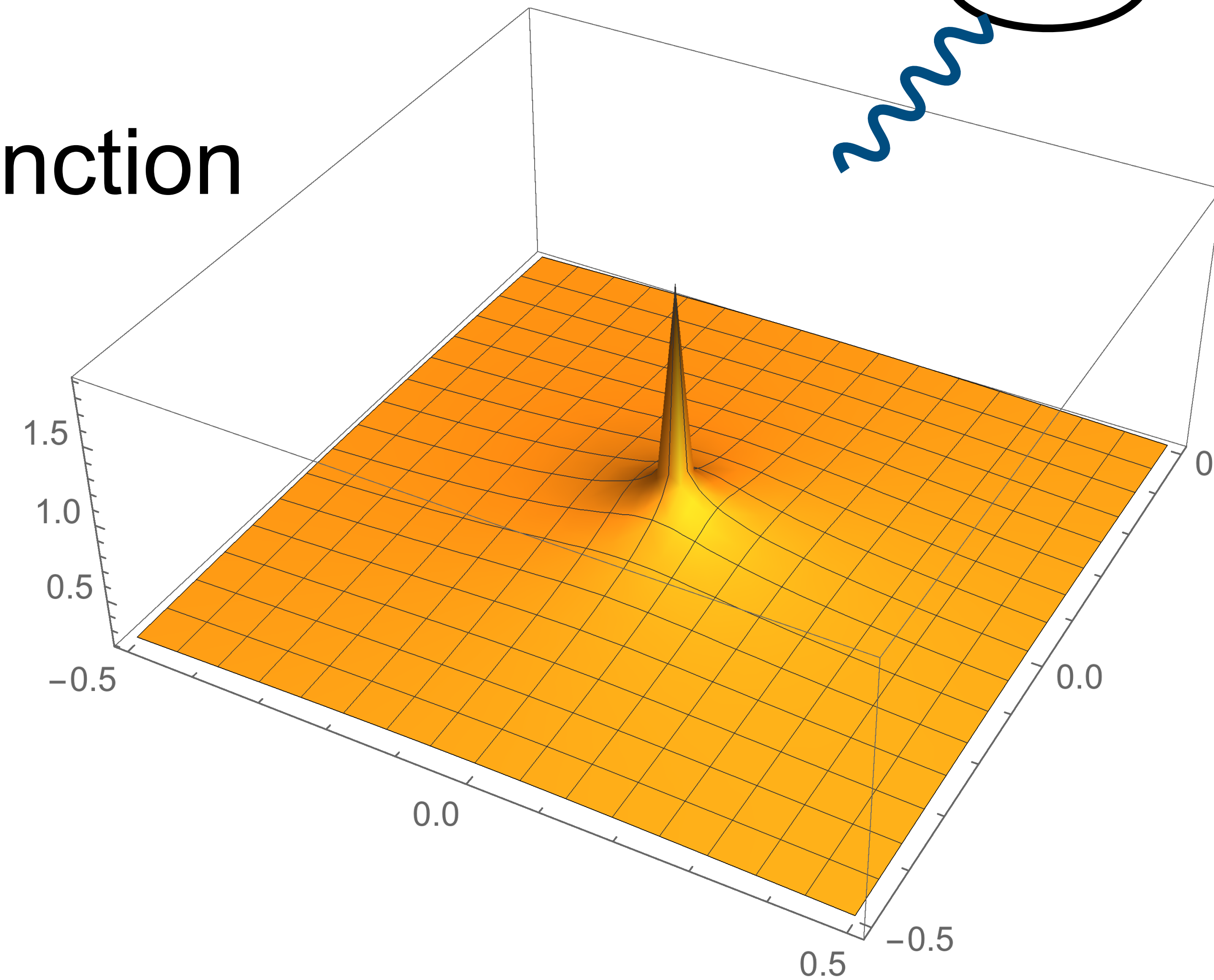
```

$\mathbb{R}^2$

$$-\frac{1}{2\pi}K_0(m|\mathbf{k}-\mathbf{x}|)$$

Bessel function






```
Integrate[m %, m]
```

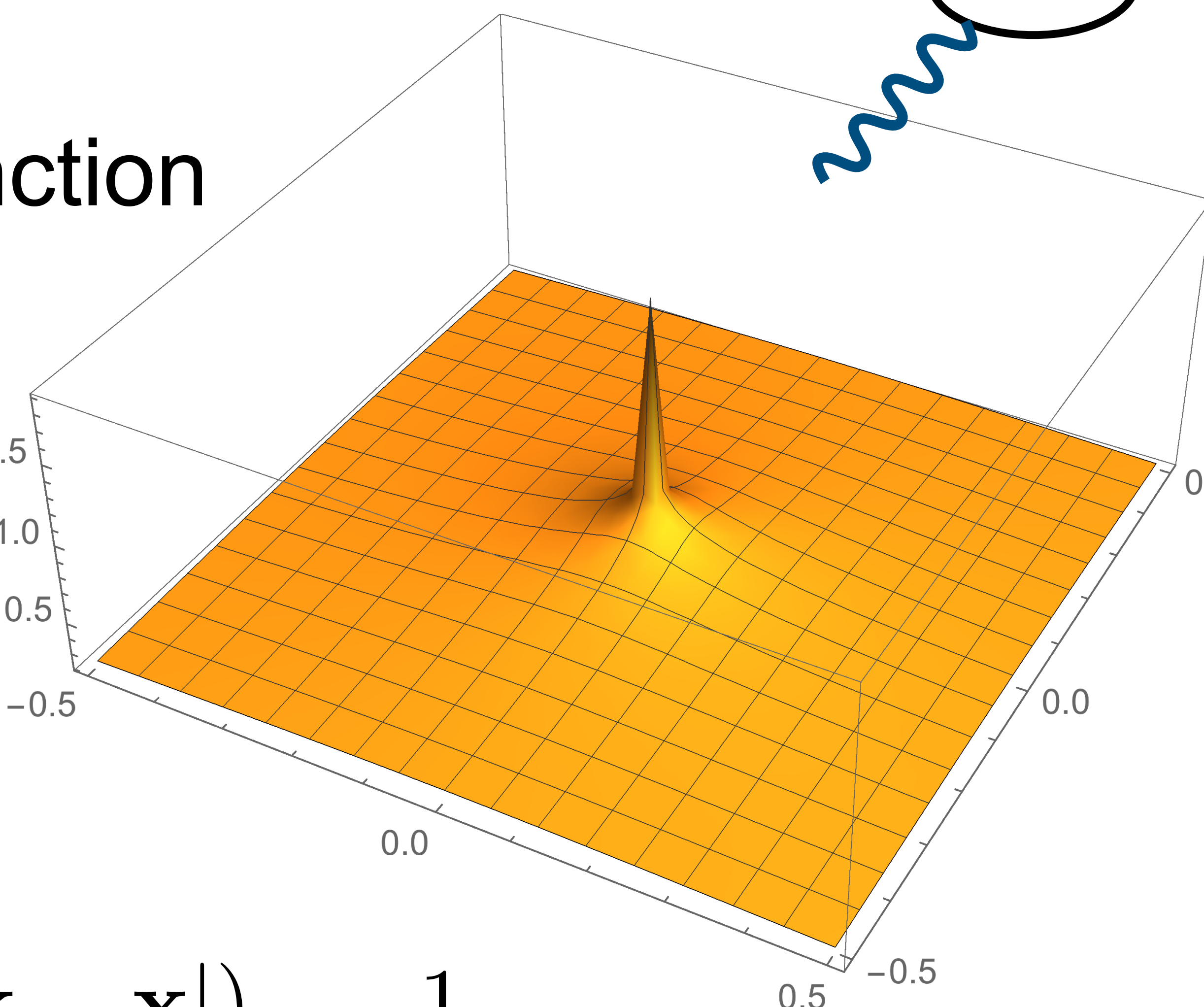
$$\frac{1}{2\pi}\frac{mK_1(m|\mathbf{k}-\mathbf{x}|)}{|\mathbf{k}-\mathbf{x}|}$$

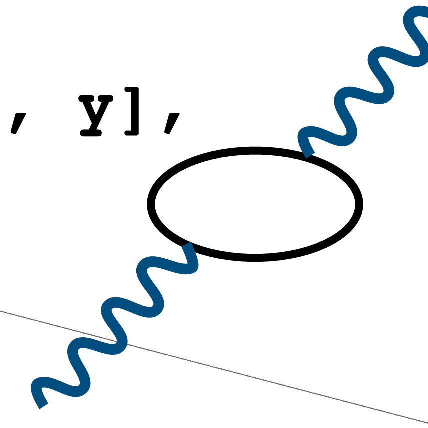
$$\sim m\cdot\frac{1}{m}$$ for small mass

```
Simplify[GreenFunction[Laplacian[G[x, y],{x, y}] - m^2 G[x, y],
        G[x, y], {x, y} ∈ FullRegion[2], {k, l}], m > 0]
```
$\mathbb{R}^2$

$$-\frac{1}{2\pi}K_0(m|\mathbf{k}-\mathbf{x}|)$$

Bessel function





```
Integrate[m %, m]
```

$$\frac{1}{2\pi}\frac{mK_1(m|\mathbf{k}-\mathbf{x}|)}{|\mathbf{k}-\mathbf{x}|}$$

```
D[%, m]/m
```

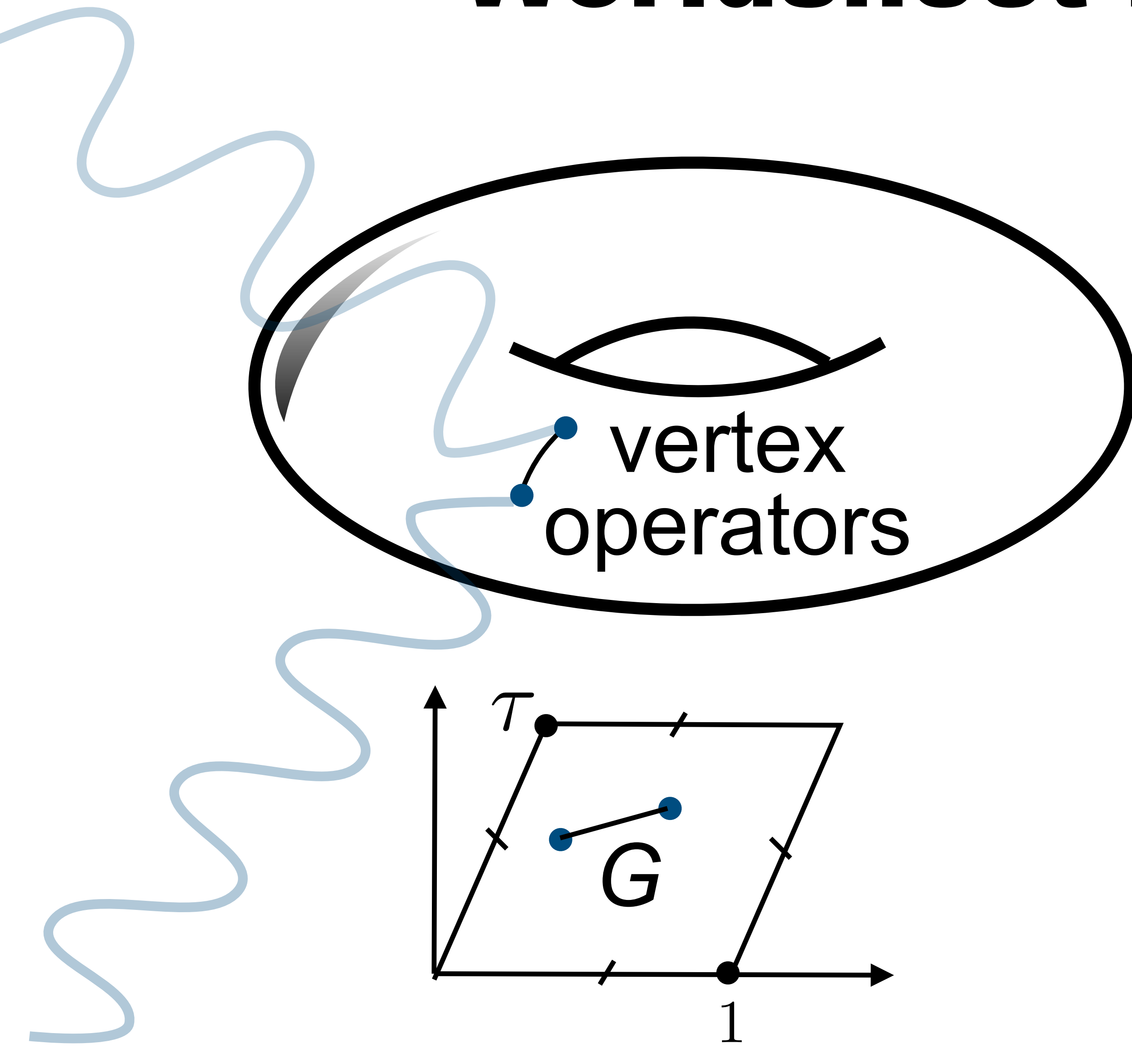
$$-\frac{1}{4\pi}K_0(m|\mathbf{k}-\mathbf{x}|)+\frac{1}{2\pi m}\frac{K_1(m|\mathbf{k}-\mathbf{x}|)}{|\mathbf{k}-\mathbf{x}|}-\frac{1}{4\pi}K_2(m|\mathbf{k}-\mathbf{x}|)$$

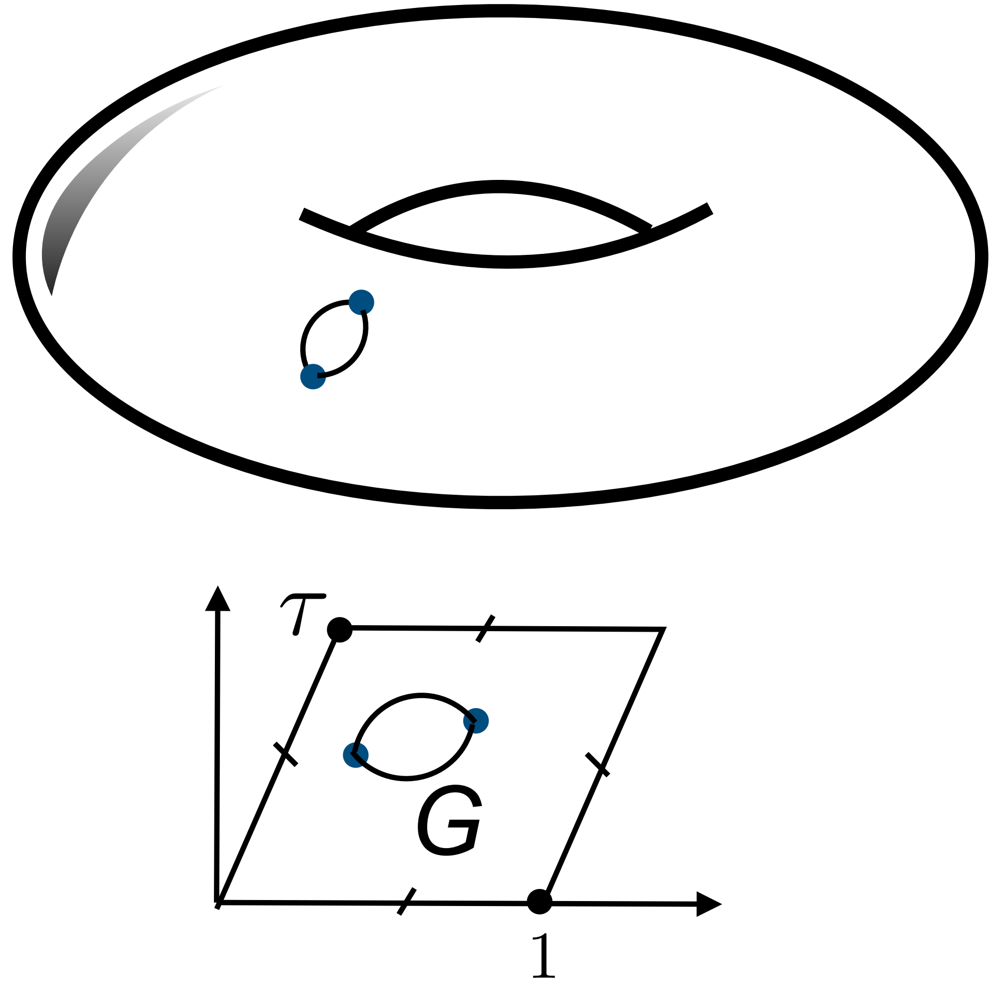
```
FullSimplify[%]
```

$$-\frac{1}{2\pi}K_0(m|\mathbf{k}-\mathbf{x}|)$$

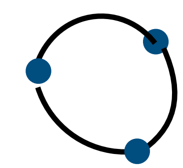
Worldsheet Feynman graphs

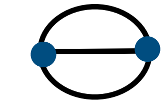
e.g. Lerche, Schellekens, Nilsson, Warner ‘86
Stieberger, Taylor ‘02





correlation functions of
worldsheet scalars/fermions

flat space: 
only 2-vertex, no 3-vertex, 4…

background field: sigma model! 
graviton vertex operators: “almost” background ⋯

$$\frac{2}{\alpha'}\partial_{\bar{z}}\partial_z G(z,z') = -2\pi\delta^2(z-z') + \frac{1}{4\pi\tau_2}$$

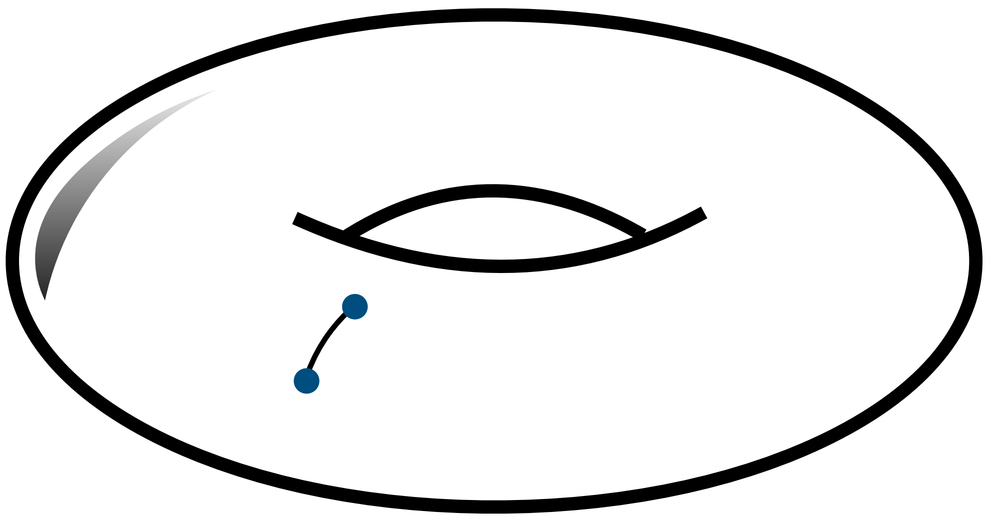
e.g. Polchinski's book, Ch.7

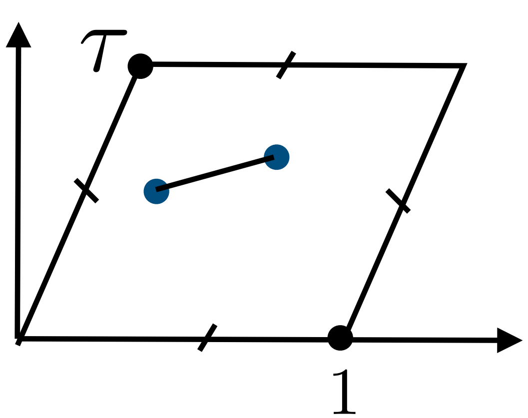
$$G(z,z') = \frac{\alpha'}{2}E_1(0,z-z')$$

$$E_1(0,z) = -\ln\left|\frac{\vartheta_1(z,\tau)}{\eta(\tau)}\right|^2 + \frac{2\pi(\operatorname{Im} z)^2}{\tau_2}$$

Kronecker-Eisenstein function

…
A.Weil, '76

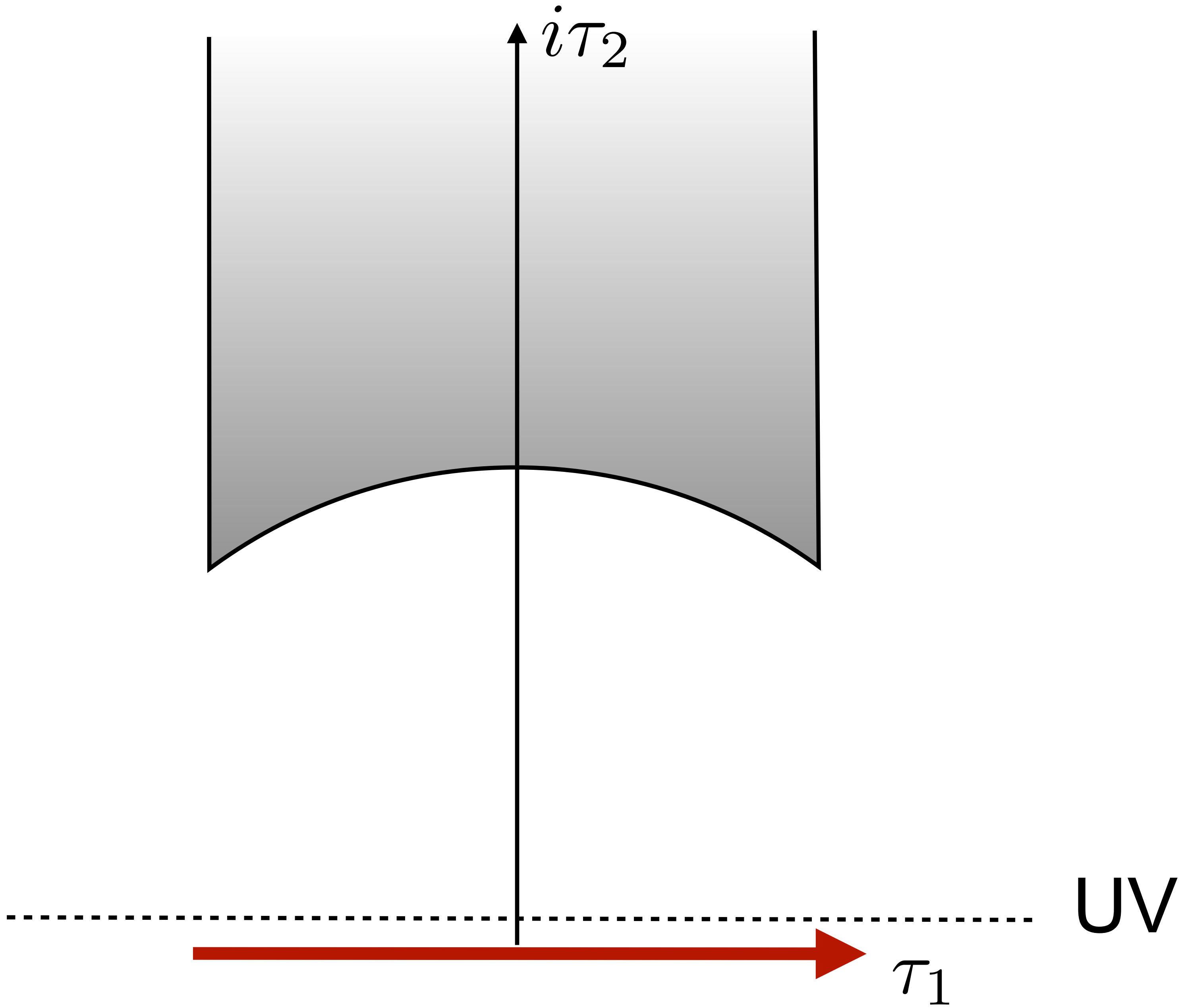




one possible choice of fundamental domain: (“*lagom*”)

$$\tau \to \tau + 1$$

$$\tau \to -1/\tau$$



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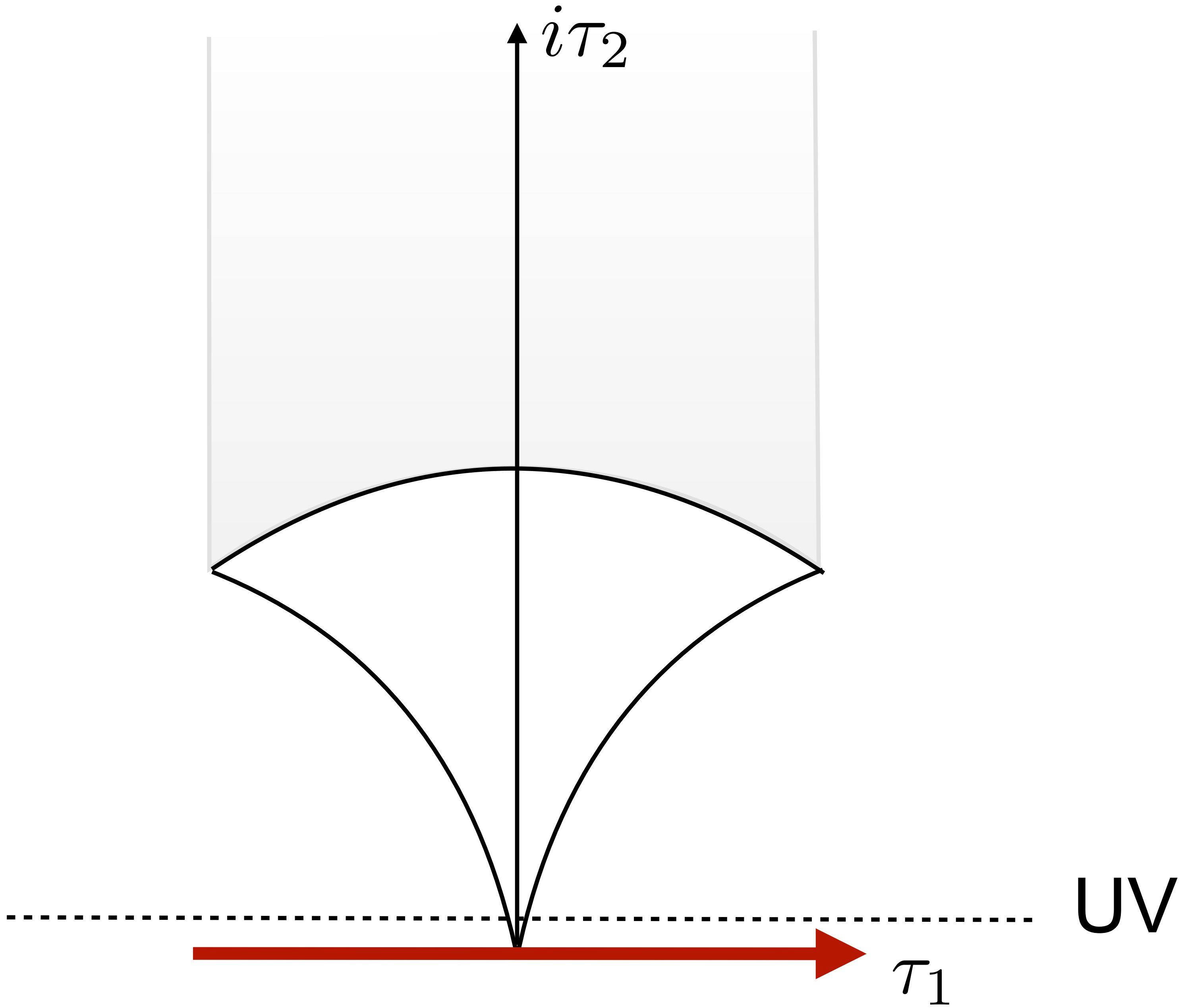
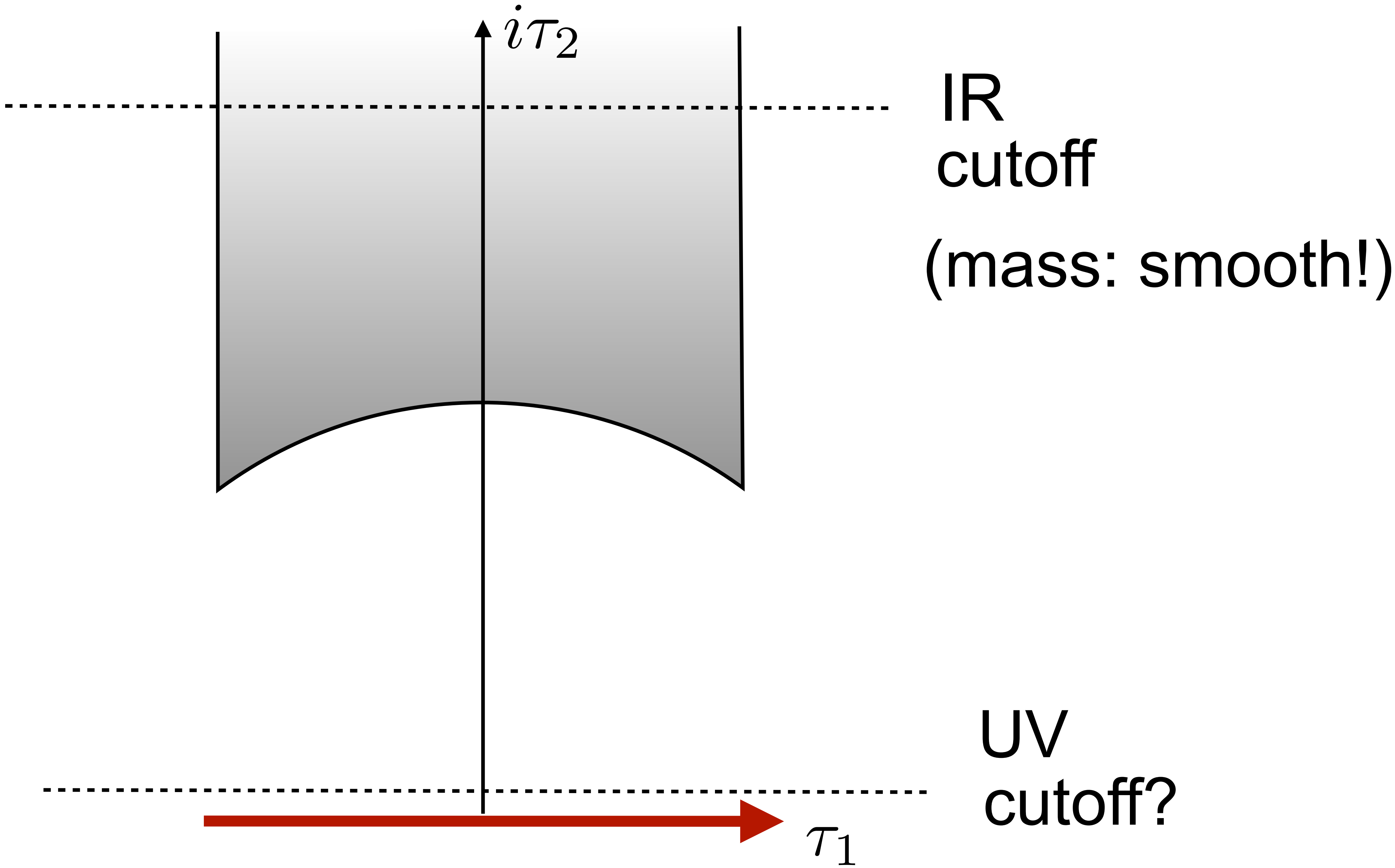
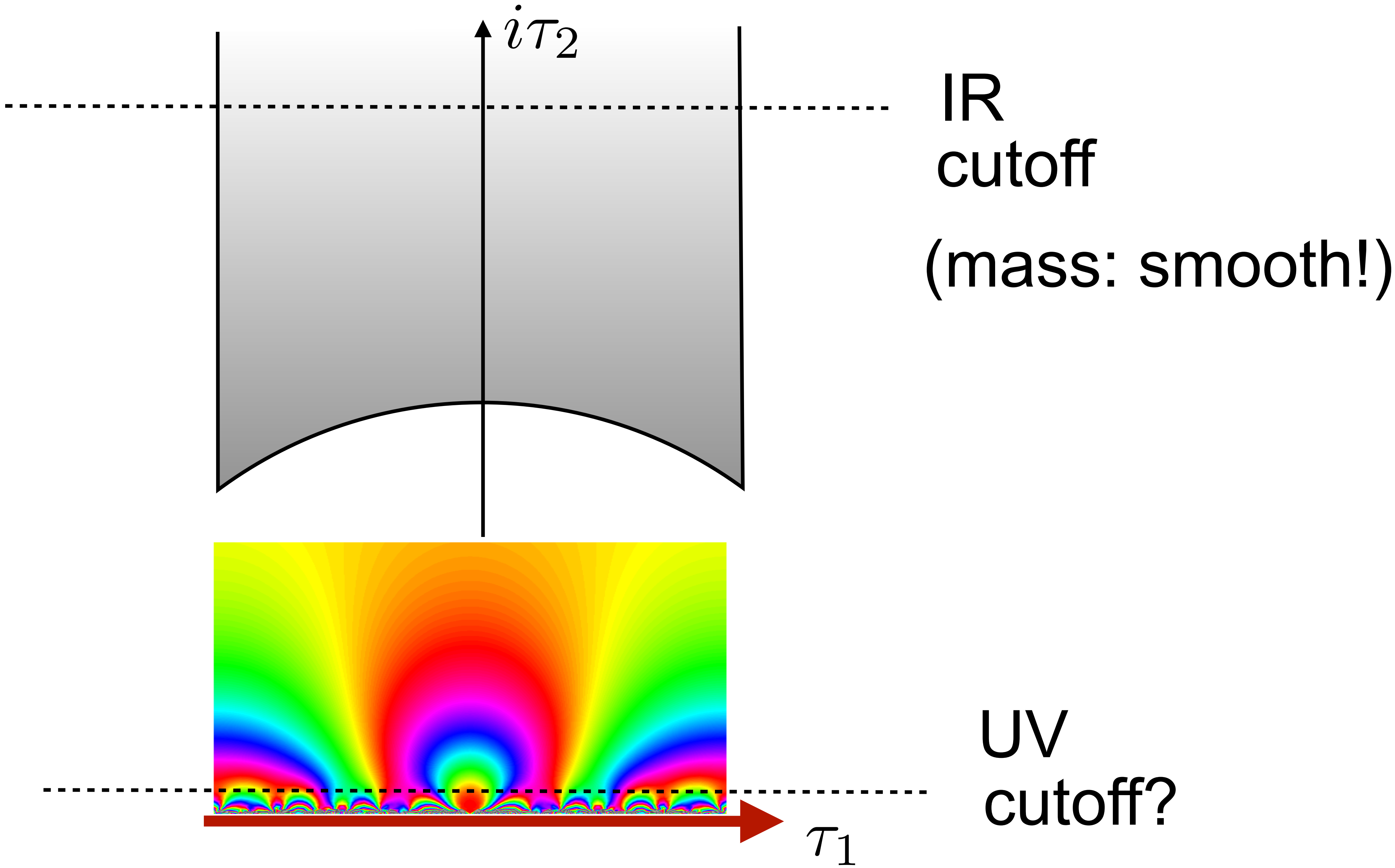


Polchinski
exercise 7.5c



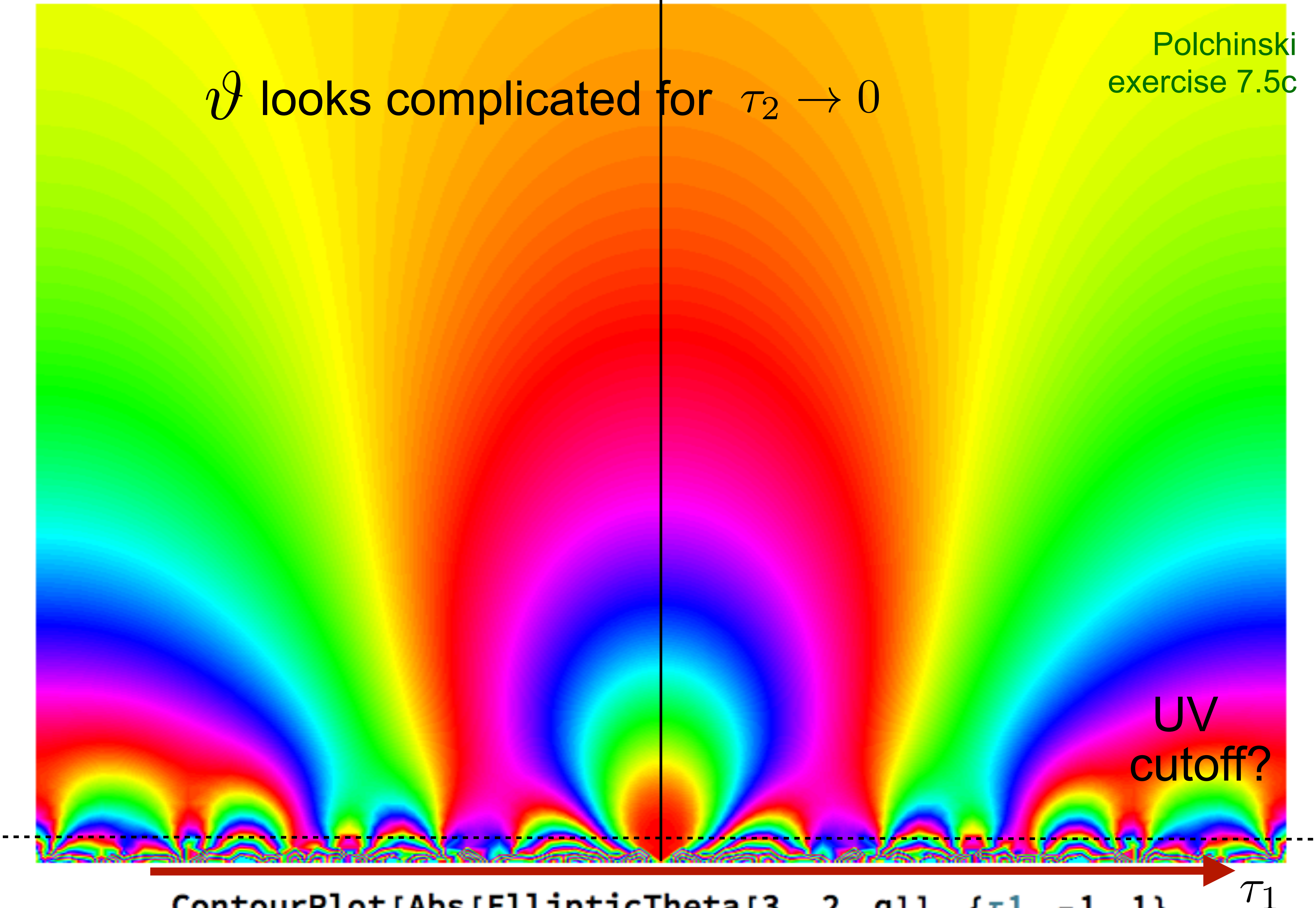
Polchinski
exercise 7.5c

ϑ looks complicated for $\tau_2 \to 0$




```
ContourPlot[Abs[EllipticTheta[3, 2, q]], {τ1, -1, 1},
 {τ2, 0.001, 0.7}, ColorFunction -> (Hue[6.1 #] &),
```

ϑ looks complicated for $\tau_2 \to 0$

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ContourPlot[Abs[EllipticTheta[3, 2, q]], {τ1, -1, 1},
 {τ2, 0.001, 0.7}, ColorFunction -> (Hue[6.1 #] &),
```
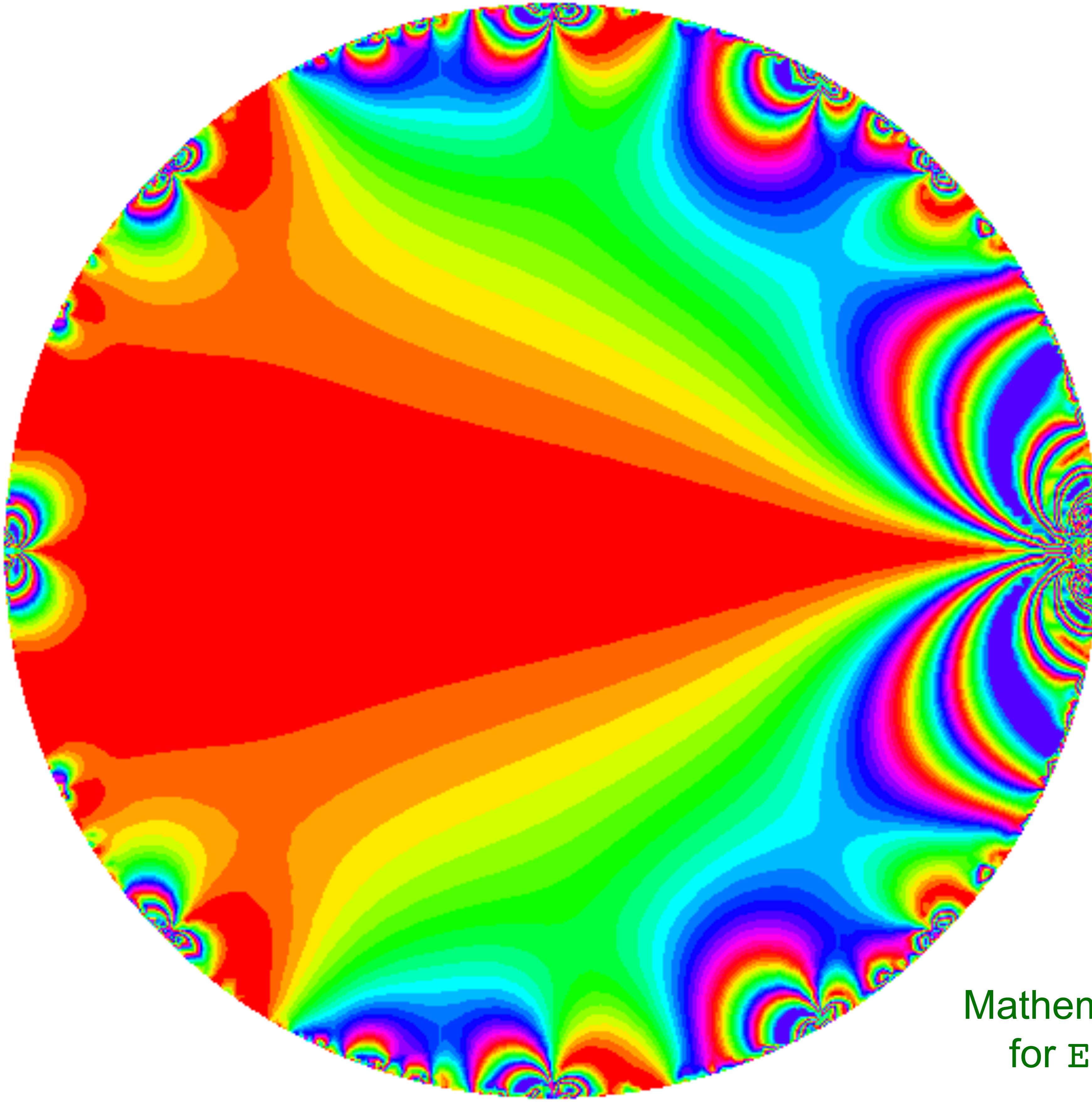

ϑ on the Poincaré disk $q = e^{2\pi i \tau}$

Polchinski
exercise 7.5c



Mathematica help page
for `EllipticTheta`

$$\frac{2}{\alpha'}\partial_{\bar{z}}\partial_z G(z,z') = -2\pi\delta^2(z-z') + \frac{1}{4\pi\tau_2}$$

e.g. Polchinski's book, Ch.7

$$G(z,z') = \frac{\alpha'}{2}E_1(0,z-z')$$

$$E_1(0,z) = -\ln\left|\frac{\vartheta_1(z,\tau)}{\eta(\tau)}\right|^2 + \frac{2\pi(\mathrm{Im}\,z)^2}{\tau_2}$$

Kronecker-Eisenstein function

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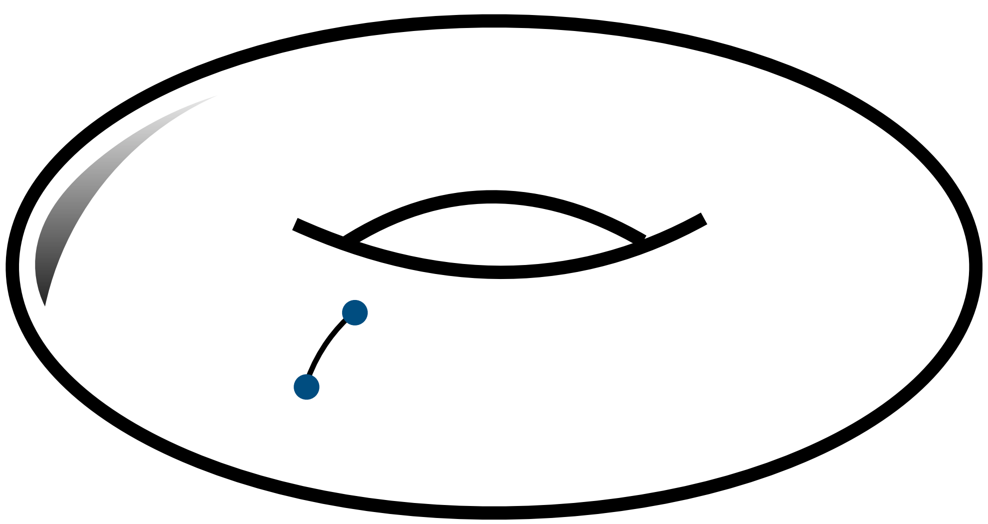
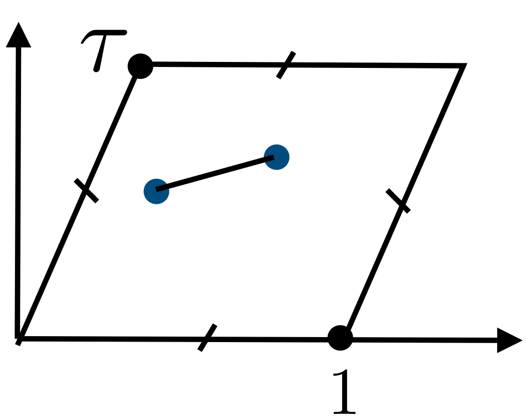
A.Weil, '76





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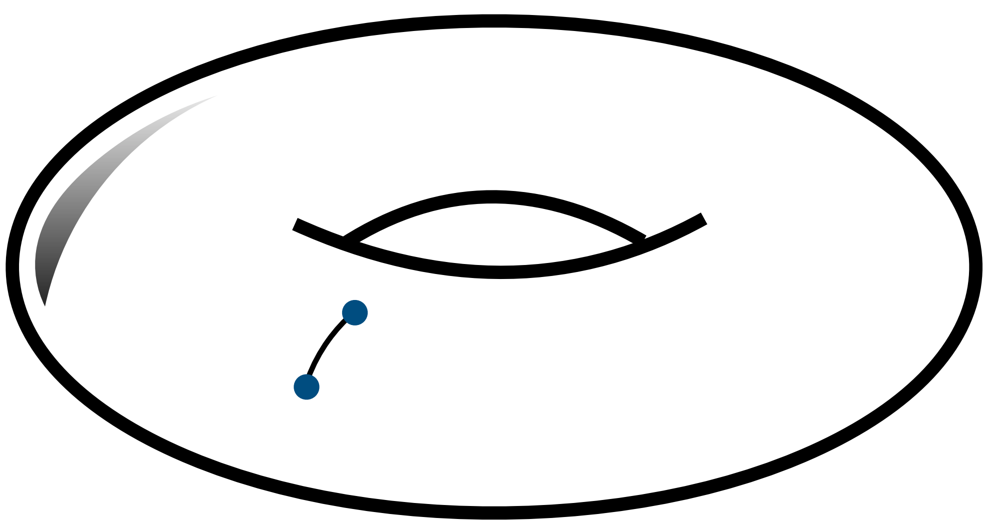
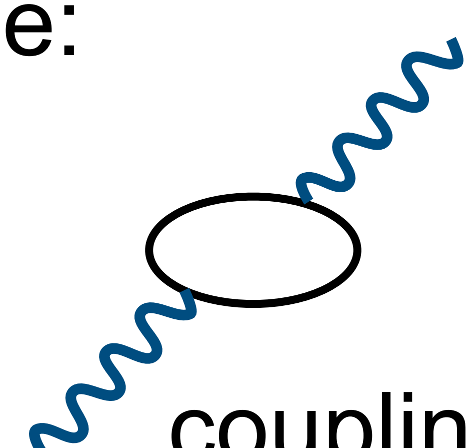
- for a *spacetime* torus we could ask the same:



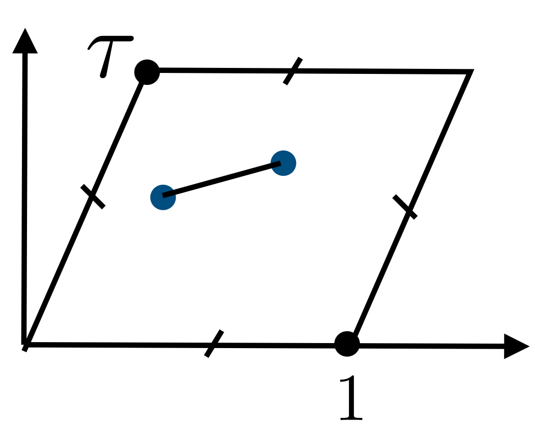
$$z \to \phi$$

$$\tau \to U$$



coupling correction

"theta lift" of genus 2 ϑ



$$\ln|f(\phi)|^2 = \ln f(\phi) + \ln\overline{f(\phi)} = 2\,\mathrm{Re}\ln f(\phi)$$

loop correction to holomorphic gauge kinetic function

M.B., Haack, Körs '04

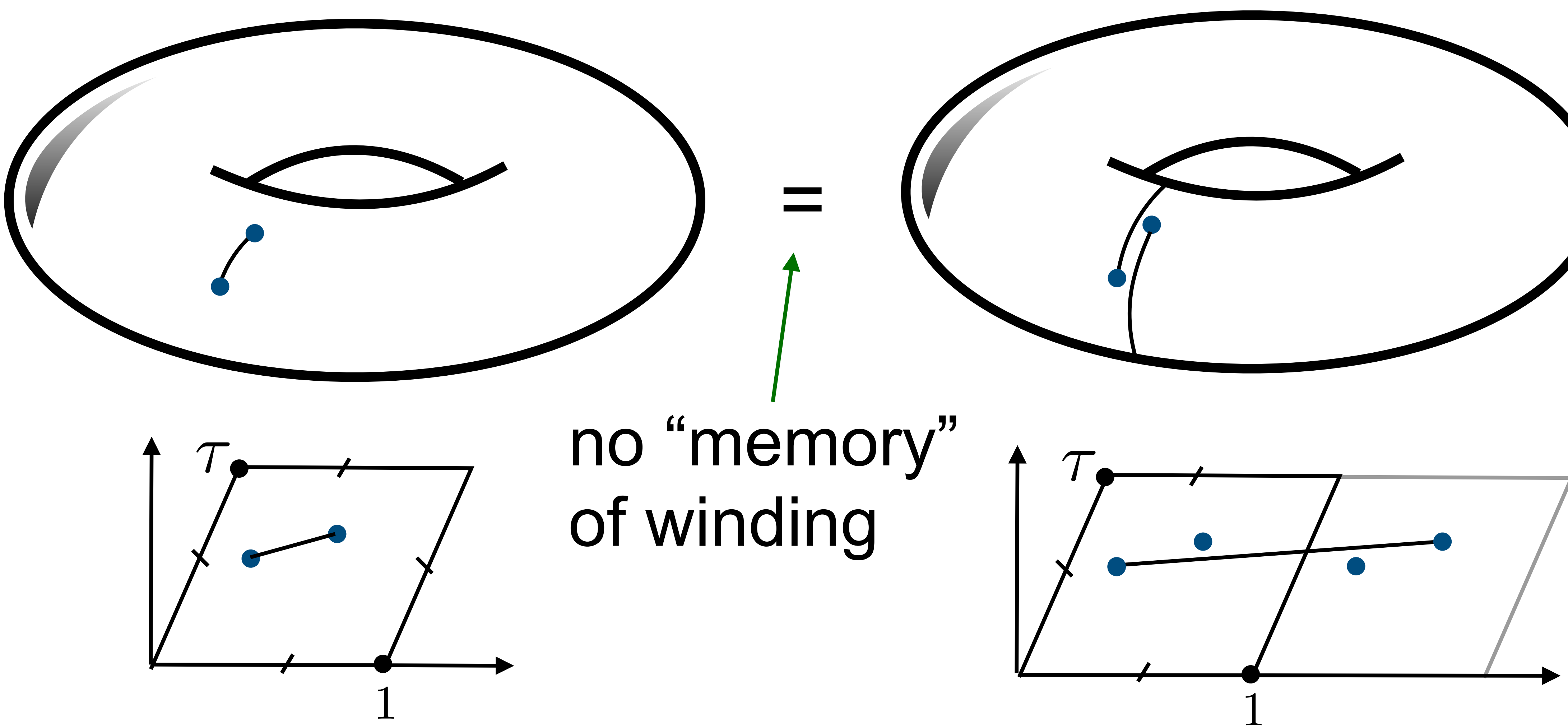
$$\frac{2}{\alpha'}\partial_{\bar{z}}\partial_z G(z,z') = -2\pi\delta^2(z-z') + \frac{1}{4\pi\tau_2}$$

e.g. Polchinski's book, Ch.7

$$G(z,z') = \frac{\alpha'}{2}E_1(0,z-z')$$

$$E_1(0,z) = -\ln\left|\frac{\vartheta_1(z,\tau)}{\eta(\tau)}\right|^2 + \frac{2\pi(\operatorname{Im} z)^2}{\tau_2}$$

- not *manifestly* independent of which fundamental cell, but cancellation between the two terms



Fourier-expand Kronecker-Eisenstein

e.g. Polchinski's book,
exercise 7.3

$$E_s(w,z) = \pi^{-s}\Gamma(s)\tau_2^s \sum_{k,\ell}{}' \frac{e^{\frac{2\pi i}{\tau_2}\mathrm{Im}((w+k+\ell\tau)\bar{z})}}{|w+k+\ell\tau|^{2s}} \qquad \mathrm{Re}\, s > 1$$

…
A.Weil, '76
…
M.B., Haack, Kang, Sjörs '14

but wanted $w = 0$, $s = 1$?

Fourier-expand Kronecker-Eisenstein

e.g. Polchinski's book,
exercise 7.3

$$E_s(w,z) = \pi^{-s}\Gamma(s)\tau_2^s {\sum_{k,\ell}}' \frac{e^{\frac{2\pi i}{\tau_2}\mathrm{Im}((w+k+\ell\tau)\bar{z})}}{|w+k+\ell\tau|^{2s}} \qquad \mathrm{Re}\, s > 1$$

integral representation: analytic continuation to all *s*, including functional relation:

… A.Weil, '76
… M.B., Haack, Kang, Sjörs '14

$$E_s(w,z) = e^{\frac{2\pi i}{\tau_2}\mathrm{Im}\, w\bar{z}} E_{1-s}(z,w)$$

"position-twist duality"
(generalizes to any dimension)

cf. Ewald method for Madelung constant:
screen ion, gives Yukawa suppression

Fourier-expand Kronecker-Eisenstein

e.g. Polchinski's book, exercise 7.3

$$E_s(w,z) = \pi^{-s}\Gamma(s)\tau_2^s \sum_{k,\ell}{}' \frac{e^{\frac{2\pi i}{\tau_2}\mathrm{Im}((w+k+\ell\tau)\bar{z})}}{|w+k+\ell\tau|^{2s}} \quad \mathrm{Re}\, s > 1$$

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A.Weil, '76
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M.B., Haack, Kang, Sjörs '14

integral representation: analytic continuation to all *s*, including functional relation:

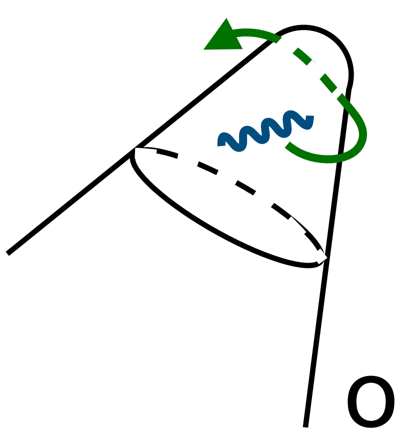
$$E_s(w+1,z) = E_s(w,z)$$

$$E_s(w+\tau,z) = E_s(w,z)$$

spacetime:

$$E_s(w,z+1) = e^{2\pi i v}E_s(w,z) \qquad w = u + v\tau$$

$$E_s(w,z+\tau) = e^{-2\pi i u}E_s(w,z)$$



orbifold

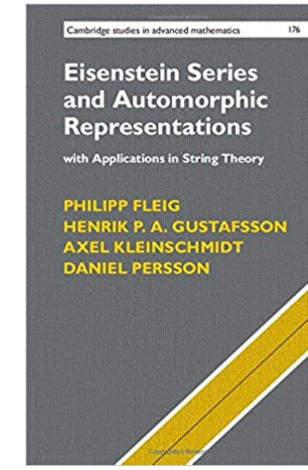
e.g. Witten '18

Z from G

1. $w = z = 0\,,\;\; s = 1+\epsilon$

$$E_{1+\epsilon}(0,0) = \frac{1}{\pi}\sum_{m,n}{}' \frac{\tau_2^{1+\epsilon}}{|m+n\tau|^{2(1+\epsilon)}}$$

Fleig et al (1511.04265, eq.10.11)



$$= \frac{1}{\epsilon} + 2(\gamma_{\rm E} - \log 2 - \log(\sqrt{\tau_2}|\eta(\tau)|^2) + \mathcal{O}(\epsilon)$$

2. $w \neq 0\,,\;\; z = 0\,,\;\; s = 1+\epsilon$ $\qquad \sqrt{\tau_2}|\eta(\tau)|^2$ per boson

$$e^{-E_0'(\beta-\frac{1}{2}+\tau(\alpha-\frac{1}{2}),0)} = e^{-2\pi(\alpha-\frac{1}{2})^2\tau_2}\left|\frac{\vartheta_1(\beta-\frac{1}{2}+(\alpha-\frac{1}{2})\tau)}{\eta(\tau)}\right|^2$$

$$= \left|\frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0,\tau)}{\eta(\tau)}\right|^2 ,$$

per fermion pair

Fourier-expand Kronecker-Eisenstein

$$E_s(w,z) = \pi^{-s}\Gamma(s)\tau_2^s \sum_{k,\ell}{}' \frac{e^{\frac{2\pi i}{\tau_2}\mathrm{Im}((w+k+\ell\tau)\bar{z})}}{|w+k+\ell\tau|^{2s}} \qquad \mathrm{Re}\, s > 1$$

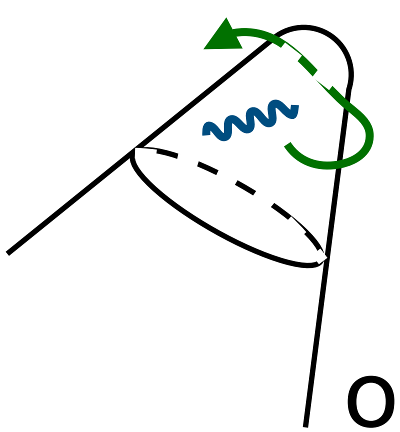
M.B., Haack, Körs ‘05

if *N* = 2 spacetime supersymmetry,

$$\frac{\tau_2}{\pi}\partial_{\bar{z}}\partial_z E_s(z,\tau) = (s-1)E_{s-1}(z,\tau)$$

gauge coupling

spacetime:

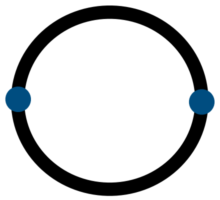


orbifold

s = 2: moduli space metric of D-brane scalars

“ladder operator” differential equation

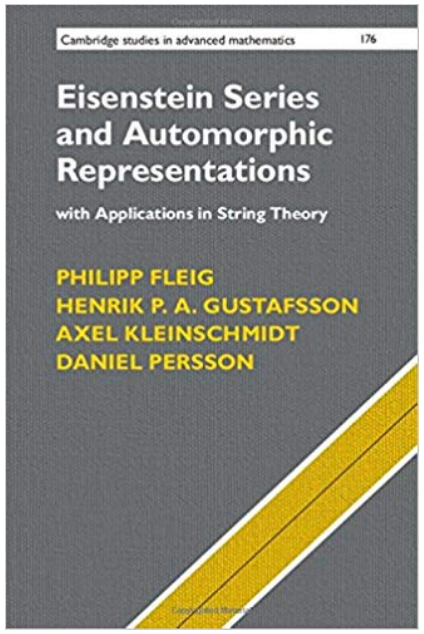
Worldsheet Feynman diagram

 $$= \int d^2z\, G(z)G(-z) = \int_0^1 dx \int_0^1 dy\; G(x,y)\, G(-x,-y)$$

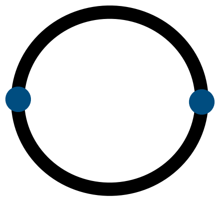
$$= \int_0^1 dx \int_0^1 dy \sum_{k_1,\ell_1}{}' \frac{\tau_2}{2\pi|\ell_1 + k_1\tau|^2}\, e^{2\pi i k_1 x} e^{2\pi i \ell_1 y} \sum_{k_2,\ell_2}{}' \frac{\tau_2}{2\pi|\ell_2 + k_2\tau|^2}\, e^{2\pi i k_2(-x)} e^{2\pi i \ell_2(-y)}$$

$$= \sum_{k_1,\ell_1}{}' \sum_{k_2,\ell_2}{}' \frac{\tau_2}{2\pi|\ell_1 + k_1\tau|^2} \frac{\tau_2}{2\pi|\ell_2 + k_2\tau|^2}\, \delta_{k_1,k_2}\delta_{\ell_1,\ell_2} \qquad \left(\int_0^1 dx\, e^{2\pi i (k_1 - k_2)x} = \delta_{k_1,k_2} \right)$$

$$= \frac{1}{4\pi^2} \sum_{k,\ell}{}' \frac{\tau_2^2}{|\ell + k\tau|^4} \propto E_2(\tau)$$ **“modular graph function”**



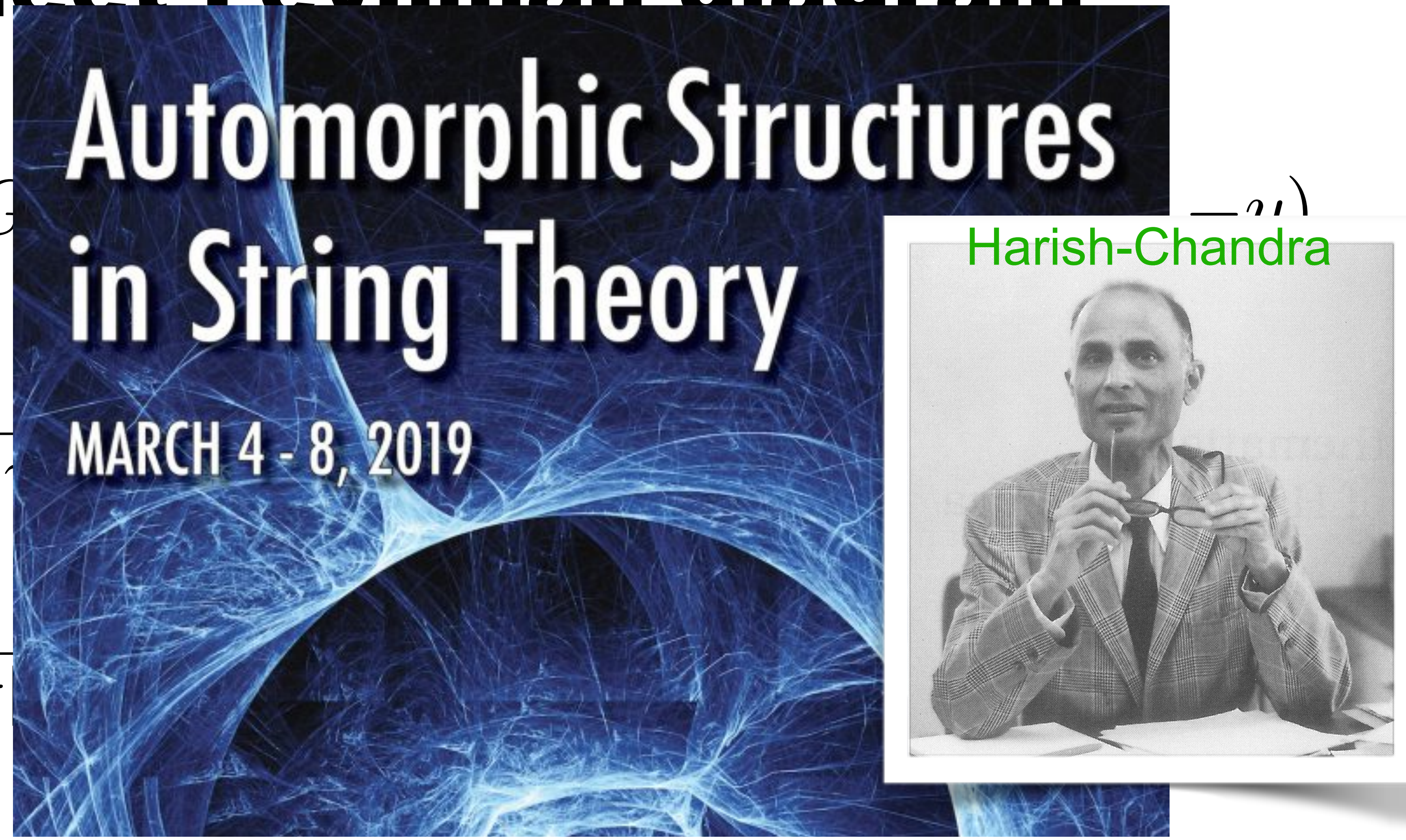
Worldsheet Feynman diagram



$$= \int d^2 z\, G(z) G \ldots$$

$$= \int_0^1 dx \int_0^1 dy \sum_{k_1,\ell_1}{}' \frac{\tau_2}{2\pi|\ell_1 + k_1 \ldots} \ldots$$

$$= \sum_{k_1,\ell_1}{}' \sum_{k_2,\ell_2}{}' \frac{\tau_2}{2\pi|\ell_1 + k_1\tau|^2} \frac{\ldots}{2\pi \ldots} \ldots$$



$$= \frac{1}{4\pi^2} \sum_{k,\ell}{}' \frac{\tau_2^2}{|\ell + k\tau|^4} \propto E_2(\tau)$$ **"modular graph function"**

eigenfunction of upper half-plane Laplacian:

$$\nabla_\tau^2 E_2 = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2) E_2(\tau) = 2E_2(\tau)$$

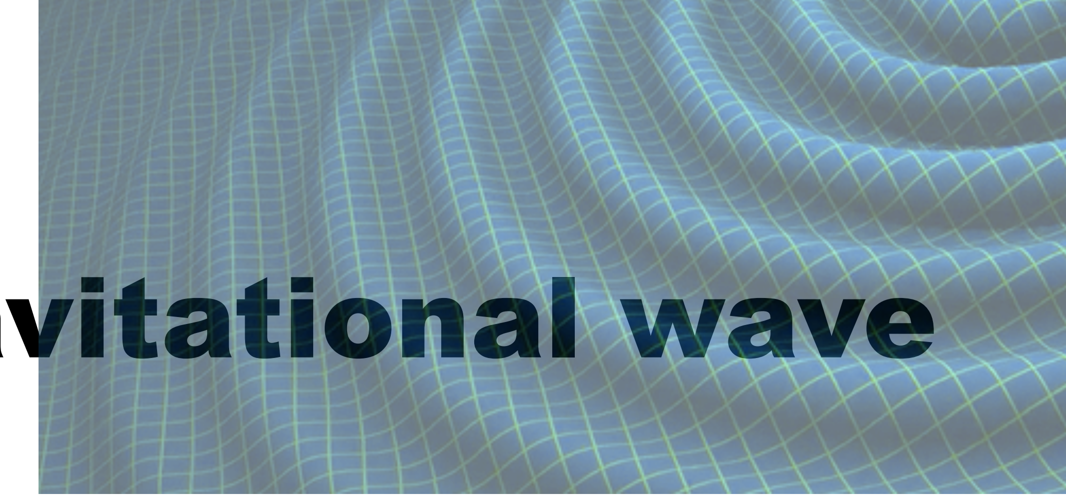
Why?

Follows from spacetime supersymmetry

…. if the torus is complexified dilaton (F-theory torus)!

Green, Sethi '98

…



Berenstein, Maldacena, Nastase ‘02

BMN: limit of $AdS_p \times S^p$ **plane gravitational wave**

$$ds^2 = -2dx^+dx^- - m^2 \sum (x^i)^2 (dx^+)^2 + \sum (dx^i)^2$$

here: $\mu \equiv m^2 \tau_2$

Bergman, Gaberdiel, Gree
…
Takayanagi
…

gauge fixing gives:

$$S = \frac{1}{2\pi\alpha'} \int d^2z \left(\frac{1}{2} \partial X_m \bar{\partial} X^m - \frac{1}{2} m^2 (X^i)^2 + \text{fermions} \right)$$

Berenstein, Maldacena, Nastase '02

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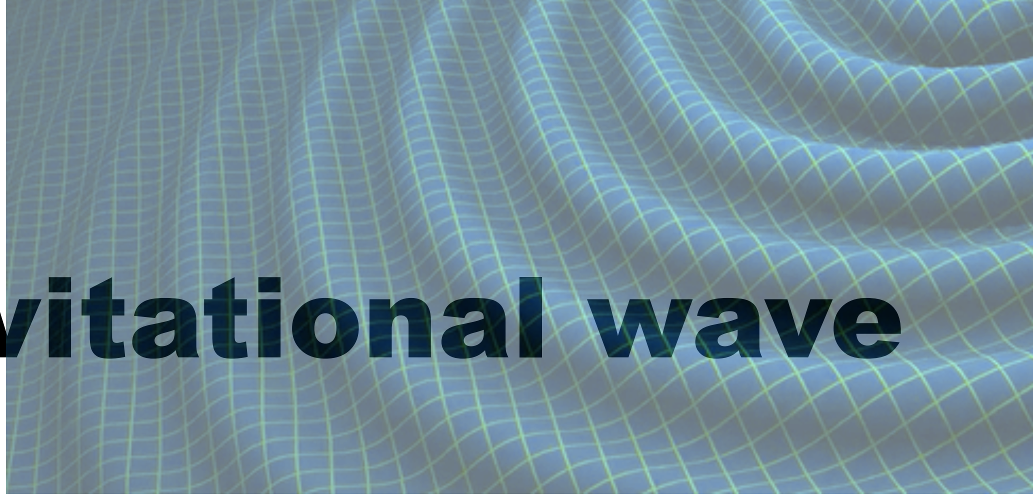

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Bergman, Gaberdiel, Gree
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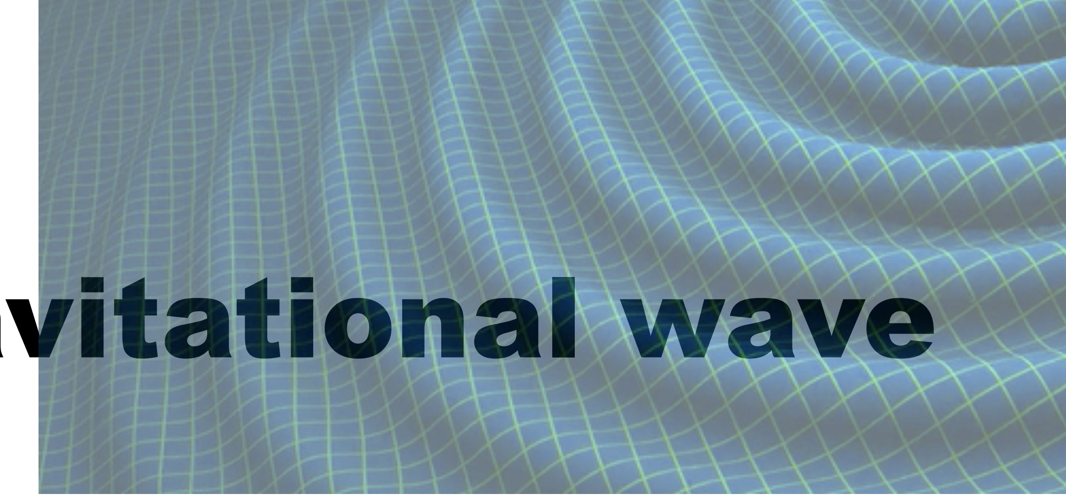
$$\ln Z^{(\mu)}_{a,b} = \underbrace{4\pi\tau_2 \Delta^{(\mu)}_b}_{} + \sum_{n=-\infty}^{\infty} \sum_{\pm} \ln \left(1 - e^{-2\pi\tau_2 \underbrace{\sqrt{\mu\tau_2^{-1} + (n \pm b)^2}}_{} + 2\pi i \tau_1 (n \pm b) \pm 2\pi i a}\right)$$

cf. $q^{n \pm b} = e^{-2\pi\tau_2 (n \pm b)} e^{2\pi i \tau_1 (n \pm b)}$

$$\frac{m}{2\pi} \sum_{k=1}^{\infty} \frac{\cos(2\pi k b)}{k} K_1(2\pi k m)$$

$$\overset{m \to 0}{\rightarrow} \frac{m}{4\pi^2} \sum_{k=1}^{\infty} \frac{\cos(2\pi k b)}{k^2} = \frac{1}{4} B_2(b) = \frac{1}{24}(1 + 6b(b-1))$$

zero-point energy



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cf. $q^{n\pm b} = e^{-2\pi\tau_2 (n\pm b)} e^{2\pi i \tau_1 (n\pm b)}$

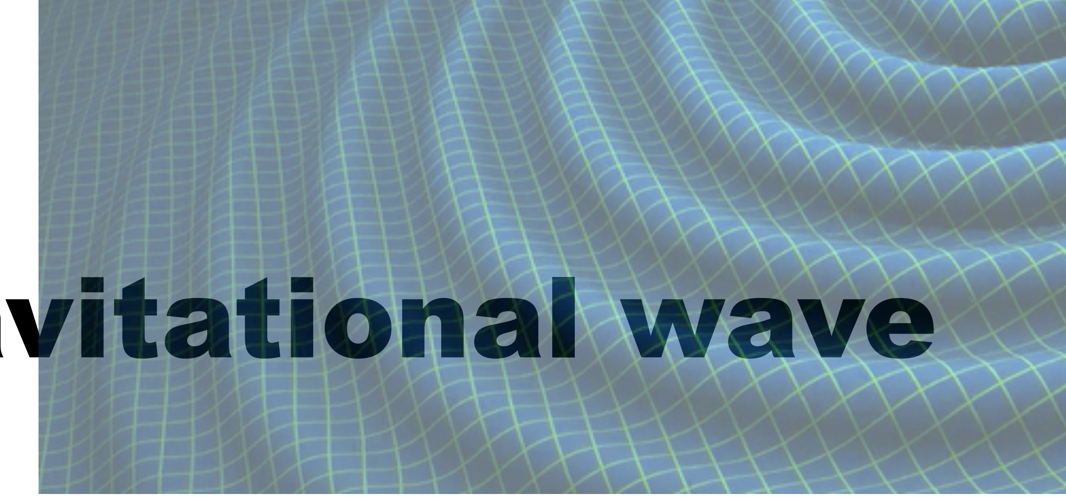
$$Z^{(\mu)}(z+1,\tau) = Z^{(\mu)}(z,\tau)$$

$$Z^{(\mu)}(z+\tau,\tau) = Z^{(\mu)}(z,\tau)$$

$$Z^{(\mu)}(z,\tau+1) = Z^{(\mu)}(z,\tau)$$

$$Z^{(\mu)}(z/\tau,-1/\tau) = Z^{(\mu)}(z,\tau)$$

$$z = a + b\tau$$



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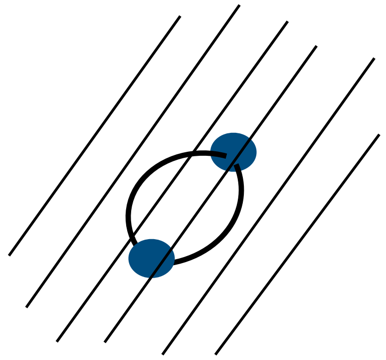
cf. $q^{n \pm b} = e^{-2\pi\tau_2 (n \pm b)} e^{2\pi i \tau_1 (n \pm b)}$

$$\int_0^\infty \cos \ell y \, e^{-\alpha\sqrt{m^2+y^2}} dy = \frac{\alpha m}{\sqrt{\ell^2 + \alpha^2}} K_1(m\sqrt{\ell^2 + \alpha^2})$$

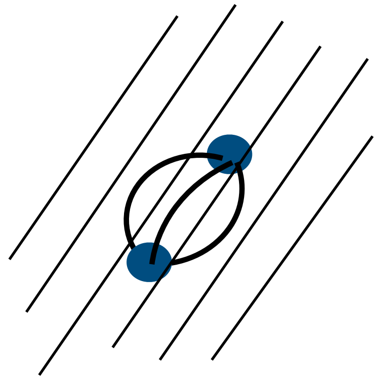
G from Z

Penrose '76

- all background metrics have plane-wave "massive" limit, so first step is solve Helmholtz equation (Laplacian + mass) to find massive Green's function on the torus



- in background field method, compute corrections due to interactions by usual Feynman diagrams e.g. with 3-vertex



plane wave (massive)
partition function
with ugly $\sqrt{\mu\tau_2^{-1} + \ldots}$

$$G^{(\mu)}(z) = \log Z^{(\mu)}(z)$$

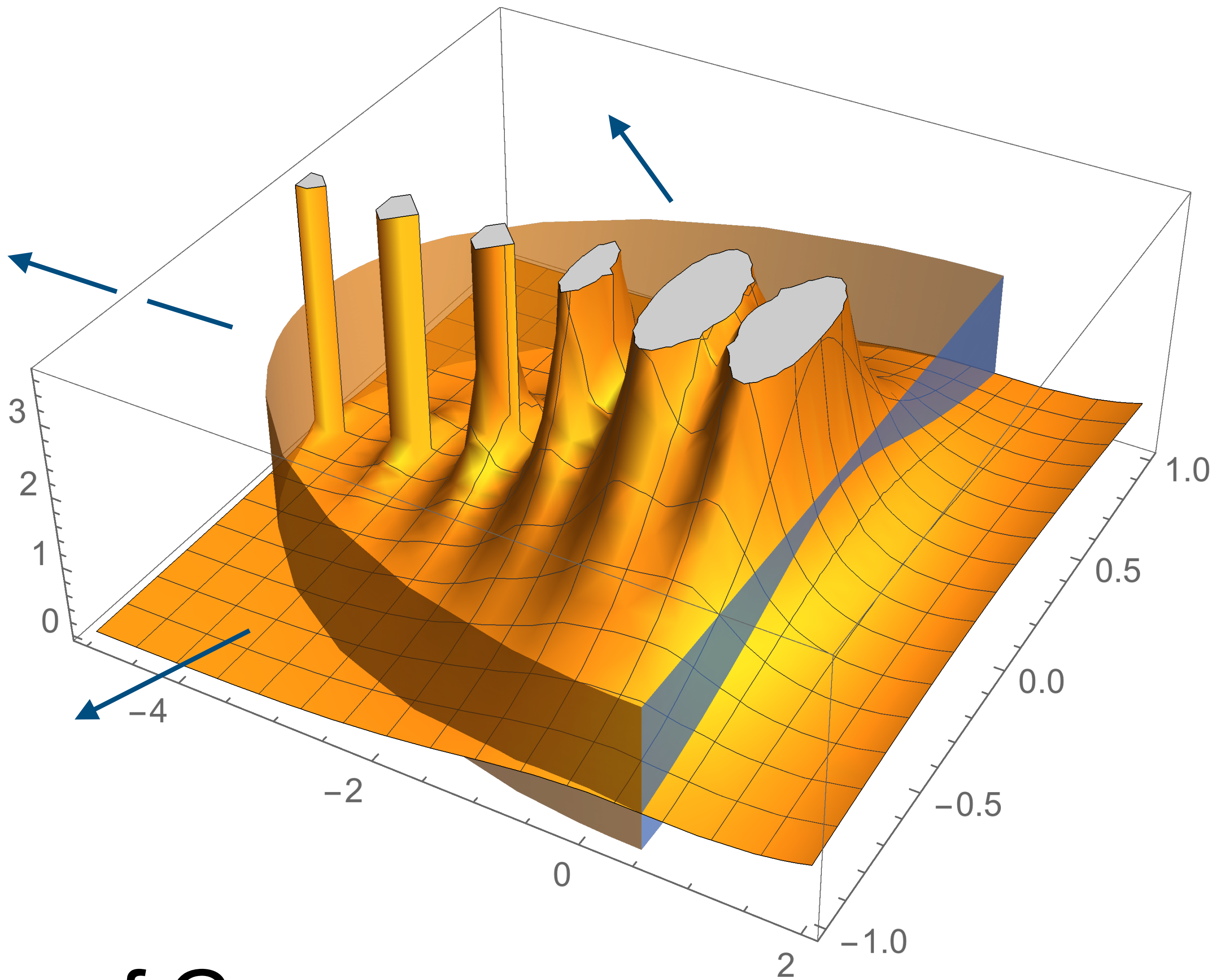
Fourier series

$$\sum_{k,\ell}{}' \frac{\tau_2 m K_1(2\pi m|\ell + k\tau|)}{|\ell + k\tau|} e^{2\pi i k x} e^{2\pi i \ell y}$$

Mellin transform
$\mu \to s$

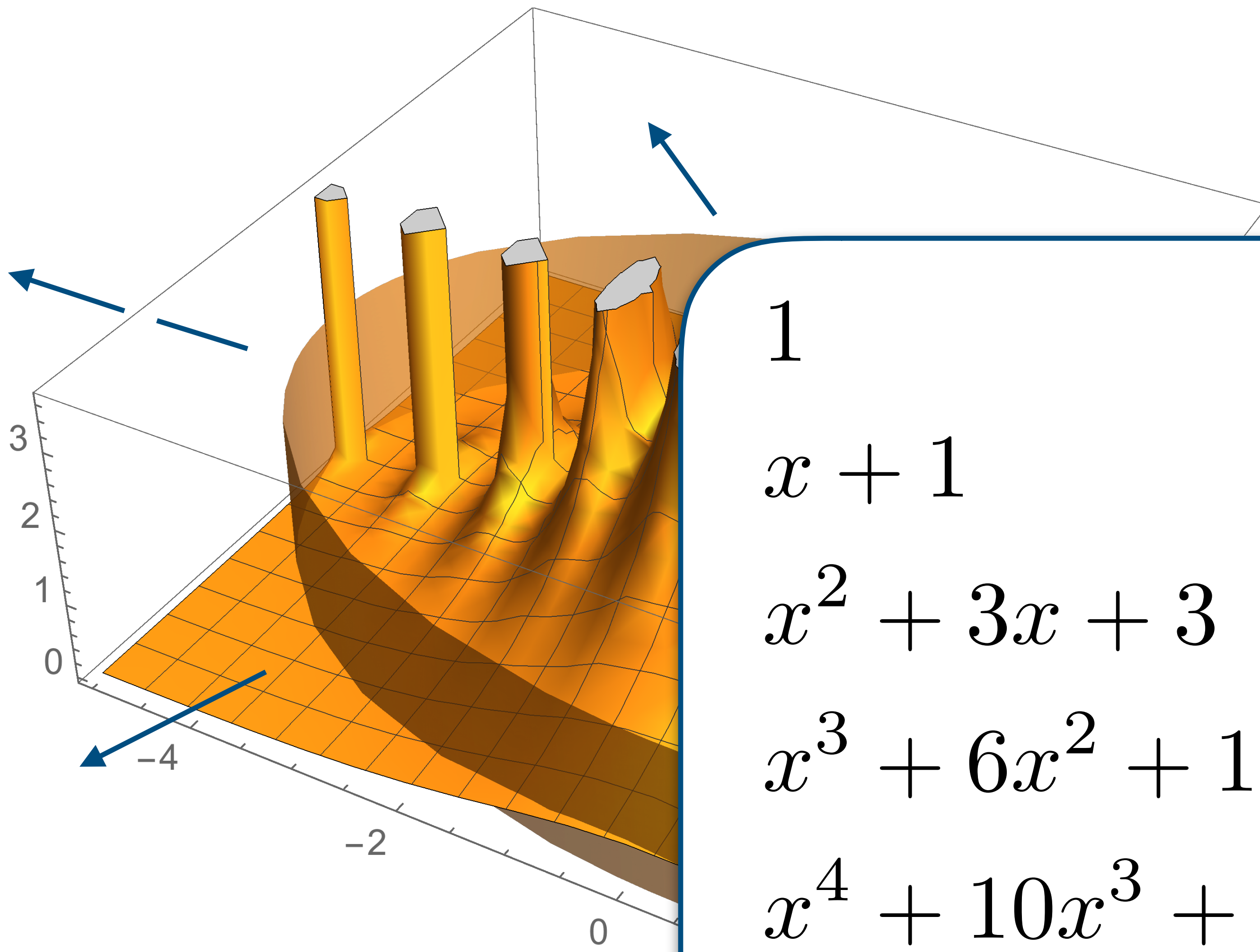
$$G_s(z) = \pi^{-s}\Gamma(s) E_{s+1}(0, z)$$

beautiful
Kronecker-
Eisenstein! (all μ)



inverse Mellin of Gamma:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}\Gamma(s)\,ds = \frac{1}{2\pi i}\oint_C x^{-s}\Gamma(s)\,ds = \sum_{n=0}^{\infty}\mathrm{Res}[x^{-s}\Gamma(s), s=-n] = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = e^{-x}$$



reverse Bessel polynomials

1

$x + 1$

$x^2 + 3x + 3$

$x^3 + 6x^2 + 15x + 15$

$x^4 + 10x^3 + 45x^2 + 105x + 105$

now take:

$$f_s(y) = \frac{1}{\sqrt{\pi}} 2^{(s-1)/2} y^{(s+1)/2} \Gamma(s/2) K_{s/2+1/2}(y)$$

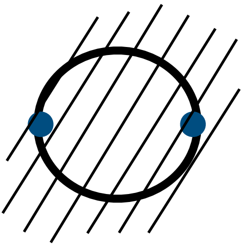
inverse Mellin:

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s} f_s(y)\, ds = \sum_{n=0}^{\infty} \mathrm{Res}[x^{-s} f_s(y), s = -2n] = e^{-\sqrt{x^2+y^2}}$$

$$e^{y(1-\sqrt{1-2t})} = 1 + y \sum_{n=1}^{\infty} \theta_{n-1}(y) \frac{t^n}{n!}$$

Worldsheet Feynman diagram

curved



$$= \int d^2z\, G^{(\mu)}(z) G^{(\mu)}(-z) = \int_0^1 dx \int_0^1 dy\; G^{(\mu)}(x,y)\, G^{(\mu)}(-x,-y)$$

$$= \int_0^1 dx \int_0^1 dy \sum_{k_1,\ell_1}{}' \frac{\tau_2 m K_1(2\pi m|\ell_1 + k_1\tau|)}{|\ell_1 + k_1\tau|} e^{2\pi i k_1 x} e^{2\pi i \ell_1 y} \sum_{k_2,\ell_2}{}' \frac{\tau_2 m K_1(2\pi m|\ell_2 + k_2\tau|)}{|\ell_2 + k_2\tau|}\, e^{2\pi i k_2(-x)} e^{2\pi i \ell_2(-y)}$$

$$= \tau_2^2 m^2 \sum_{k_1,\ell_1}{}' \sum_{k_2,\ell_2}{}' \frac{K_1(2\pi m|\ell_1 + k_1\tau|)}{|\ell_1 + k_1\tau|} \frac{K_1(2\pi m|\ell_2 + k_2\tau|)}{|\ell_2 + k_2\tau|} \delta_{k_1,k_2}\delta_{\ell_1,\ell_2} \qquad \left(\int_0^1 dx\, e^{2\pi i(k_1-k_2)x} = \delta_{k_1,k_2}\right)$$

$$= \underbrace{\tau_2^2 m^2 \sum_{k_1,\ell_1}{}' \frac{K_1(2\pi m|\ell_1 + k_1\tau|)^2}{|\ell_1 + k_1\tau|^2}}_{E_2^{(\mu)}(\tau)} \quad \overset{m\to 0}{\to} \quad \propto \sum_{k,\ell}{}' \frac{\tau_2^2}{|\ell + k\tau|^4} = E_2(\tau)$$

Integral of Green's function of 2D Helmholtz w.r.t μ

"massive modular graph function"

(exponential convergence)

Worldsheet conformal field theory

with Ramond-Ramond-supported curved background?

$$\underbrace{AdS_3 \times S^3}_{\text{6 dimensions}} \times K3$$

Berkovits-Vafa-Witten '99
"harmonic hybrid"

superfield $\Phi = \theta^a u_a^- + \theta^a \theta^b \sigma^m_{ab} e_m + (\theta^3)_a \chi^a_+$ $\quad \chi = \not{k} u$

frame (vielbein) σ^m_{ab} , τ^{mab} $\quad m = 1, \ldots, 6$

$$\{\sigma^m, \tau^n\} = 2\eta^{mn} \qquad a = 1, \ldots, 4$$

flat space

$$\mathrm{Tr}\, \tau_m \sigma_n = 4\eta_{mn}$$

curved space

$$\bar{g}_{mn} = \frac{1}{4} \mathrm{Tr}\, \tau_m \sigma_n$$

vertex operators

Dolan, Witten '99

Cadabra

Kasper Peeters

3-point gauge boson 1-loop amplitude in hybrid:

```
@(J)@(A1)@(W2)@(W3);
```

$$\theta\sigma_s\theta\ell_m(e_1^m - \theta\sigma^m\chi_1)(\xi_2 + f_{2np}\sigma^{np} - k_{2p}\theta\sigma^p\theta)\sigma^s (\xi_3 + f_{3qr}\sigma^{qr}\theta - k_{3q}\theta\sigma^q\theta)$$

```
substitute(_,$\xi2->0, \xi3->0, \chi1->0, \chi2->0, \chi3->0$);
```

$$\theta\sigma_s\theta\ell_m e_1^m f_{2np}\sigma^{np}\theta\sigma^s f_{3qr}\sigma^{qr}\theta$$

```
explicit_indices(_);
```

$$\theta^a\sigma_{sab}\theta^b\ell_m e_1^m f_{2np}\sigma^{np\,c}{}_d\theta^d\sigma^s_{ce}f_{3qr}\sigma^{qr\,e}{}_f\theta^f$$

```
substitute(_,ruleZeroMode)
```

$$\theta^a\theta^b\theta^c\theta^d \to \epsilon^{abcd}$$

$$\epsilon^{abdf}\sigma_{sab}\ell_m e_1^m f_{2np}\sigma^{np\,c}{}_d\sigma^s_{ce}f_{3qr}\sigma^{qr\,e}{}_f$$

Cadabra

Kasper Peeters

gamma matrix algebra:

```
\Gamma{#}::GammaMatrix(metric=\eta).
ex:= \Gamma^{m n} \Gamma^{p q};
```

$$\Gamma^{mn}\Gamma^{pq}$$

```
join_gamma(_);
```

$$\Gamma^{mnpq} + \Gamma^{mq}\eta^{np} - \Gamma^{mp}\eta^{nq} - \Gamma^{nq}\eta^{mp} + \Gamma^{np}\eta^{mq} + \eta^{np}\eta^{mq} - \eta^{mp}\eta^{nq}$$

Plane wave in pure spinor/hybrid

Berkovits, “Conformal Field Theory for the Superstring in a Ramond-Ramond Plane Wave Background”, hep-th/0203248

$$S_{(0)} = \frac{1}{2\pi\alpha'} \int d^2z \left(\frac{1}{2} \partial X_m \bar{\partial} X^m + \frac{1}{2} m^2 \partial X^+ \bar{\partial} X^+ (X^i)^2 + \text{fermions} \right)$$

“It has therefore been proven that the action for the superstring in an R-R plane wave background is an exact conformal field theory”

Massive torus Green's functions

Classical modular forms are holomorphic, mass is not.
There are an *infinite* number of automorphic forms of weight 0, and an *infinite* number of ways to generalize Green's functions.

What are the partial differential equations that replace holomorphy when there is a mass scale?
("Renormalization group equations"?)

M.B., Bringmann, Gannon '19

"Lowest" form:

"eigenvalue operator"
(spacetime supersymmetry?)

$$\left(\tau_2^2 \partial_\tau \partial_{\bar{\tau}} - \mu^2 \frac{\partial^2}{\partial \mu^2}\right) G^{(\mu)}(z, \tau) = 0$$

"ladder operator"?

$$\left(\tau_2 \partial_z \partial_{\bar{z}} + 4\mu \frac{\partial^2}{\partial \mu^2}\right) G^{(\mu)}(z, \tau) = -2\pi \delta^{(2)}(z)$$

Massive Automorphic Green's functions

caveat

Marcus Berg
Karlstad University, Sweden

work in progress with W. Linch, C.Mafra, O.Schlotterer
work in progress with K.Bringmann, T.Gannon

Dreams

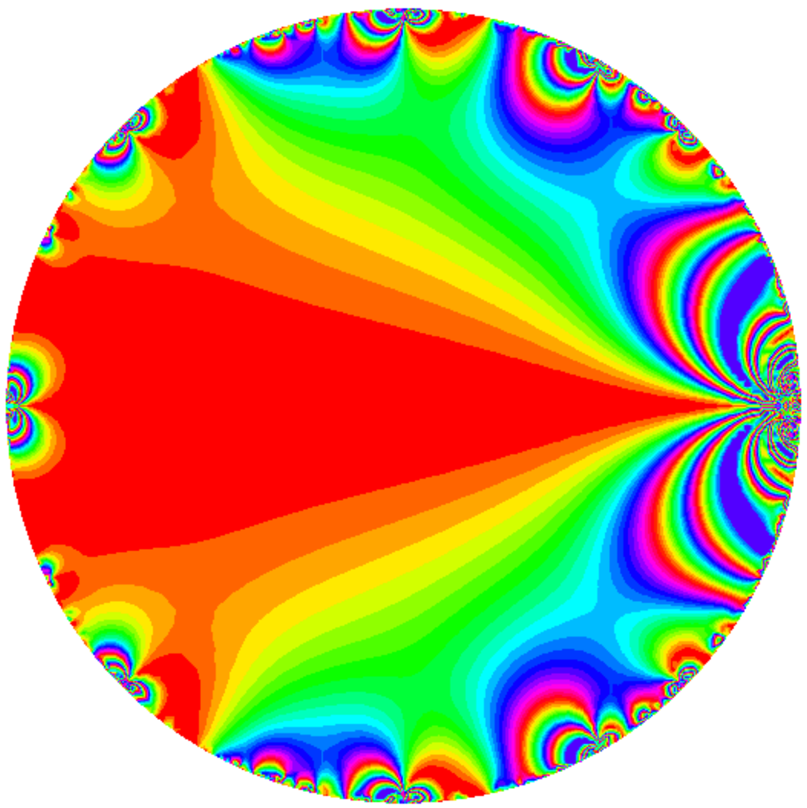
covariant pure spinor/hybrid AdS or plane wave
worldsheet equation of motion + spacetime supersymmetry

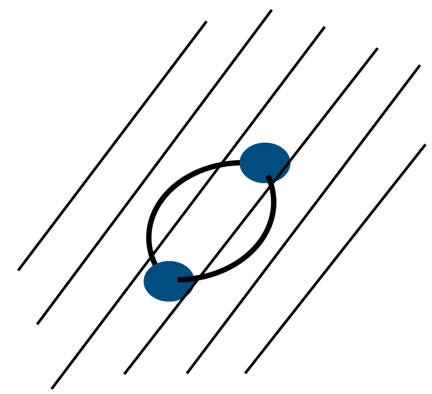
↓

partial differential equations

↑

Kaluza-Klein reduction?
of e.g. genus-2 moduli space:
automorphic on bigger group *G*





Thank you!

Marcus Berg
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