Soft theorems for the Gravity Dilaton and the Nambu-Goldstone Boson Dilaton

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❖ Talk based on:

- P. Di Vecchia, R. M., M. Mojaza, JHEP 1901 (2019) 038 + work in progress.
- P. Di Vecchia, R. M., M. Mojaza, JHEP 1710(2017)017
- P. Di Vecchia, R.M. , M. Mojaza, J. Nohle, Phys. Rev D93 (2016) n.8, 080515.

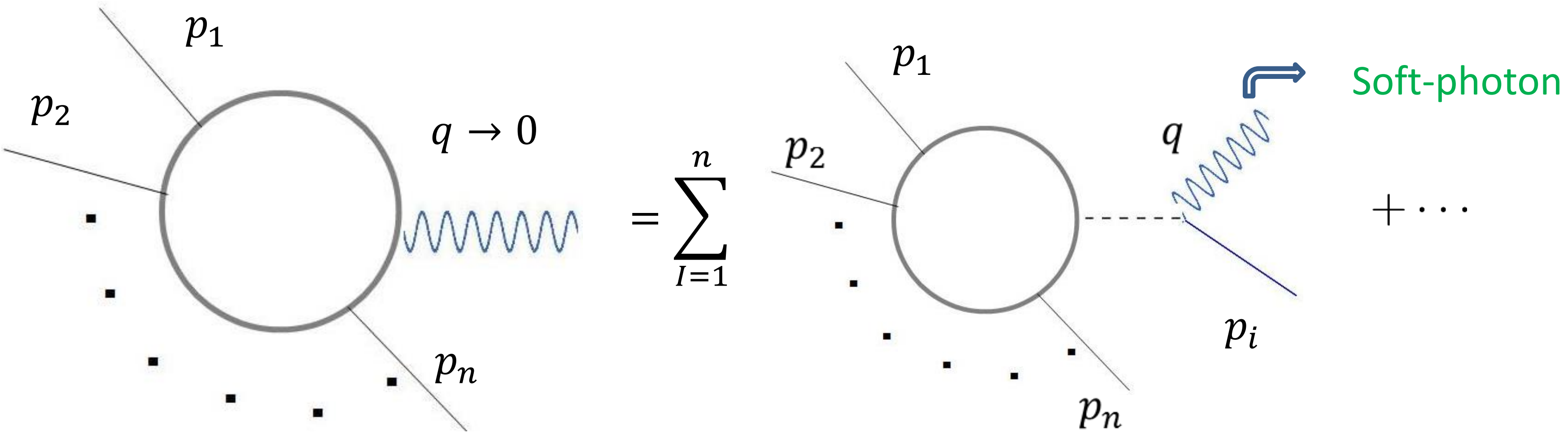
Plan of the talk

- Motivations
- Soft theorems in spontaneously broken conformal field theories.
- Soft theorems from tree level string amplitudes
- Multiloop soft theorems in bosonic string theory
- Conclusions.

Motivations

- It is well know that scattering amplitudes in the deep infrared region (or soft limit) satisfy interesting relations.

Low’s theorem: Amplitudes with a soft photon are determined from the corresponding amplitude without the soft particle



Soft-photon polarization

$$\mathcal{M}_{n+1}(q,\, p_1,\, \ldots p_n) \sim \sum_{i=1}^{n} e_i \frac{\epsilon_q \cdot p_i}{q \cdot p_i} \mathcal{M}_n(p_1, \ldots p_n) + N_{n+1}(q,\, p_1, \ldots p_n)$$

e_i charge of the particle i

smooth in the soft limit

Weinberg: Amplitudes involving gravitons and matter particles show and universal behavior when one graviton becomes soft.

$$\mathcal{M}^{\mu\nu}_{n+1}(q,\, p_1,\, \ldots p_n) \sim \sum_{i=1}^{n} \frac{p_i^{\mu} \cdot p_i^{\nu}}{q \cdot p_i} \mathcal{M}_n(p_1, \ldots p_n) + N_{n+1}(q,\, p_1 \ldots p_n)$$

$$\Downarrow$$

They were recognized to be a consequence of the gauge invariance

$$q^{\mu} M^{\mu}_{n+1}(q,\, p_1, \ldots p_n) = 0 \quad ; \quad q_{\mu} M^{\mu\nu}_{n+1}(q,\, p_1, \ldots p_n) = q_{\nu} M^{\mu\nu}_{n+1}(q,\, p_1, \ldots p_n) = 0$$

$\Downarrow$ Soft-Photon

$\Downarrow$ Soft-graviton

Adler's zero

(Weiberg, The Quatum Theory of Fields Vol .II.)

- Goldstone theorem: When a symmetry G is spontaneously broken to a sub-group H the spectrum of the theory contains as many Goldston bosons π^a, parametrizing the coset space ${}^{G}/_{H}$.

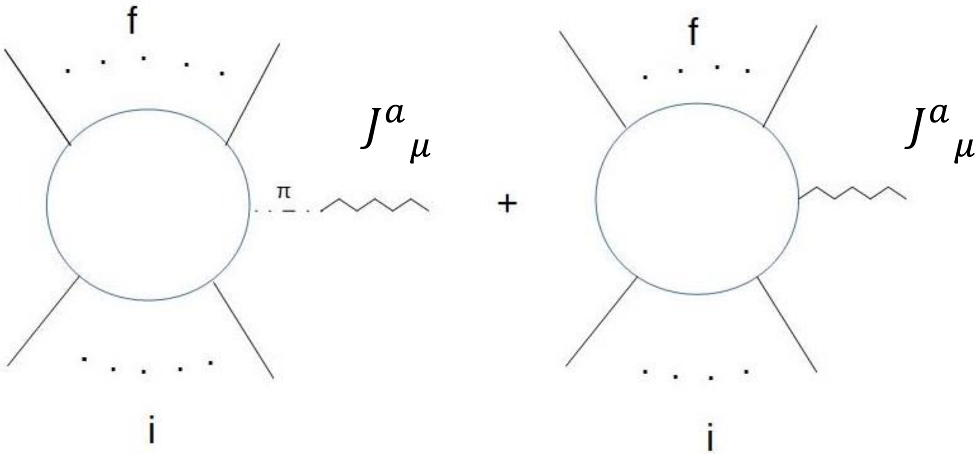
$$[T^i, T^j] = f^{ijk}T^k \quad ; \quad [T^i, X^a] = f^{iab}X^b \quad ; \quad [X^a, X^b] = f^{abi}T^i$$

- T^i, X^a are the unbroken and broken generators, respectively. $J^i{}_{\mu,} J^a{}_{\mu}$ are the corresponding currents.

- The matrix elements of the broken currents $J_\mu{}^a$ are:

$$\langle 0|J^a_\mu|\pi^b(p)\rangle = ip_\mu\, F\, \delta^{ab}$$

- The matrix element of a broken current between arbitrary states I, j, has two contributions:



$$\langle f|J^a_\mu(0)|i\rangle = \frac{iF\,q_\mu}{q^2}\langle f+\pi|i\rangle + N^\mu_{if}$$

Goldstone pole dominance for $q^2 \to 0$

No-pole contribution to the matrix element of the current.

F is the "decay constant"

- The conservation law $q_\mu J^\mu{}_a$=0 requires: $\langle f+\pi|i\rangle = i\frac{q_\mu\ N^\mu_{if}}{F}$

- Unless $N^\mu{}_{if}$ has a pole at $q \to 0$, the matrix element $\langle f+\pi|i\rangle$ for emitting a Goldstone boson in a transition $i \to f$ vanishes as $q \to 0$ (Adler's zero).

- The interest in the argument has been renewed by a proposal of A. Strominger (arXiv:1312.2229) and T. He, V. Lysov, P. Mitra and A. Strominger (arXiv:1401.7026) asserting that soft-theorems are nothing but the Ward-identities of the BMS-symmetry of asymptotic flat metrics.

- These theorems to subleading order for gluons and sub-subleading order for gravitons have been proved in arbitrary dimensions by using Poincarré and on-shell gauge invariance of the amplitudes.
- (J. Broedel, M. de Leeuw, J. Plefka, M. Rosso , arXiv: 1406.6574 and Z. Bern, S. Davies, P. Di Vecchia, and J. Nohle, arXiv:1406.6987)

- Furthermore, the soft graviton theorem has been extended to generic theories of quantum gravity and it has been proposed a soft theorem for multiple soft gravitons.[Laddha, Sen, 170600759; Chakrabarti, Kashyap, Sahoo, Sen, Verma, 1706.00759 .]
- Many papers on the subject.

Spontaneosly breaking of the Conformal symmetry

- A conformal transformation of the coordinates is an invertible mapping, $x \to x'$, leaving the metric invariant up to a local scale factor:

$$g_{\mu\nu} \to \Lambda(x) g_{\mu\nu}$$

✓ The group is an extension, with dilatations, $\mathcal{D}$, and special conformal transformations, $\mathcal{K}_\mu$, of the Poincaré group which belong to $\Lambda = 1$.

✓ Infinitesimally, the group transforms the space-time coordinates as follows:

D : space-time dimension

$$x^\mu \to x'^\mu = x^\mu + \epsilon^{MN} f^\mu_{MN}(x) \;\; ; \;\; \partial^\mu f^\nu_{MN} + \partial^\nu f^\mu_{MN} = \frac{2}{D} g^{\mu\nu} \partial_\rho f^\rho_{MN}$$

- For D>2 and in flat space, the generators are:

$$\mathcal{D} = i(d_\Phi + x_\mu \partial^\mu)\,, \quad \mathcal{J}^{\mu\nu} = -i(x^\mu \partial^\nu - x^\nu \partial^\mu) - \mathcal{S}^{\mu\nu}\,,$$
$$\mathcal{P}^\mu = i\partial^\mu\,, \qquad \mathcal{K}^\mu = i(2x^\mu x_\nu \partial^\nu - x^2 \partial^\mu + 2d_\Phi x^\mu) + 2x_\nu \mathcal{S}^{\mu\nu}$$

$\mathcal{S}^{\mu\nu}$: Spin angular momentum operator.

- The Nöther currents associated to the dilatations and special conformal transformations are conserved, because of the traceless of the improved energy momentum tensor:

$$\partial_\mu J^\mu_D = \partial_\mu [x^\nu T^{\mu\nu}] = T^\mu_\mu\,, \qquad \partial_\mu J^\mu_{K,\rho} = \partial_\mu [(2x_\nu x_\rho - \eta_{\rho\nu} x^2) T^{\mu\nu}] = 2x_\rho T^\mu_\mu$$

- Let’s consider now a situation where the conformal symmetry is spontaneously broken due to a scalar field getting a nonzero vev:

$$\langle 0|\phi|0\rangle = v^{d_\phi}$$

d_ϕ is the scaling dimension of the field

- v is the only scale mass of the theory and the vacuum remains invariant under the Poincaré group.
- When the conformal group is spontaneously broken to the Poincaré group, although the broken generators are the dilatations and the special conformal transformations, only one massless mode, the Dilaton, is needed. From the conformal algebra:

$$[P_\mu, \mathcal{K}_\nu]\langle\phi(x)\rangle = -2i\eta_{\mu\nu}\mathcal{D}\langle\phi(x)\rangle$$

$$\hookrightarrow \quad \begin{cases} \mathcal{D}\langle\phi\rangle \neq 0 \\ P_\mu\langle\phi\rangle = 0 \end{cases} \Rightarrow \mathcal{K}_\mu\langle\phi\rangle \neq 0$$

I. Low and A.V. Manohar hep-th/0110285

- The modes associated to the breaking of $\mathcal{K}_\mu$ can be eliminated leaving only the dilaton $\xi(x)$ which is the fluctuation of the field around the vev.
- The dilaton couples linearly to the trace of the energy momentum tensor:

$$T_{\mu\nu} = -d_\xi v^{d_\xi} \left(\frac{\eta_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu}{D-1} \right) \xi(x) \dots$$

Space-time dimension

✓ Taking the trace: $T^\mu_\mu = -d_\xi v^{d_\xi} \partial^2 \xi(x)$

✓ An observable consequence of spontaneously broken symmetry are the so called soft-theorems. i.e. identities between amplitudes with and without the Nabu-Goldstone boson carrying low momentum.

- ➢ Soft-Theorems follow from the Ward-identities of the spontaneously broken symmetry.
- ✓ The starting point is the derivative of the matrix element of Nöther currents $J_i^{\mu_i}(y_i)$ and scalar fields $\phi(x_i)$.

$$\frac{\partial}{\partial y^{\mu_1}} \cdots \frac{\partial}{\partial y^{\mu_m}} T^* \langle J_1^{\mu_1}(y_1) \ldots J_m^{\mu_m}(y_m) \phi(x_1) \ldots \phi(x_n) \rangle$$

- T^* denotes the T-product with the derivatives placed outside of the time-ordering symbol.
- Single soft theorem is obtained by considering only one current:

$$-\partial_\mu T^* \langle 0 | j^\mu(x) \phi(x_1) \ldots \phi(x_n) | 0 \rangle + T^* \langle 0 | \partial_\mu j^\mu(x) \phi(x_1) \ldots \phi(x_n) | 0 \rangle$$

$$= -i \sum_{i=1}^{n} \delta^D(x - x_i) T^* \langle 0 | \phi(x_1) \ldots \delta\phi(x_i) \ldots \phi(x_n) | 0 \rangle ,$$

- The infinitesimal transformation of the field under the action of the symmetry is:

$$\delta\phi(x) = i[Q, \phi(x)] = i\int d^D y[J^0(y), \phi(x)]\delta(y^0 - x^0)$$

✓ If the current is unbroken, $\partial_\mu J^\mu = 0$, one gets the usual Ward-identity of conserved symmetries.

✓ If the current is spontaneously broken, by transforming in the momentum space the matrix elements and taking the limit of small transferred momentum of the current q_μ, we get (in the absence of poles):

$$iq_\mu\langle 0|J^\mu(q)\phi(x_1)\cdots\phi(x_n)|0\rangle = 0 + O(q)$$

✓ This leads to the single soft Ward-identity:

$$\int d^D x\, \mathrm{e}^{-iqx} T^* \langle 0|\partial_\mu J^\mu(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$$
$$= -\sum_{i=1}^n \mathrm{e}^{-iqx_i} T^* \langle 0|\phi(x_1)\cdots\delta\phi(x_i)\cdots\phi(x_n)|0\rangle + O(q)$$

✓ For spontaneously broken scale transformations

$$\partial_\mu J^\mu_{\mathcal{D}}(y) = T^\mu_\mu = -d_\xi v^{d_\xi} \partial^2 \xi(y)$$

$$\delta\phi(x) = [\mathcal{D}, \phi(x)] = i\,(d + x^\mu \partial_\mu)\,\phi(x)\,,$$

✓ The relation between correlation functions is translated in a relation between amplitudes through the LSZ reduction:

$$[\mathrm{LSZ}] \equiv i^n \left(\prod_{i=1}^{n} \lim_{k_j^2 \to -m_j^2} \int d^D x_j \, e^{-ik_j x_j} (-\partial_j^2 + m_j^2) \right)$$

$\Rightarrow$ On-shell limit

- Applying the LSZ-reduction on the first term of the single soft Ward identity:

$$\Big[\mathrm{LSZ}\Big] \int d^D x \, \mathrm{e}^{-iq\cdot x} \, T^* \langle 0 | \partial_\mu j^\mu_{\mathcal{D}}(x) \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

$$= (-i) \, d_\xi v^{d_\xi} \, (2\pi)^D \delta^{(D)} (\textstyle\sum_{j=1}^n k_j + q) \mathcal{T}_{n+1}(q; k_1, \dots, k_n) \, ,$$

- and on the second term of the Ward-identity:

$$\left[\mathrm{LSZ}\right] \left(-\sum_{i=1}^{n} \mathrm{e}^{-iq\cdot x_i}\, T^* \left\langle 0\right| \phi(x_1)\cdots\delta\phi(x_i)\cdots\phi(x_n)\left|0\right\rangle\right)$$

$$= -\sum_{i=1}^{n}\Bigg[\lim_{k_i^2\to -m_i^2}(k_i^2+m_i^2)\, i\left(d-D-(k_i+q)^\mu\frac{\partial}{\partial k_i^\mu}\right)$$

$$\times\frac{(2\pi)^D\delta^{(D)}\left(\sum_{j=1}^{n}k_j+q\right)}{(k_i+q)^2+m_i^2}\mathcal{T}_n(k_1,\ldots,k_i+q,\ldots,k_n)\Bigg]$$

- By commuting the delta-function with the differential operators:

$$d_\xi v^{d_\xi}\,\mathcal{T}_{n+1}(q;k_1,\ldots,k_n) = \left\{-\sum_{i=1}^{n}\frac{m_i^2}{k_i\cdot q}\left(1+q^\mu\frac{\partial}{\partial k_i^\mu}\right)+D-nd-\sum_{i=1}^{n}k_i^\mu\frac{\partial}{\partial k_i^\mu}\right\}\mathcal{T}_n(k_1,\ldots,\bar{k}_n)$$
$$+\mathcal{O}(q)\,.$$

We can repeat the same calculation with the current associated to the special conformal transformations.

$$j^{\mu}{}_{(\lambda)} = T^{\mu\nu}(2x_\nu x_\lambda - \eta_{\nu\lambda}x^2) \;\; ; \;\; \partial_\mu j^{\mu}{}_{(\lambda)} = 2\, x_\lambda\, T^{\mu}{}_{\mu} = 2\, d_\xi v^{d_\xi}\, x_\lambda(-\partial^2)\,\xi(x)$$

whose action on the scalar fields is:

$$\delta_{(\lambda)}\phi(x) = [\mathcal{K}_\lambda, \phi(x)] = i\left((2x_\lambda x_\nu - \eta_{\lambda\nu}x^2)\partial^\nu + 2\, d\, x_\lambda\right)\phi(x)\,.$$

The single Ward-identity is a relation between the derivative of the amplitude with the soft dilaton and the amplitude without the dilaton.

$$d_\xi v^{d_\xi}\frac{\partial}{\partial q^\lambda}\mathcal{T}_{n+1}(q; k_1, \ldots, -q - \sum_{j=1}^{n-1} k_j) = \sum_{i=1}^{n}\left\{\frac{m_i^2}{k_i \cdot q}\left(1 - \frac{1}{2}q^\mu q^\lambda \frac{\partial^2}{\partial k^\mu \partial k^\lambda}\right)\right.$$
$$\left. - \left[k_i^\mu\left(\frac{\partial^2}{\partial k_i^\mu \partial k_i^\lambda} - \frac{1}{2}\eta_{\mu\lambda}\frac{\partial^2}{\partial k_{i\nu}\partial k_i^\nu}\right) + d\frac{\partial}{\partial k_i^\lambda}\right]\right\}\mathcal{T}_n(k_1, \ldots, k_{n-1}) + \mathcal{O}(q)$$

- The two single Ward-identities can be combined and in total one gets:

$$d_\xi v^{d_\xi} \, \mathcal{T}_{n+1}(q;k_1,\ldots,k_{n-1}) = \Bigg\{ -\sum_{i=1}^{n-1} \frac{m_i^2}{k_i\cdot q}\left(1+q^\mu \frac{\partial}{\partial k_i^\mu} + \frac{1}{2} q^\mu q^\nu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\nu}\right) + D - nd - \sum_{i=1}^{n-1} k_i^\mu \frac{\partial}{\partial k_i^\mu}$$

$$-q^\lambda \sum_{i=1}^{n-1}\left[\frac{1}{2}\left(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\lambda} - k_{i\,\lambda}\frac{\partial^2}{\partial k_{i\nu}\partial k_i^\nu}\right) + d\,\frac{\partial}{\partial k_i^\lambda}\right]\Bigg\} \mathcal{T}_n(k_1,\ldots,k_{n-1}) + \mathcal{O}(q^2)$$

- The Ward-identities of the scale and special conformal transformations determine completely the low-energy behavior, through the order $O(q^1)$, of the amplitude with a soft dilaton in terms of the amplitude without the dilaton.

Multi-soft Dilaton Behaviour

- Double soft behavior is obtained starting from the matrix elements with the insertion of two broken currents.

$$\frac{\partial}{\partial y^{\mu_1}} \frac{\partial}{\partial y^{\mu_2}} T^* \langle J_1^{\mu_1}(y_1) \dots J_m^{\mu_2}(y_2) \phi(x_1) \dots \phi(x_n) \rangle$$

- Three different combinations of scale and special conformal currents can be considered.

- Ward-identity with two Dilaton currents and with all scalar fields with the same dimension $d_i = d$:

$$f_\xi^2 T_{n+2}(q,k,k_1,\ldots,\bar{k}_n) = \left[D-(n+1)d-\sum_{j=1}^{n} k_j\cdot\partial_{k_j}\right]\left[D-nd-\sum_{i=1}^{n} k_i\cdot\partial_{k_i}\right] \times T_n(k_1,\ldots,\bar{k}_n) + O(q,k)$$

$$\boxed{f_\xi = d_\xi\, v^{d_\xi}}$$

- q and k are the momenta of the two soft-dilatons.

- Ward-identity with one Dilaton and one Special conformal current.

$$f_\xi^2 \frac{\partial}{\partial k^\lambda} T_{n+2}(q,k,k_1,\ldots,\bar{k}_n) = \sum_{j=1}^{n} \hat{K}_{k_j,\lambda}\left(D-d+\sum_{i=1}^{n}\hat{D}_i\right) T_n(k_1,\ldots,\bar{k}_n) + O(q,k)$$

with:

$$\boxed{\hat{D}_i = -(d_i + k_i\cdot\partial_{k_i})} \qquad \boxed{\hat{K}^\lambda_{k_i} = \frac{1}{2}k_i^\lambda \partial^2_{k_i} - (d_i + k_i\cdot\partial_{k_i})\partial^\lambda_{k_i}}$$

- Ward-identity with two Special conformal currents is still an open problem.
- The two Ward-identities can be combined in a single expression giving, up to $O(q)$, the double soft behavior of an amplitude with two soft dilatons:

$$f_\xi^2 T_{n+2}(q,k,k_1,\ldots,\bar{k}_n) = \left[\left(D - d + \sum_{i=1}^{n} \hat{D}_i \right) \left(D + \sum_{i=1}^{n} \hat{D}_i \right) \right.$$
$$\left. +(q^\lambda + k^\lambda) \sum_{i=1}^{n} \hat{K}_{k_i,\lambda} \left(D - d + \sum_{i=1}^{n} \hat{D}_i \right) \right] T_n(k_1,\ldots,\bar{k}_n) + O(q^2,k^2,qk)$$

- ✓ It can be easily seen that the double soft theorem can be obtained by making two consecutive emissions of soft dilatons, one after the other.
- ✓ We conjecture that the amplitude for emission of any number of soft dilatons is fixed by the consecutive soft limit of single dilatons emitted one after the other.

- The single and double soft theorem have been explicitly verified by computing four, five and six point amplitudes in two different models:

- A conformal invariant version of the Higgs potential:

$$L = -\frac{1}{2}\partial_\mu \xi \partial^\mu \xi - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\left(\frac{\lambda \xi}{a}\right)^{\frac{4}{D-2}} \phi^2 a^{\frac{4}{D-2}}$$

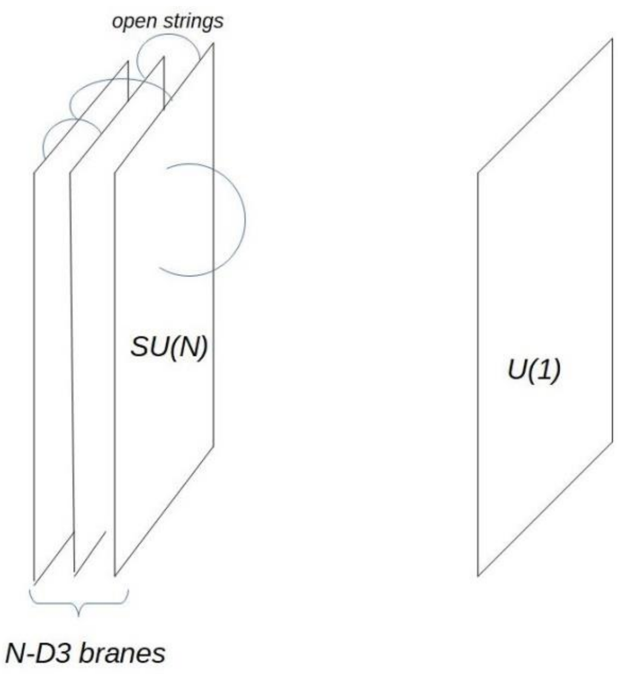
R.Boels, W. Woermsbecker, arXiv:1507.08162

- Expanding around the flat direction $\xi = a + r$, the field ϕ acquire mass $m^2 = (\lambda a)^{\frac{4}{D-2}}$ and the conformal invariance is spontaneously broken. The field r remains massless (dilaton).

- Gravity dual of $\mathcal{N} = 4$ super Yang Mills on the Coulomb branch. ⇨ N=4 in the strongly coupled regime.

(Elvang, Freedman, Hung, Kiermaier, Myers, Theisen, JHEP 1210 011(2012))

- D3 brane probe in the background of N D3-branes



In the Large N limit the backreaction of the probe on the background can be neglected and the dynamics of the D3-brane is governed by the Dirac-Born-Infeld action and the Wess-Zumino term on $AdS_5 \times S^5$.

$$S = -\frac{1}{\kappa^2}\int d^4x\,\frac{r^4}{L^4}\left(\sqrt{-\det\left(\eta_{\mu\nu} + \frac{L^4}{r^4}\frac{\partial x^i}{\partial x^\mu}\frac{\partial x^i}{\partial x^\nu} + \kappa\frac{L^2}{r^2}F_{\mu\nu}\right)} - 1\right)$$

L is the AdS_5 radius. $r^2 = \sum_{i=1}^{6}(x^i)^2$ is the S^5 radius.

Coulomb branch: $\phi_i = \kappa x_i = v\delta_{i6} + \tilde{\phi}_i\,, \quad \tilde{\phi}_6 \equiv \xi$

Dilaton

Expanding the action one gets the following Lagrangian for the Dilaton:

$$\mathcal{L}^{\xi}_{4,5,6} = \frac{\lambda^2}{8v^4}\left[1 - \frac{4\xi}{v} + 10\frac{\xi^2}{v^2}\right](\partial_\mu\xi\partial^\mu\xi)^2 - \frac{\lambda^4}{16v^8}(\partial_\mu\xi\partial^\mu\xi)^3 + \ldots$$

Soft-Theorems in String theory

- Soft theorems and their connections with gauge symmetries have been extensively studied in String-theories.
- ✓ Tree-level: Ademollo et al (1975) and J. Shapiro (1975);
- B.U.W. Schwab 1406.4172 and 1411.6661;
- M. Bianchi, Song He, Yu-tin Huang and Congkao Wen arXiv:1406.5155;
- P. Di Vecchia, R. M. and M. Mojaza 1507.00938, 1512.0331, 1604.03355, 1610.03481,1706.02961 .
- M. Bianchi and A. Guerrieri 1505.05854, 1512.00803, 1601.03457.
- A. Sen et al. 1702.03934, 1707.06803, 1804.09193.
- ✓ Loops: A. Sen 1703.00024; Laddha and Sen 1706.00754.
- How much universal are the soft theorems? Naively they should be modified in any theory with a modified three point interaction. String theories are a good arena where to explore the universality of low energy theorems.

Amplitudes in Bosonic, Heterotic and Superstring theories

- In closed bosonic, heterotic and superstring theory, amplitudes with a graviton or a dilaton with soft momentum q and n hard particles with momentum k_i, are obtained from the same two index tensor $M^{\mu\nu}_{n+1}(q,\, k_1, \ldots k_n)$.

z_l are complex coordinates parametrizing the insertion on the string world-sheet of the "hard vertex operators".

$F^{\mu\nu}$ is a function of all Koba-Nielsen variables having pole for $z \sim z_i$

$$M^{\mu\nu}_{n+1}(q,\, k_1, \ldots k_n) = M_n(\epsilon_i,\, k_i) * \int dz \prod_{l=1}^{n} |z - z_l|^{\alpha' q \cdot k_l} \mathcal{F}^{\mu\nu}(q,\, \{\epsilon_i\},\, \{k_i\};\, z,\, \{z_i\})$$

Model dependent n point amplitude

z Koba-Nielsen variable of the soft particle

One soft-graviton and n-hard particles

Soft-graviton amplitude is obtained by saturating the n+1-amplitude with the polarization:

$$\epsilon^g_{\mu\nu} = \frac{1}{2}(\epsilon_{\mu\nu} + \epsilon_{\nu\mu}) \quad ; \quad \eta^{\mu\nu}\epsilon^g_{\mu\nu} = 0$$

$$\epsilon^g_{\mu\nu} M^{\mu\nu}_{n+1}(q,\, k_1 \ldots k_n) = \left(S^{(0)} + S^{(1)} + S^{(2)}\right) M_n(k_1 \ \ldots k_n)$$

- $S^{(0)}$ is the standard Weinberg leading soft behavior.
- $S^{(1)}$ in bosonic, heterotic and superstring theory is:

$$S^{(1)} = -i\epsilon^g_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho\, k_i^\mu\, J_i^{\nu\rho}}{k_i \cdot q} \qquad\qquad J_i^{\nu\rho} = L_i^{\mu\rho} + \mathcal{S}_i^{\mu\rho} + \bar{\mathcal{S}}_i^{\mu\rho}$$

$$-iL_i^{\mu\rho} = k_i^\mu \frac{\partial}{\partial k_{i\rho}} - k_i^\rho \frac{\partial}{\partial k_{i\mu}} \quad ; \quad -i\mathcal{S}_i^{\mu\rho} = \epsilon_i^\mu \frac{\partial}{\partial \epsilon_{i\mu}} - \epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\mu}}$$

$$\boxed{\epsilon_{\mu\nu} = \epsilon_\mu \ \bar{\epsilon}_\nu}$$

Subsubleading order is different in bosonic, heterotic and superstring amplitudes.

$$S^{(2)} = -\frac{\epsilon^{g}_{\mu\nu}}{2}\left[\frac{q_\rho : J_i^{\mu\rho}\, q_\sigma J_i^{\nu\sigma} :}{k_i \cdot q} - \alpha' \left(q_\sigma k_{i\nu}\eta_{\rho\mu} + q_\rho k_{i\mu}\eta_{\sigma\nu} - \eta_{\rho\mu}\eta_{\sigma\nu}(k_i \cdot q) - q_\rho q_\sigma \frac{k_{i\nu}k_{i\nu}}{k_i \cdot q}\right)\right]\Pi^{\rho\sigma}$$

The colons denote that the action of one operator on the other is excluded.

String correction.

$$\Pi^{\rho\sigma} = \epsilon_i^{\rho}\frac{\partial}{\partial \epsilon_{i\sigma}} + \bar{\epsilon}_i^{\rho}\frac{\partial}{\partial \bar{\epsilon}_{i\sigma}}$$

- String corrections are present only in the bosonic and heterotic amplitudes. They are due to coupling between the dilaton and the Gauss-Bonnet term which is present in the bosonic and heterotic string effective action but not in superstring.

- The complete three point amplitude with massless states (graviton + dilaton + Kalb-Ramond) in string theories is:

$$M_3^{\mu\nu;\mu_i\nu_i;\alpha\beta} = 2\kappa_D \left(\eta^{\mu\mu_i}q^\alpha - \eta^{\mu\alpha}q^{\mu_i} + \eta^{\mu_i\alpha}k_i - \frac{\alpha'}{2}k_i^\mu q^{\mu_i}q^\alpha\right) \times \left(\eta^{\nu\nu_i}q^\beta - \eta^{\nu\beta}q^{\nu_i} + \eta^{\nu_i\beta}k_i^\nu - \frac{\alpha'}{2}k_i^\nu q^{\nu_i}q^\beta\right)$$

- All these soft theorems can be obtained by imposing the gauge invariance of the stripped amplitude $q^\mu M_{\mu\nu}(q,k_i) = q^\nu M_{\mu\nu}(q,k_i) = 0$. (More in Di Vecchia's talk)

Soft-theorem for the gravity dilaton.

- Soft-dilaton amplitude is obtained by saturating the n+1-string amplitude with a soft graviton/dilaton with the dilaton projector:

$$\epsilon_d^{\mu\nu} = \frac{1}{\sqrt{D-2}}\left(\eta^{\mu\nu} - q^\mu \bar{q}^\nu - q^\nu \bar{q}^\mu\right) \quad ; \quad \bar{q} \text{ lightlike vector } q \cdot \bar{q} = 1$$

D is the space-time dimension

The soft behavior with n-hard Tachyons is:

$$m_i^2 = -\frac{4}{\alpha'}$$

$$\epsilon_d^{\mu\nu} M_{\mu\nu}(q;k_i) = \frac{\kappa_D}{\sqrt{D-2}}\Bigg\{ -\sum_{i=1}^{n} \frac{m_i^2}{k_i \cdot q}\left(1 + q^\rho \frac{\partial}{\partial k_i^\rho} + \frac{q^\rho q^\sigma}{2}\frac{\partial^2}{\partial k_i^\rho \partial k_i^\sigma}\right)$$

$$+2 - \sum_{i=1}^{n} k_i^\mu \frac{\partial}{\partial k_i^\mu} + \frac{q^\rho}{2}\sum_{i=1}^{n}\left(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho}\frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}}\right)\Bigg\}\mathcal{M}_n + \mathcal{O}(q^2)$$

- n-tachyon amplitude:

$$\mathcal{M}_n = \frac{8\pi}{\alpha'}\left(\frac{\kappa_D}{2\pi}\right)^{n-2} \int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}} \prod_{i \neq j} |z_i - z_j|^{\frac{\alpha'}{2} k_i k_j}$$

Similarly, the soft-behavior with n-hard massless particles, is:

Scale transformation generator

Special conformal transformation generator

$$\epsilon_d^{\mu\nu} M_{\mu\nu}(q; k_i) = \frac{\kappa_D}{\sqrt{D-2}} \Big\{ 2 - \sum_{i=1}^{n-1} k_i^\mu \frac{\partial}{\partial k_i^\mu}$$

$$+ \frac{q^\rho}{2} \sum_{i=1}^{n-1} \left[\left(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho} \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} \right) - i S_{\mu\rho}^{(i)} \frac{\partial}{\partial k_{i\mu}} \right]$$

$$+ \sum_{i=1}^{n-1} \frac{q^\rho q^\sigma}{2 k_i \cdot q} \left((S_{\rho\mu}^{(i)}) \eta^{\mu\nu} (S_{\nu\sigma}^{(i)}) + D \epsilon_{i\rho} \frac{\partial}{\partial \epsilon_i^\sigma} \right) \Big\} M_n + \mathcal{O}(q^2)\,.$$

- It can be obtained via gauge invariance from a string inspired three point vertex describing graviton/dilaton interactions.
- No-string corrections in the soft-dilaton behavior.
- Universal, it is the same in bosonic, heterotic and superstring theories.
- It depends on the scale and special conformal generators in D-dimensions. Why?

Multiloop extension

- The h-loop amplitude involving one graviton/dilaton and n-tachyons in bosonic string is:

$$\mathcal{A}^{(h)}{}_{N+1} = C_h\,(N_0)^{h+1}\epsilon_{q\,NM}\int dV_N \prod_{i<j=1}^{N} e^{\frac{\alpha'}{2}k_i k_j \mathcal{G}_h(z_i,z_j)} \star \int d^2 z \prod_{l=1}^{N} e^{\frac{\alpha'}{2}k_i q \mathcal{G}_h(z,\,z_l)}$$

$$\times \left[\frac{\alpha'}{2}\sum_{i,j} k_i^M\, k_j^N\, \partial_z \mathcal{G}_h(z,\, z_i)\partial_{\bar{z}}\mathcal{G}_h(z,\, z_j) + \frac{1}{2}\eta^{MN}\omega^\rho(z)(2\pi Im\tau)^{-1}_{\rho\sigma}\bar{\omega}^\sigma(z)\right]$$

- It has been obtained with the formalism of the *N*-string vertex that has the advantage of not requiring the external states to be on the mass shell.

[Di Vecchia, Pezzella, Frau, Hornfeck, Lerda, Sciuto, Nucl. Phys. B322(1989),317; Petersen and Sidenius Nucl Phys. B301 (1988) 247; Mandelstam, (1985)]

- The measure of the moduli, in the Schottky parametrization of the Riemann surface, is:

$$dV_N = \prod_{i=1}^{N}\left(\frac{d^2 z_i}{|V_i'(0)|^2}\right)\frac{1}{dV_{abc}}\prod_{I=1}^{h}\left[\frac{d^2\kappa_I d^2\xi_I d^2\eta_I}{|\kappa_I|^4|\xi_I-\eta_I|^4}|1-\kappa_I|^4\right](\det 2\pi \mathrm{Im}\tau)^{-\frac{D}{2}}$$

$$\times \prod_{\alpha}{}' \left[\prod_{n=1}^{\infty}\left|\frac{1}{1-\kappa_\alpha^n}\right|^{52}\prod_{n=2}^{\infty}|1-\kappa_\alpha^n|^4\right][\sum_{(\mathbf{m},\mathbf{n})\in\mathbb{Z}^{2h}} e^{i\pi(\mathbf{p_R}\tau\mathbf{p_R}-\mathbf{p_L}\bar{\tau}\mathbf{p_L})}]^{26-D}$$

- dV_{abc} is the volume of the Möbius group.
- $(\xi_a, \eta_a, \kappa_a)$ are the two fixed points and the multiplier, respectively, of the projective transformations S_a defining the Schottky group:

$$\frac{S_a(z)-\eta_a}{S_a(z)-\xi_a} = \kappa_a \frac{z-\eta_a}{z-\xi_a}$$

- Left and right momenta along the compact directions depending on the compactification radii R:

$$p_{R;L} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{\alpha'}}{R} \mathbf{n} \pm \frac{R}{\sqrt{\alpha'}} \mathbf{m} \right)$$

- The h-loops string Green's function:

$$\mathcal{G}_h(z_i, z_j) = \log \frac{|E(z_i, z_j)|^2}{|V_i'(0) V_j'(0)|} + \Re \left(\int_{z_j}^{z_i} \omega_I \right) (2\pi \mathrm{Im}\tau)^{-1}_{IJ} \Re \left(\int_{z_i}^{z_j} \omega_J \right)$$

- $\omega^{\mu}(z)$, $E(z, w)$ and τ_{IJ} are the abelian differentials, the Prime-form and the period matrix, respectively.

- $V_i^{-1}(z) = w_i$ are local coordinates vanishing defined around the punctures w_i.
- On-shell the final result of the amplitude will not depend on the local coordinates.
- The string Green-function satisfies:

$$\partial_z \partial_{\bar{z}} \log \mathcal{G}_h(z, w) = \pi \delta^{(2)}(z - w) - \frac{1}{4\pi} \omega_I(z) (\mathrm{Im}\tau)^{-1}_{IJ} \bar{\omega}_J(\bar{z}) - \partial_z \partial_{\bar{z}} \log |V_i'(0)|$$

- The Green's function, $< x(z)x(w) >$=G(z, w), on a Riemann surface with metric $ds^2 = 2g_{z\bar{z}} dz d\bar{z}$, satisfies

$$\partial_z \partial_{\bar{z}} G(z,\, w) = \pi \delta(z - w) - \frac{2\pi g_{z\bar{z}}}{\int d^2 z \sqrt{g}}$$

$$\int d^2 z \sqrt{g}\, G(z,\, w) = 0$$

- By identifying the two expressions we get:

$$\frac{1}{4\pi}\omega_I(z)(\mathrm{Im}\tau)^{-1}_{IJ}\bar{\omega}_J(\bar{z}) + \partial_z\partial_{\bar{z}}|V_i'(0)| = \frac{2\pi\, g_{z\bar{z}}}{\int d^2z\,\sqrt{g}}$$

What represents this expression?

- The Jacobian variety is the torus $\mathbb{C}^h/\Lambda$ with:

$$\Lambda := \left\{\sum_{I=1}^{h}\left[n_I\int_{a_I}\vec{\omega} + n_{I+h}\int_{b_I}\vec{\omega}\right],\ n_I\in\mathbb{Z}\right\} \equiv \mathbb{Z}^h + \tau\mathbb{Z}^h$$

- $(a_I, b_I), I = 1, \ldots h,$ are the homology cycles.
- The Jacobian variety is a Kähler manifold and its Kähler form is:

$$\kappa = \frac{1}{4\pi h}\sum_{IJ}(2\pi\mathrm{Im}\tau)^{-1\,IJ}\, d\xi_I \wedge d\xi_J$$

[Jost, Geometry and Physics; D'Hoker, Green , Pioline, Comm. Math Phys. 366, Isuue 3, (2019) 927]

- The pull-back, under the embedding (Abelian map) $\vec{\xi} = \int_{z_0}^{z} \vec{\omega}$, of the Kähler form, defines the metric:

$$\kappa = \frac{1}{4\pi h} \sum_{IJ} (2\pi \mathrm{Im}\tau)^{-1\,IJ} \omega_I(z)\,\bar{\omega}_J(\bar{z})\,dz \wedge d\bar{z} \equiv g^B_{z\bar{z}}\,dz \wedge d\bar{z}$$

$$-\partial_z \partial_{\bar{z}} \log |V_i'(0)| = 2\pi(h-1) g^B_{z_i \bar{z}_i}$$

- The $V_i'(0)$ under coordinate transformations transform as a metric, we identify them with the Arakelov's metric:

$$\frac{1}{2 g^A_{z_i \bar{z}_i}} = |V_i'(0)|^2$$

- Def. of Arakelov metric

$$g^A_{z\bar{z}}\, R^A = -\partial_z \partial_{\bar{z}} \ln g^A_{z\bar{z}} = 4\pi(1-h)\kappa_{z\bar{z}}$$

- The Green's function becomes invariant around the homology cycles and it coincides with the Arakelov Green's function
- By using the general properties of the Arakelov Green's function,

$$\partial_z \partial_{\bar{z}} \mathcal{G}_h^A(z,w) = -\pi \delta^2(z,w) + \frac{g^A_{z\bar{z}}\, R^A}{4(h-1)}$$

$$\int d^2 z g^A_{z\bar{z}}\, R^A\, \mathcal{G}_h^A(z,w) = 0 \quad \int d^2 z \partial_z \partial_{\bar{z}} \mathcal{G}_h^a(z,w) = 0$$

all the integrals are calculable and one can see that the graviton soft theorem is the same as at three level.

- The dilaton soft-theorem becomes:

$$M^{(h)}_{N;\phi}(k_i;q) = \frac{\kappa_D}{\sqrt{D-2}}\left[-\sum_{i=1}^{N}\frac{m^2}{k_i q}e^{q\partial_{k_i}} + 2 - \sum_{i=1}^{N}\hat{D}_i + h(D-2) + q_\mu\sum_{i=1}^{N}\hat{K}_i^\mu\right]M_N^{(h)} + O(q^2)$$

$$\hat{D}_i = k_i\cdot\partial_{k_i} \qquad \hat{K}_i^\mu = \frac{1}{2}k_i^\mu\left(\partial_{k_i}\cdot\partial_{k_i}\right) - \left(k_i\cdot\partial_{k_i}\right)\partial^\mu_{k_i}$$

- Because of the dependence on h, we cannot immediately write the all-loop soft behaviour.

- However from the scaling properties:

$$M_N^{(h)} = \frac{\kappa_D^{2(h-1)+N}}{\sqrt{\alpha'}^{-(2-D)h+2}} F\left(\sqrt{\alpha'}k_i, R/\sqrt{\alpha'}\right) \quad ; \quad \kappa_D = \frac{(2\pi)^{\frac{D-3}{2}}}{\sqrt{2^{-9}}}\, g_s\, \sqrt{\alpha'}^{\frac{D-2}{2}} \left(\frac{\sqrt{\alpha'}}{R}\right)^{\frac{26-D}{2}}$$

we deduce,

$$\left[2 - \sum_{i=1}^{N} k_i \cdot \frac{\partial}{\partial k_i} + (D-2)h\right] M_N^{(h)} = \left[\frac{D-2}{2} g_s \frac{\partial}{\partial g_s} - \sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}} - R\frac{\partial}{\partial R}\right] M_N^{(h)}$$

- Getting the loop independent expression:

$$\mathcal{M}_{N;\phi}(k_i;q) = \frac{\kappa_D}{\sqrt{D-2}}\left[-\sum_{i=1}^{N}\frac{m^2}{k_i q}\mathrm{e}^{q\partial_{k_i}} + \frac{D-2}{2} g_s \frac{\partial}{\partial g_s} - \sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}} - R\frac{\partial}{\partial R} + q_\mu \sum_{i=1}^{N} \hat{K}_i^\mu\right] \mathcal{M}_N(k_i)$$
$$+O(q^2)$$

Infrared divergences

- In bosonic string, with $(26 - D)$ compact dimensions, there are three kinds of infrared (IR) divergences:

1. When massless states are involved, IR-divergences may appear for low values of non-compact dimensions. In D=4 field theories they are known as soft and collinear divergences.

✓ It has been proved to all orders in perturbation theory that such divergences do not appear in the full amplitude involving gravitons and other massless states.

[Weinberg, Phys. Rev. 140, B516 (1965); Akhoury, Saotome, Sterman, Phys. Rev D 84, (2011), 104040)]

- ✓ In string theory at 1-loop they appear when Im $\tau \rightarrow \infty$. [Green, Schwarz, Brink, NPB 198, (1982) 474]
- ✓ They depend on the number of external legs and therefore prevent the soft theorem to be valid at the loop level in the same form as at tree level . [Bern, Davies, Nohle PRD 90, 085015]
- ✓ The peculiarities of D=4 are discussed by Sahoo and Sen JHEP 1902 (2019)086 and Ladda and Sen JHEP 10 (2018) 56.

2. String amplitudes involving massive external states are also plagued by divergences that requires mass-renormalization [Weinberg (1985); Pius, Rudra, Sen, JHEP 1401.7014] .

- ✓ These divergences can be regularized by not allowing the Koba-Nielsen variables to get close each other in certain configurations. Since they depend only on the number of external massive legs, we do not expect that they modify the soft operators. [Weinberg, Talk August (1985) preprint UTG-22-85; Cohen, Kluber-Stern, Navelet, Peschanski, NP B347,(1990), 802.]

3. The third kind of divergences, peculiar of the bosonic string theory, are Dilaton and Tachyon tadpoles.

- ✓ Let us discuss these divergences in the case of *N*-tachyon amplitude at 1-loop.

$$T_N^{(1)} = \quad C_1 N_0^N \int_{\mathcal{F}} d^2\tau\ \mu(\tau,\bar\tau) \prod_{i=1}^{N-1} \left[\int d^2\nu_i\right]$$
$$\times \prod_{i<j} \left| \frac{\sin \pi\nu_{ij}}{\pi} \prod_{n=1}^{\infty} \frac{(1-\kappa^n \mathrm{e}^{2\pi i\nu_{ij}})(1-\kappa^n \mathrm{e}^{-2\pi i\nu_{ij}})}{(1-\kappa^n)^2} \mathrm{e}^{-\pi \frac{(\mathrm{Im}\,\nu_{ij})^2}{\mathrm{Im}\tau}} \right|^{\alpha' k_i k_j},$$

- Where $\eta = 0, \xi = \infty, \nu_N = 0, \kappa = e^{2\pi i\tau}$. $\mathcal{F}$ denotes the fundamental integration region of τ, and

$$\mu(\tau,\bar\tau) = (2\pi)^2 \mathrm{e}^{4\pi \mathrm{Im} t} \prod_{n=1}^{\infty} \left[\frac{1}{|1-\mathrm{e}^{2\pi i\tau n}|^{48}}\right] \frac{(F(\tau,\bar\tau))^{26-D}}{(\mathrm{Im}\tau)^{D/2}}.$$

- We consider the region of the moduli space where all the ν_i are very close:

$$\mathrm{e}^{i\phi}\varepsilon\eta_i = \nu_i \ ,\ i = 1\ldots N-2 \ ;\ \varepsilon\mathrm{e}^{i\phi} = \nu_{N-1} \ ;\ \eta_{N-1} = 1\,.$$

- By using these variables and keeping the terms divergent for $\varepsilon \to 0$, we find: $\Rightarrow$

$$T_n^{(1)} = \quad C_1 N_0^N \int_{\mathcal{F}} d^2\tau \mu(\tau,\bar\tau) \prod_{i=1}^{N-2} \int d^2\eta_i \int_0^{2\pi} d\phi \int_0^1 \frac{d\varepsilon}{\varepsilon^{3-\frac{\alpha'}{2}p^2}} \prod_{i<j} |\eta_{ij}|^{\alpha' k_i k_j}$$
$$\times \left[1 - \alpha' \sum_{i<j} k_i k_j \frac{\pi\varepsilon^2}{\mathrm{Im}\tau} \left(\sin\phi\,\mathrm{Re}(\eta_{ij}) + \cos\phi\,\mathrm{Im}(\eta_{ij})\right)^2 + O(\varepsilon^4) \right] ,$$

- The divergence at ε=0 is regularized with the substitution -3→-3+$\frac{\alpha'}{2}p^2$, with p a finite momentum. The integrals in ε and ϕ can be performed

$$T_N^{(1)} = \frac{C_1}{C_0 N_0} \int_{\mathcal{F}} d^2\tau \mu(\tau,\bar\tau) \left[-\frac{2\pi T_{(N+1)\mathrm{tach}}}{2-\frac{\alpha'}{2}p^2} + \frac{(2\pi)^2}{\alpha' p^2 \mathrm{Im}\tau} T_{N\mathrm{tach}+1\mathrm{dil}} \right] + \cdots .$$

- The introduction of the finite momentum p, has regularized the tachyon and dilaton contribution. Also the integral in τ has to be regularized for Im τ→∞.
- The IR-divergences are independent on N and therefore don't affect the soft-theorem.

Conclusions

- ➢ We have studied the Ward-identities of spontaneously broken conformal theories and derived single and double soft theorems for the dilaton; i.e. the Nabu-Goldstone boson of the broken symmetry.
- ✓ The two soft-theorems are valid up to the subleading order.
- ✓ The double soft theorem is equivalent to two consecutive single soft limits performed one after the other.
- ✓ Guess for a multi-soft dilaton theorem.
- ✓ The Nambu-Goldstone dilaton soft theorems, being a consequence of the symmetry, are universal.

- In bosonic string theory with a cut-off on the Infrared divergences, the graviton soft theorem is valid, for $D > 4$, at any order of the perturbative expansion.
- ✓ The dilaton soft theorem is modified by loop corrections already at leading order.
- ✓ The corrections when rewritten in terms of the coupling and string slope turn out to be the same as at three level.
- ❑ The Gravity dilaton and the Nambu-Goldstone boson of the broken conformal symmetry satisfy similar soft theorems, why?