Asymptotic safety v. strings: UV completion on the world line

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w/ Nicola Dondi (CP3), Daniel Lewis

Overview

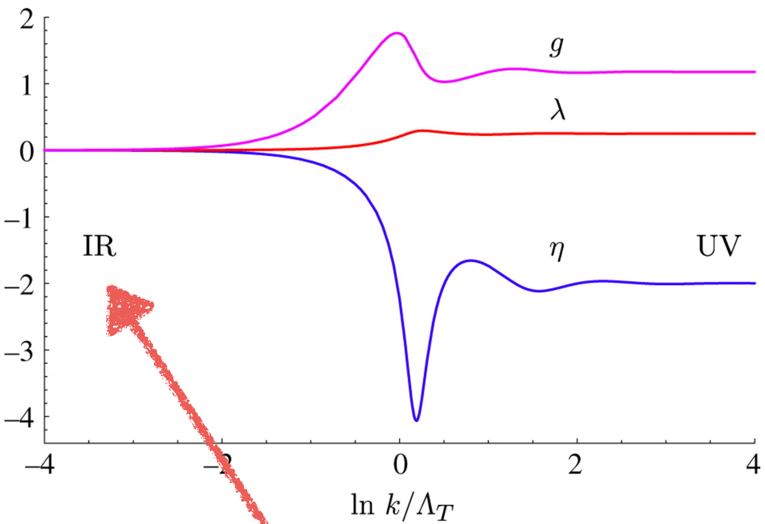
- Thoughts on asymptotic safety in a messy UV
- RG in a messy UV: the string case
- UV completion on the world line

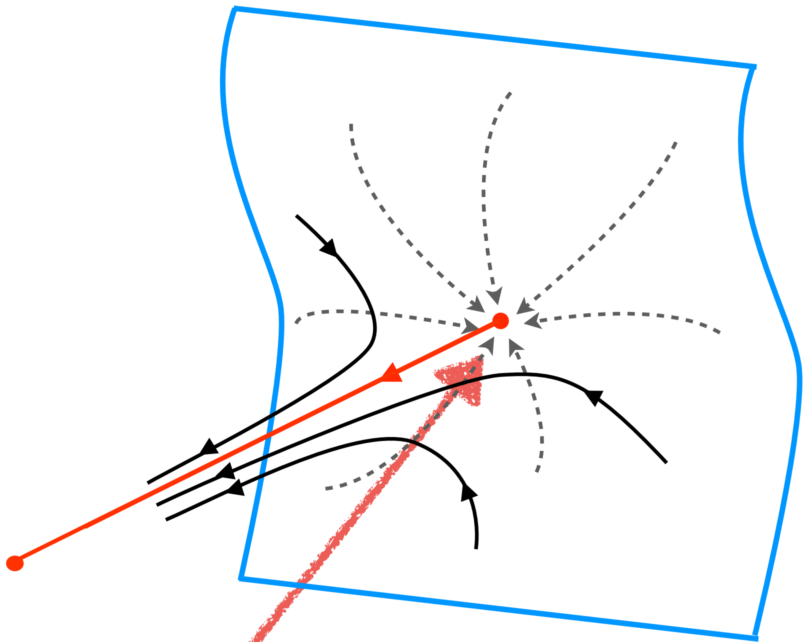
Asymptotic safety in a messy UV?

Gastmans et al ’78
Weinberg ’79
Peskin
Reuter, Wetterich
Gawedski, Kupiainen
Kawai et al,
de Calan et al ’,
Litim
Morris

AS as a UV completion

Weinberg et al’s basis for a proposal of UV complete theories





Gaussian IR fixed point => perturbative

Interacting UV fixed point => finite anomalous dimensions
In a field theory replace 1/e with 1/c => divergences of marginal operators (which affect the fixed point), some cured

Categorise the possible content of a theory as follows:

Irrelevant operators: would disrupt the fixed point - therefore asymptotically safe theories have to emanate precisely from UV fixed point where they are assumed zero (exactly renormalizable trajectory)

Marginal operators: can be involved in determining the UV fixed point where they become *exactly* marginal. Or can be marginally relevant (asymptotically free) or irrelevant.

Relevant operators: become "irrelevant" in the UV but may determine the IR fixed point.

Dangerously irrelevant operators: grow in both the UV and IR (common in e.g. SUSY)

Harmless relevant operators: shrink in both the UV and IR

Note relevant or marginally relevant operators still have "infinities" at the FP - just as quark masses, they still run at the FP just like any other relevant operator: but being relevant they do not affect the FP. (By definition they become unimportant at in the UV.)

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***B)* Yes! (Wetterich)** String theory has only one dimensionful parameter (which goes into defining the units by which we measure energy). A second energy scale is needed to observe scale violation. This could be the Planck scale, or the dynamical scale of some field theory. But well above the physics at which this second scale is generated, the theory should return to scale invariance(a.k.a. a UV fixed point for operators)

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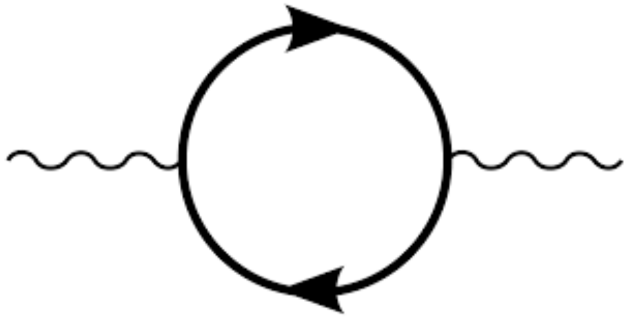
It would be interesting to know if it is B) and if so how string theory does it.

- ***A meaningful RG procedure with a messy UV: attempt 1)***



IR cut-off

$$\frac{16\pi^2}{g^2}\mathcal{A}^{(2)}_{\text{gauge}}(s) = -\frac{22C_A}{3}(p_\mu p_\nu - p^2 g_{\mu\nu})\left(\frac{1}{\epsilon} - \gamma_E + \log 4\pi + \log\left(-\frac{\mu^2}{s}\right)\right) \ ,$$

$$\frac{16\pi^2}{g^2}\mathcal{A}^{(2)}_{\text{ferm}}(s) = \frac{4N_f}{3}(p_\mu p_\nu - p^2 g_{\mu\nu})\left(\frac{1}{\epsilon} - \gamma_E + \log 4\pi + \log\frac{\mu^2}{m_f^2} + \left(1+\frac{2m_s^2}{s}\right)\Lambda(s;m_f,m_f)\right) \ ,$$

$$\frac{16\pi^2}{g^2}\mathcal{A}^{(2)}_{\text{scalar}}(s) = \frac{2N_s}{3}(p_\mu p_\nu - p^2 g_{\mu\nu})\left(\frac{1}{\epsilon} - \gamma_E + \log 4\pi + \log\frac{\mu^2}{m_s^2} + \left(1-\frac{4m_s^2}{s}\right)\Lambda(s;m_s,m_s)\right) \ ,$$

Interested in s dependence at a particular mu. Normally count UV divergences

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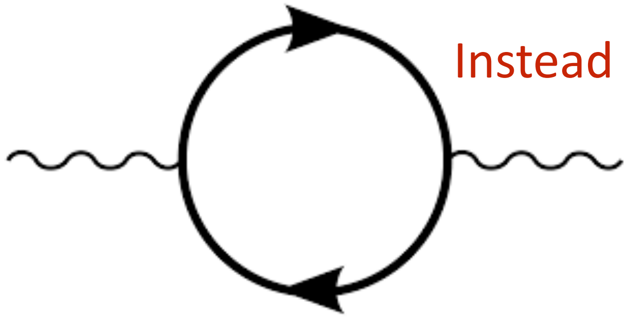

Instead count branch cuts as a function of s

$$\frac{16\pi^2}{g^2}\mathcal{A}^{(2)}_{\rm gauge}(s) = -\frac{22C_A}{3}(p_\mu p_\nu - p^2 g_{\mu\nu})\left(\frac{1}{\epsilon} - \gamma_E + \log 4\pi + \log\left(-\frac{\mu^2}{s}\right)\right) ,$$

$$\frac{16\pi^2}{g^2}\mathcal{A}^{(2)}_{\rm ferm}(s) = \frac{4N_f}{3}(p_\mu p_\nu - p^2 g_{\mu\nu})\left(\frac{1}{\epsilon} - \gamma_E + \log 4\pi + \log\frac{\mu^2}{m_f^2} + \left(1 + \frac{2m_s^2}{s}\right)\Lambda(s; m_f, m_f)\right) ,$$

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The most physical picture: Total s branch cuts just tell us how many states above threshold (s > 4m^2) (but hard to get without doing the actual integral)

$$\beta_{\frac{16\pi^2}{g^2}}(s) = -\frac{1}{\pi}\left[\frac{16\pi^2}{g^2}{\rm Im}\tilde{\mathcal{A}}^{(2)}(s)\right]$$

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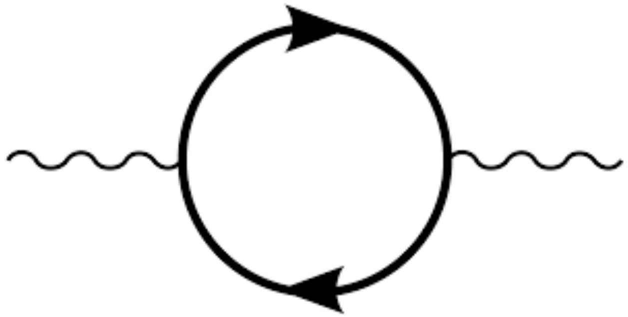

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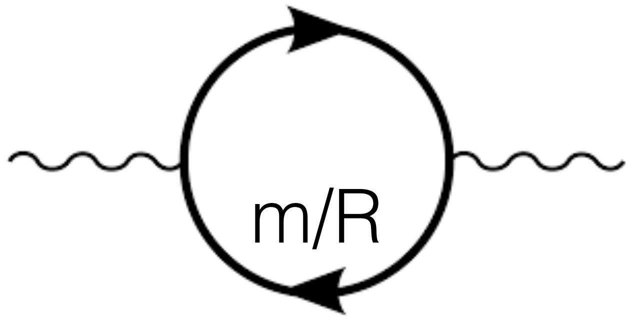
Or impose IR cut-off on Schwinger integral: equivalent to deep Euclidean s, and then..

$$\beta_{\frac{16\pi^2}{g^2}}(s) = \text{Re}\frac{\partial\left(\frac{16\pi^2}{g^2}\tilde{A}^{(2)}\right)}{\partial \log s}$$

- ***Toy example: KK theory***



$$\beta_{\frac{8\pi^2}{g^2}}(s) = \beta^{(\text{non-KK})}_{\frac{8\pi^2}{g^2}} + \text{Im} \sum_{\vec{m}} \int_0^\infty \int_0^1 d\tau dx \tau^{-1} \Delta b \quad \exp\left(\tau(s\,x(1-x) - \frac{\vec{m}\cdot\vec{m}}{R^2}\right)$$

$$= \beta^{(\text{non-KK})}_{\frac{8\pi^2}{g^2}} + \text{Im} \int_0^\infty \int_0^1 d\tau dx \frac{1}{\tau^{1+\frac{d}{2}}} \Delta b \sum_{\vec{\ell}} R^d \pi^{d/2} \exp\left(\tau(s\,x(1-x) - \frac{\vec{\ell}\cdot\vec{\ell}}{\tau}\pi^2 R^2\right)$$

Poisson resum then to get the branch cut expand the exponential until you get the pole —> log —> power law running beta function:

$$\beta_{\frac{8\pi^2}{g^2}}(s) = \beta^{(\text{non-KK})}_{\frac{8\pi^2}{g^2}} + \frac{\Delta b}{\Gamma(3+d/2)} \frac{\pi^{(d+3)/2}}{2^{d+1}} \left(R\sqrt{s}\right)^d + \mathcal{O}\left(\left(R\sqrt{s}\right)^{d-1}\right)$$

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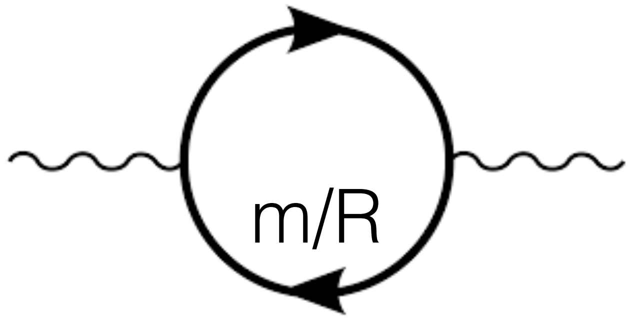
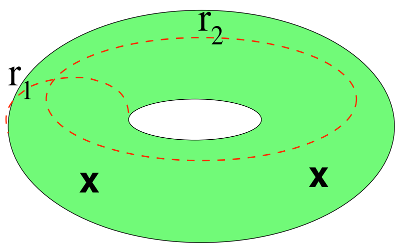
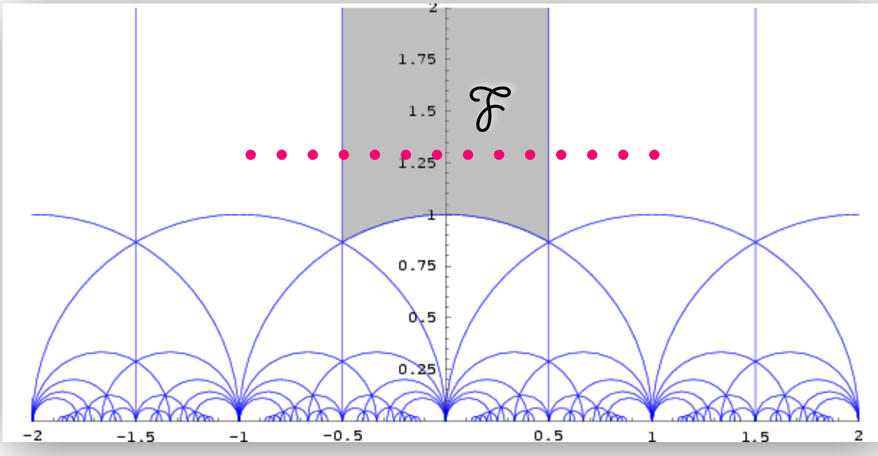


Note that the answer averages over the UV states and is *not the same as a naive rigid cut-off* at the scale s. (e.g. can introduce Scherk-Schwarz splitting of N=4 theory — the KK modes still give zero, even though the naive beta function would oscillate as ~ +- $\left(R\sqrt{s}\right)^d$)

RG in a messy UV: the string case

- *Can we do the same thing in a string theory?*
- *Kaplunovsky + \infty … calculate threshold corrections by doing the same diagram:*





$$\Pi^{\mu\nu} \approx \frac{g_{YM}^2}{16\pi^2}(k_1^\mu k_2^\nu - k_1.k_2\eta^{\mu\nu})\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\frac{1}{4\pi^2|\eta(\tau)|^4}\sum_{\alpha,\beta,Z_2}\mathcal{Z}_{B_{int}}^{Z_2}\mathcal{Z}_F^{\alpha,\beta,Z_2}$$

$$\times\int\frac{d^2z}{\tau_2}\left(4\pi i\partial_\tau\log(\frac{\vartheta_{\alpha\beta}(0|\tau)}{\eta(\tau)}\right)|\vartheta_1(z)|^{2k_1.k_2}\exp\left[-k_1.k_2\frac{2\pi}{\tau_2}\Im(z)^2\right]\delta^{ab}\mathrm{Tr}\left[\frac{k}{4\pi^2}\partial_{\bar z}^2\log\vartheta_1(\bar z)+Q^2\right]$$

$$\approx\frac{g_{YM}^2}{16\pi^2}\delta^{ab}(k_1^\mu k_2^\nu - k_1.k_2\eta^{\mu\nu})\int\frac{d\tau_2}{\tau_2}e^{-\pi s\tau_2}\frac{1}{4\pi^2}\mathrm{Tr}\left(4\pi i\partial_\tau\log\frac{\vartheta_{\alpha\beta}(0|\tau)}{\eta(\tau)}\left[-\frac{1}{4\pi\tau_2}+Q^2\right]\right)$$

This is the scale s — the answer will go like log(s) — so this gives the correct running in the field theory limit (s << 1) where the cut-off is at tau_2 >> 1.

$$
\begin{aligned}
\Pi^{\mu\nu} \quad &\approx \quad \frac{g_{YM}^2}{16\pi^2}(k_1^\mu k_2^\nu - k_1.k_2\eta^{\mu\nu})\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\frac{1}{4\pi^2|\eta(\tau)|^4}\sum_{\alpha,\beta,Z_2}\mathcal{Z}_{B_{int}}^{Z_2}\mathcal{Z}_F^{\alpha,\beta,Z_2} \\
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This is the one you want:

$$G(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m\tau+n|^2} e^{2\pi i(mu-nv)}$$

$$\equiv \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m\tau+n|^2} e^{2\pi i(m(z_1-\tau_1 z_2/\tau_2)-nz_2/\tau_2)}$$

$$\equiv \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m\tau+n|^2} e^{\frac{\pi}{\tau_2}(\bar z(m\tau+n)-z(m\bar\tau+n))}$$

$$G(z|\tau) = -\log\left|\frac{\theta_1(z|\tau)}{\theta_1'(\tau)}\right|^2 + 2\pi\frac{z_2^2}{\tau_2}$$

$$G(z|\tau) = \frac{2\pi z_2^2}{\tau_2} - \log(\left|\frac{\sin(\pi z)}{\pi}\right|^2) - 4\sum_{m=1}^{\infty}\left\{\frac{q^m}{1-q^m}\frac{\sin^2(\pi m z)}{m} + c.c.\right\}$$

$$G(z|\tau) = -2\left(\sum_{n,m\in\mathbb{Z}}\log|z+m+n\tau| - \sum_{(m,n)\neq(0,0)}\log|m+n\tau|\right) + \frac{2\pi z_2^2}{\tau_2}$$

$$\hat G(z|\tau) = \sum_{p=1}^{\infty}\frac{1}{p^2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\psi(\gamma(z),\gamma(\tau)), \qquad \text{with } \psi(z,\tau) = \frac{\tau_2}{\pi}e^{-2\pi i p z_2/\tau_2}$$

$$\hat G(z|\tau) = \underbrace{\frac{\tau_2}{\pi}\sum_{n\neq 0}\frac{1}{n^2}e^{2\pi i n z_2/\tau_2}}_{=\hat G^\infty(z|\tau)=2\pi\tau_2(z_2^2/\tau_2^2-|z_2/\tau_2|+\frac{1}{6})} + \sum_{\substack{m\neq 0\\ k\in\mathbb{Z}}}\frac{1}{|m|}e^{2\pi i m(k\tau_1+z_1)}e^{-2\pi\tau_2|m||k-z_2/\tau_2|}$$

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$$\boxed{\approx 2\pi\tau_2(z_2^2/\tau_2^2 - |z_2/\tau_2| + \frac{1}{6}) + e^{-2\pi\tau_2} + \dots}$$

c.f. the the factor $e^{\tau(sx(1-x)-m^2)}$ that appeared in the field theory two-point fn.

Takes the form of the one-loop ***world-line*** Green's function + stringy corrections.

However: string theory is defined on-shell — can use tricks but probably not very meaningful at scales well above s>>1.

- ***A meaningful RG procedure with a messy UV: attempt 2)***

Instead focus on amplitudes we can calculate on-shell: 4pt gluon amplitude in the Euclidean region s>>1, t,u<0 and add contributions from t channel and u channel. Also gives corrections to the Yang-Mills action, but can now put gluons on-shell.

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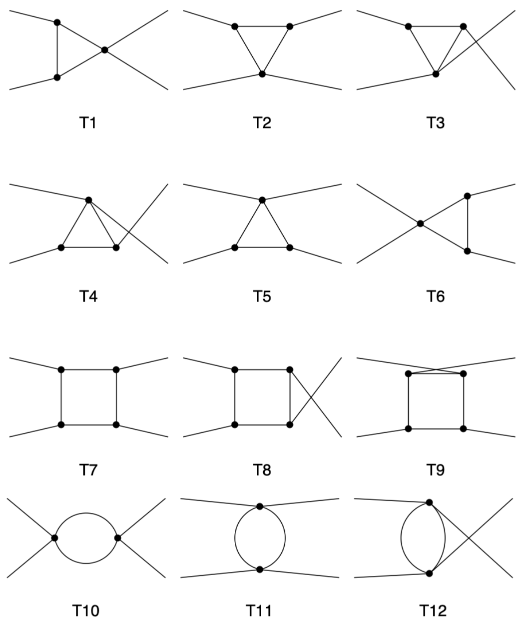
In field theory: in principle we need to calculate about 1000 diagrams. However can use various tricks to extract the divergences, or branch-cuts. e.g. only need to populate these topologies …

Adding the diagrams in s,t,u channel gives correct answer!



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In string theory: The fixed angle scattering amplitude and region of phase space was done by Gross-Mende: dominated by saddle at

$$\left(\frac{\theta_2}{\theta_3}\right)^4 = -\frac{t}{s} \simeq \sin^2\phi/2 \, ,$$
$$\left(\frac{\theta_4}{\theta_3}\right)^4 = -\frac{u}{s} \simeq \cos^2\phi/2 \, .$$

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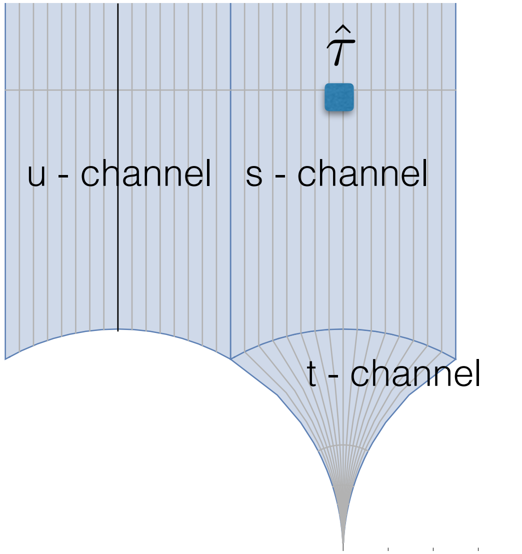
$$\left(\frac{\theta_4}{\theta_3}\right)^4 = -\frac{u}{s} \simeq \cos^2 \phi/2 \ .$$

$$\hat{\tau} = \frac{F(\frac{1}{2},\frac{1}{2};1;\cos^2(\phi/2))}{F(\frac{1}{2},\frac{1}{2};1;\sin^2(\phi/2))}$$



$\hat{\tau} \longrightarrow i\infty$ in the zero angle limit logarithmically … $\exp(-\pi\hat{\tau}_2) = -\dfrac{t}{s}$

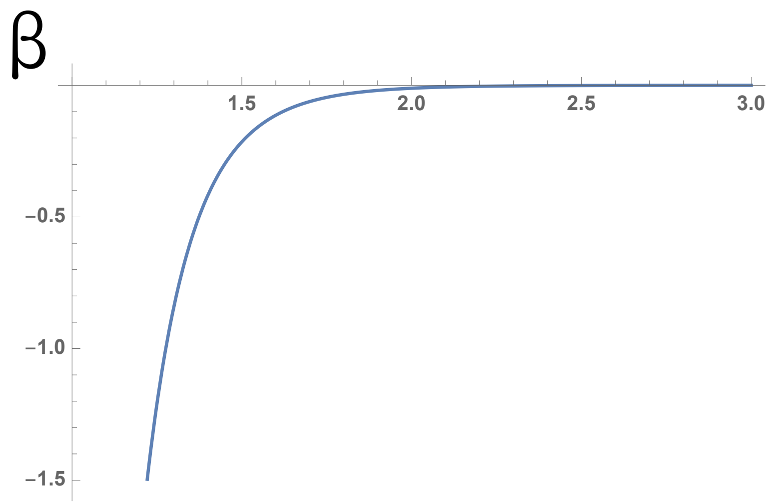
If we add the s,t,u parts equally, the definition is modular invariant

- ***A meaningful RG procedure with a messy UV: attempt 2)***

The integrand has a well defined saddle point which gives the amplitude

$$g^4 2^{10} \pi^{-24} (stu)^{-8/3} e^{-(s\log s + t\log t + u\log u)/8} \left| \prod_{\alpha=2}^{4} \frac{\vartheta''_\alpha}{\vartheta_\alpha} \left(\frac{\vartheta''_\alpha}{\vartheta_\alpha} + \frac{2\pi}{\Im(\hat\tau)} \right) \right|^{-\frac{1}{2}} \Im(\hat\tau)^{-13} \left(\frac{\vartheta'_1}{\pi} \right)^{40/3}$$

Adding the 3 channels we get a “beta function” that goes to zero in the UV:



UV completion on the world-line

- ***So what just happened? How does string theory quench amplitudes in the UV?***

As we saw the saddle point obeys

$$\exp(-\pi\hat{\tau}_2) = -\frac{t}{s}$$

But small angles is the particle limit. So we could have just used the modified world-line Green's function: the saddle in the vertex positions is *entirely* determined by the unmodified Green's function. Then you are left with a factor in the one-loop integrand of

$$\exp(-\pi\tau_2\frac{tu}{2s} + 4ue^{\pi\tau_2 u/s})$$

Replacing $u \sim -s$ this gives the correct saddle

Conclusion: string theory amplitudes can be mimicked by adding the leading exponential term into the world-line propagator!

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This was the only work string theory had to do

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Conclusion: string theory amplitudes can be mimicked by adding the leading exponential term into the world-line propagator!

- ***Conversely: contemplate simply defining a world-line theory with a G that has similar properties.***

Although the WL formalism emerges in the particle limit of string theory, a first quantised particle theory can be built from the bottom up.

Feynman;
Affleck, Alvarez, Manton;
Bern, Kosower;
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Schmidt, Schubert

Normally would have e.g. the tree-level propagator in a scalar theory:

$$\Delta(p^2) = \frac{1}{p^2+m^2} = \int_0^\infty \mathrm{d}T e^{-T(p^2+m^2)}$$

Here T is the Schwinger proper-time — essentially G(T)

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To mimic string amplitudes, copy the only Moebius transformation that matters:

$$\Delta(p^2) \; = \; \int_0^\infty \mathrm{d}t \, e^{-T(t)(p^2+m^2)}$$

$$\begin{aligned} T(t) \quad &= \quad T(t^{-1}) \ , \\ T(t) \; &\overset{\tau\to\infty}{\longrightarrow} \; t \ . \end{aligned}$$

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$$T(t) \;=\; T(t^{-1}) \;,$$
$$T(t) \;\overset{\tau\to\infty}{\longrightarrow}\; t \;.$$

Simple example: $T \;=\; t + t^{-1}$ gives infinite derivative field theory

Siegel;
Biswas, Mazumdar,
Gerwick, Koivisto
Buoninfante, Lambiase

$$\Delta(p^2) \;=\; 2K_1(2(p^2+m^2))$$

$$\longrightarrow \begin{cases} \frac{1}{p^2+m^2} & ; \quad p^2 \ll 1 \;, \\ \frac{\sqrt{\pi}e^{-2(p^2+m^2)}}{\sqrt{p^2+m^2}} & ; \quad p^2 \gg 1 \;. \end{cases}$$

Importantly only single pole: ghost-free (c.f. siegel et al. exponentially dressed props.)

Th'm: *Any theory for which* $tT(t^{-1})$ *is entire is ghost-free (at tree-level)*

e.g. the trivial case $T(t) = t + 1$ gives precisely the Siegel et al theory:

$$\Delta(p^2) = e^{-(p^2+m^2)}/(p^2+m^2)$$

This case is indistinguishable from imposing a cut-off on proper time (by reparam'n):

$$\Delta(p^2) \;=\; \int_{T_0}^{\infty} \mathrm{d}T \frac{1}{T'} e^{-T(p^2+m^2)}$$

The previous case corresponds to a weighted sum over paths that diverges "nicely" at T=2

Th'm: *Any theory for which* $tT(t^{-1})$ *is entire is ghost-free*

e.g. the trivial case $T(t) = t + 1$ gives precisely the Siegel et al theory:

$$\Delta(p^2) = e^{-(p^2+m^2)}/(p^2+m^2)$$

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$$\Delta(p^2) \;=\; \int_{T_0}^{\infty} \mathrm{d}T \frac{1}{T'} e^{-T(p^2+m^2)}$$

In target space the Bessel function has introduced minimal length:

$$\Delta(x,y) \;=\; \int_0^{\infty} \mathrm{d}t \,\frac{1}{(4\pi T)^{d/2}} e^{-\left[\frac{(x-y)^2}{4T}+Tm^2\right]}$$

Solutions to heat equation with in our example $D(t) = (1 - 1/t^2)$

Generic trees: written like the string version (or rather vice-versa)

e.g. scalar QED: write as a world-line theory, with Wilson line for photon emission

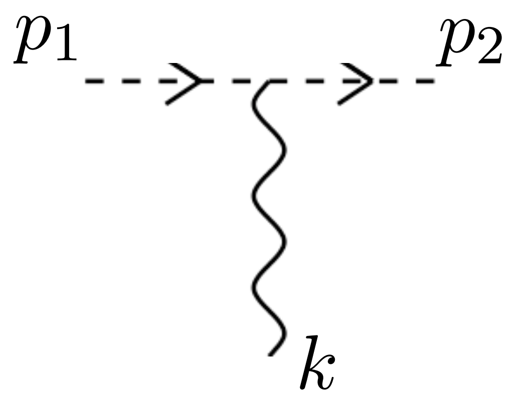
$$\begin{aligned} \Delta(x,y) &= \int_0^\infty \mathrm{d}t e^{-Tm^2} \int_{x(0)=x}^{x(T)=y} \mathcal{D}x e^{-S[x,A_\mu]} , \\ S[x,A_\mu] &= \int_0^T \mathrm{d}\tau \, \frac{\dot{x}^2}{4} + iq\,\dot{x}\cdot A(x) , \end{aligned}$$

expand photon as plane waves: $A_\mu(x(\tau)) = \sum_{i=1}^{n} \varepsilon_{i,\mu} e^{ik_i\cdot x}$

$$\begin{aligned} \mathcal{A}^{(n)} &= q^n \delta^4(p_1+p_2+\textstyle\sum_i k_i) \displaystyle\int_0^\infty \mathrm{d}t\, e^{-T(p_1^2+m^2)} \\ &\quad \times \int_0^T \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n \, e^{(p_1-p_2)\cdot\sum_i(-\tau_i k_i - i\varepsilon_i)} e^{(k_i\cdot k_j G_{ij} - 2i\varepsilon_i\cdot k_j \dot{G}_{ij} + \varepsilon_i\cdot\varepsilon_j \ddot{G}_{ij})} \end{aligned}$$

with $G_{ij} = \frac{1}{2}|\tau_i - \tau_j|$, and extract term in n-polarization vectors.

e.g. gauge coupling …



$$\mathcal{A}^{(1)} \;=\; iq\,\delta^4(p_1+p_2+k)\;\varepsilon\cdot(p_1-p_2)\,\frac{\Delta_{12}}{p_1^2-p_2^2}$$

$$\frac{\Delta_{12}}{p_1^2-p_2^2} \to \begin{cases} \frac{-1}{(p_1^2+m^2)(p_2^2+m^2)} & ;\ p^2 \ll 1, \\ \frac{\sqrt{\pi}}{p_1^2-p_2^2}\left(\frac{e^{-2(p_1^2+m^2)}}{\sqrt{p_1^2+m^2}} - \frac{e^{-2(p_2^2+m^2)}}{\sqrt{p_2^2+m^2}}\right) & ;\ p^2 \gg 1, \end{cases}$$

Generic one-loop diagrams written like the string version (or rather vice-versa)

$$\mathcal{A}^{(n)}_{1\ell}(\{p_i\}) \;=\; \int \mathrm{d}t\, e^{-m^2 T(t)} \int \mathcal{D}x\; V[p_1]...V[p_n]e^{-S[x,0]}$$

$$V_A[p] \;=\; \int_0^T \mathrm{d}\tau \varepsilon \cdot \dot{x}\, e^{ip\cdot x}$$

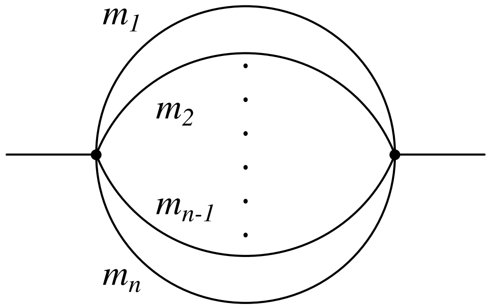
Can always rearrange it so propagators are treated democratically: e.g. 2 point

$$\mathcal{A}^{(2)}_{1-loop}(\{p_i\}) = (p^\mu p^\nu - g^{\mu\nu}p^2)\frac{1}{(4\pi)^{d/2}}\int \frac{\mathrm{d}t_1\,\mathrm{d}t_2}{(T_1+T_2)^{d/2}} \times e^{-m^2(T_1+T_2)+s\frac{T_1T_2}{T_1+T_2}}$$

$$\sim \quad (p^\mu p^\nu - g^{\mu\nu}p^2)\;\frac{\pi}{(16\pi)^{d/2}}\frac{e^{-s}}{s}$$

Dominated by the saddle at t=1: but this is not surprising, because we built it in. All UV sensitive amplitudes are dominated by saddles.

N-loop 2 point Sunset diagram — (perturbative control?)



$$\mathcal{A}_s(p) = \int \prod_i^N \mathrm{d}t_i \frac{1}{(4\pi \sum_i^N W_{N-1}^i)^{(N-1)D/2}} e^{-m^2 \sum_i^N t_i - p^2 \left[t_N - t_N^2 \frac{\sum_i^{N-1} W_{N-2}^{i \neq N}}{\sum_i^N W_{N-1}^i} \right]}$$

$\sum_i^N W_{N-1}^i$ is sum of all words of length N-1 that can be made with the symbols $\{t_i\}_{i=1\ldots N}$

$$\mathcal{A}_s(p) \sim \frac{1}{(16\pi^2)^N N!} e^{-2s/N}$$

Conclusions

- The behaviour of perturbative amplitudes (e.g. Gross Mende) can be understood by perturbing world line Green's functions without string theory clutter/beauty
- The lowest corrections to G recovers the attenuation of string amplitudes in the UV
- Can define sensible RG at scales much higher Ms in terms of physical amplitudes, in which string theory seems to have a Gaussian UV fixed point
- Inspired by this to look at new class of UV-complete world-line theories
- Correspond to infinite derivative field theories, but much nicer properties — e.g. amplitudes dominated by saddle points
- Gravity? Macrocausality? Unitarity at level of S-matrix?