Soft Theorem and its Classical Limit

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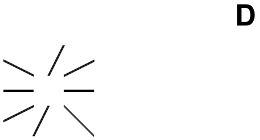
A partial preview of the results

Alok Laddha, A.S.   arXiv:1806.01872

Biswajit Sahoo, A.S.   arXiv:1808.03288

+ earlier papers

Consider an explosion in space



A bound system at rest breaks apart into fragments carrying four momenta $\mathbf{p}_1, \mathbf{p}_2, \cdots$ with

$$\mathbf{p}_a^2 + \mathbf{m}_a^2 = 0, \quad \mathbf{a} = 1, 2, \cdots, \quad \mathbf{p}_a^2 \equiv -(\mathbf{p}_a^0)^2 + \vec{\mathbf{p}}_a^2$$

This process emits gravitational waves

Detector D placed far away detects $\quad \mathbf{h}_{\mu\nu} \equiv (\mathbf{g}_{\mu\nu} - \eta_{\mu\nu})/2$

Physical components: $\mathbf{h}_{ij}^{TT}$ (transverse, traceless)

We shall be interested in the late time tail of the radiation – the radiation at a large time u after the passage of the peak

It has the form

$$h_{ij}^{TT} = A_{ij}^{TT} + \frac{1}{u} B_{ij}^{TT} + \cdots$$

$$A^{\mu\nu} = -\frac{2\,G}{R} \sum_a \frac{p_a^\mu p_a^\nu}{p_a.n}, \quad u.v = -u^0 v^0 + \vec{u}.\vec{v}$$

$$B^{\mu\nu} = \frac{2\,G^2}{R} \Bigg[2 \sum_{a,b} \frac{n.p_b}{n.p_a}\, p_a^\mu p_a^\nu$$

$$+ \sum_{\substack{a,b \\ b\neq a}} \frac{n_\rho p_a^{(\nu}}{p_a.n} \left(p_a^{\mu)} p_b^\rho - p_b^{\mu)} p_a^\rho\right) \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \, \{2(p_b.p_a)^2 - 3 m_a^2 m_b^2\} \Bigg]$$

n=(1, $\hat{n}$), $\hat{n}$: unit vector towards the detector

R: distance to detector, G: Newton's constant

$$h_{ij} = A^{TT}_{ij} + \frac{1}{u} B^{TT}_{ij}, \quad \text{for large } u$$

$$A^{\mu\nu} = -\frac{2\,G}{R} \sum_a \frac{p_a^\mu p_a^\nu}{p_a.n}$$

$$B^{\mu\nu} = \frac{2\,G^2}{R} \Bigg[2 \sum_{a,b} \frac{n.p_b}{n.p_a} p_a^\mu p_a^\nu$$

$$+ \sum_{\substack{a,b \\ b\neq a}} \frac{n_\rho p_a^{(\nu}}{p_a.n} (p_a^{\mu)} p_b^\rho - p_b^{\mu)} p_a^\rho) \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \left\{ 2(p_b.p_a)^2 - 3 m_a^2 m_b^2 \right\} \Bigg]$$

Note: The result is given completely in terms of the momenta of final state particles without knowing what caused the explosion or how the particles moved during the explosion

– consequence of soft graviton theorem (a result for quantum S-matrix)

$$h_{ij} = A_{ij}^{TT} + \frac{1}{u} B_{ij}^{TT}, \quad \text{for large } u$$

A_{ij}: memory term

– a permanent change in the state of the detector after the passage of gravitational waves

Zeldovich, Polnarev; Braginsky, Grishchuk; Braginsky, Thorne; $\cdots$

– connected to leading soft theorem

Strominger; $\cdots$

B_{ij}: tail term

– connected to logarithmic terms in the subleading soft theorem

Laddha, A.S.

A similar result exists for a general scattering process

– gives the gravitational wave-form $\tilde{h}_{ij}$ at low frequency in terms of momenta of incoming and outgoing particles.

$$\tilde{h}^{TT}_{ij}(x,\omega) = C^{TT}_{ij}(\omega)$$

$$\begin{aligned} C^{\mu\nu} \;=\; & \frac{2\,G}{i\,R}\sum_a \eta_a \frac{p_a^\mu p_a^\nu}{p_a.k}\left\{1 + 2\,i\,G\,\ln(\omega^{-1}R^{-1}) \sum_{b,\eta_b=-1} k.p_b\right\} \\ & + 2\,\frac{G^2}{R}\,\ln\omega^{-1}\sum_a \sum_{\substack{b\neq i\\ \eta_a\eta_b=1}} \frac{k_\rho p_a^{(\nu}}{p_a.k}\,(p_a^{\mu)}p_b^\rho - p_b^{\mu)}p_a^\rho) \\ & \times \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\} + \text{finite}\,. \end{aligned}$$

$k = -\omega\left(1, \vec{x}/|\vec{x}|\right)$, η_a: +1 if a is incoming, -1 if a is outgoing.

– Matches explicit results in special cases

Peters; Ciafaloni, Colferai, Veneziano; Addazi, Bianchi, Veneziano

In the rest of the talk we shall try to explain the origin of this result from soft graviton theorem

– combination of logic + guesswork + test

Units: $\hbar = \mathbf{c} = \mathbf{8}\,\pi\,\mathbf{G} = \mathbf{1}$

Some earlier references:

Weinberg; . . .

White; Cachazo, Strominger; Bern, Davies, Di Vecchia, Nohle; Elvang, Jones, Naculich; . . .

Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha

Bianchi, Guerrieri; Di Vecchia, Marotta, Mojaza; . . .

Strominger; He, Lysov, Mitra, Strominger; Strominger, Zhiboedov; Campiglia, Laddha; . . .

Bern, Davies, Nohle; Cachazo, Yuan; He, Kapec, Raclariu, Strominger

What is soft graviton theorem?

Take a general coordinate invariant quantum theory of gravity coupled to matter fields

Consider an S-matrix element involving

– arbitrary number N of external particles of finite momentum $p_1, \cdots p_N$

– M external gravitons carrying small momentum $k_1, \cdots k_M$.

Soft graviton theorem: Expansion of this amplitude in power series in $k_1, \cdots k_M$ in terms of the amplitude without the soft gravitons.

Under some assumptions one can give a completely general derivation of soft graviton theorem

A.S.; Laddha, A.S.; Chakrabarti, Kashyap, Sahoo, A.S., Verma

– generic theory (including string theory)

– generic number of dimensions

– arbitrary mass and spin of elementary / composite finite momentum external states

e.g. gravitons, photons, electrons, massive string states, nuclei, molecules, planets, stars, black holes

Main ingredient: Graviton coupling with zero or one derivative is fixed completely by general coordinate invariance.

Assumptions

1. The scattering is described by a general coordinate invariant one particle irreducible (1PI) effective action

– tree amplitudes computed from this give the full quantum results

2. The vertices do not contribute powers of soft momentum in the denominator

– breaks down in D=4

Work in D $>$ 4 for now

– to be rectified at the end

Result:

Let Γ be the scattering amplitude of any set of finite energy (hard) particles.

Scattering amplitude of the same set of states with M additional soft gravitons of polarization $\{\varepsilon_r\}$ and momentum $\{k_r\}$ ($1 \le r \le M$) takes the form

$$S(\{\varepsilon_r\}, \{k_r\})\,\Gamma$$

up to subleading order in expansion in powers of soft momentum.

$S(\{\varepsilon_r\}, \{k_r\})$: Known, universal operator, involving derivatives with respect to momenta of hard particles and matrices acting on the polarization of the hard particles.

Classical limit

Laddha, A.S.

We take the limit in which

1. Energy of each hard particle becomes large (compared to M_{pl})

–represented by wave-packets with sharply peaked distribution of position, momentum, spin etc.

2. The total energy carried by the soft particles should be small compared to the energies of the hard particles

– a necessary criterion for how small the momenta should be so as to be declared soft.

In this limit the multiple soft theorem takes the form

$$\left\{\prod_{r=1}^{M} S_{gr}(\varepsilon_r, k_r)\right\} \Gamma, \qquad S_{gr} = S^{(0)} + S^{(1)} + \cdots$$

$$S^{(0)}(\varepsilon, k) \equiv \sum_{a=1}^{N} (p_a \cdot k)^{-1} \, \varepsilon_{\mu\nu} \, p_a^\mu \, p_a^\nu$$

$$S^{(1)}(\varepsilon, k) = i \sum_{a=1}^{N} (p_a \cdot k)^{-1} \, \varepsilon_{\mu\nu} \, p_a^\mu \, k_\rho \, J_a^{\rho\nu}$$

$J_a^{\mu\nu}$: classical angular momentum of the a-th hard particle

All (angular) momenta are counted positive if ingoing.

If in the far past / future the object has trajectory

$$x_a^\mu = c_a^\mu + m_a^{-1} \, p_a^\mu \tau$$

then

$$J_a^{\mu\nu} = (x_a^\mu p_a^\nu - x_a^\nu p_a^\mu) + \text{spin} = (c_a^\mu p_a^\nu - c_a^\nu p_a^\mu) + \text{spin}$$

S_{gr} is large in the classical limit since p_a and $J_a^{\mu\nu}$ are large.

Amplitude: $\Gamma_{soft} \equiv \left\{ \prod_{r=1}^{M} S_{gr}(\epsilon_r, k_r) \right\} \Gamma$

Probability of producing M soft gravitons of

- **polarisation** ε**,**
- **energy between** ω **and** $\omega(1+\delta)$
- **within a small solid angle** Ω **around a unit vector** $\hat{n}$

$$\frac{1}{M!}\,|\Gamma_{soft}|^2 \times \left\{ \frac{1}{(2\pi)^{D-1}}\,\frac{1}{2\omega}\,\omega^{D-2}\,(\omega\,\delta)\,\Omega \right\}^M = |\Gamma|^2\, A^M / M!\,,$$

$$A \;\equiv\; |S_{gr}(\varepsilon, k)|^2\,\frac{1}{(2\pi)^{D-1}}\,\frac{1}{2\omega}\,\omega^{D-2}\,(\omega\,\delta)\,\Omega\,, \quad k = -\omega(1, \hat{n})$$

Note: A is large in the classical limit

$$|\Gamma|^2 \, A^M / M!$$

is maximised at

$$\frac{\partial}{\partial M} \ln \left\{ |\Gamma|^2 \, A^M / M! \right\} = 0$$

Assuming that M is large,

$$\Rightarrow \frac{\partial}{\partial M} (M \ln A - M \ln M + M) = 0$$

$$\Rightarrow M = A$$

In the classical limit M is large since A is large

Probability distribution of M is sharply peaked

Note: the value of M does not change if we allow soft radiation in other bins.

$$\text{no. of gravitons} = A = \frac{1}{2^D \pi^{D-1}} |S_{gr}(\varepsilon, k)|^2 \omega^{D-2}\, \Omega\, \delta$$

Total energy radiated in this bin

$$A\, \omega = \frac{1}{2^D \pi^{D-1}} |S_{gr}(\varepsilon, k)|^2 \omega^{D-1}\, \Omega\, \delta$$

This can be related to the radiative part of the metric field

$\Rightarrow$ gives a prediction for the low frequency radiative part of the metric field during classical scattering (up to overall phase and gauge transformation)

$$(h^{\mu\nu}(\vec{x}, \omega))^{TT} = \frac{1}{2\omega^2} \left(\frac{\omega}{2\pi i R}\right)^{(D-2)/2} \sum_{a=1}^{N} (p_a \cdot n)^{-1} \left[p_a^\mu p_a^\nu - i\, \omega\, n_\rho\, J_a^{\rho(\nu} p_a^{\mu)} \right]^{TT}$$

$n = \left(1, \vec{x}/|\vec{x}|\right), \quad R = |\vec{x}|$

– tested in many explicit examples involving classical scattering

D=4

The S-matrix suffers from IR divergence, making soft factor ill-defined.

However we can still use the radiative part of the gravitational field during classical scattering to define soft factor.

Naive guess: Soft factor defined this way is still given by the same formulæ:

$$\mathbf{S}^{(0)} \equiv \sum_{a=1}^{N} (p_a \cdot k)^{-1} \varepsilon_{\mu\nu} \, p_a^\mu \, p_a^\nu$$

$$\mathbf{S}^{(1)} = i \sum_{a=1}^{N} (p_a \cdot k)^{-1} \varepsilon_{\mu\nu} \, p_a^\mu \, k_\rho \, J_a^{\rho\nu}$$

Problem: Due to long range force on the initial / final trajectories due to other particles, the trajectory of the a-th particle takes the form:

$$\mathbf{x_a^\mu} = \mathbf{c_a^\mu} + \mathbf{m_a^{-1}}\,\mathbf{p_a^\mu}\,\tau + \mathbf{b_a^\mu}\ln|\tau|$$

for some constants $\mathbf{b_a^\mu}$.

$$\mathbf{J_a^{\mu\nu}} = (\mathbf{x_a^\mu p_a^\nu} - \mathbf{x_a^\nu p_a^\mu}) = (\mathbf{c_a^\mu p_a^\nu} - \mathbf{c_a^\nu p_a^\mu}) + (\mathbf{b_a^\mu p_a^\nu} - \mathbf{b_a^\nu p_a^\mu})\ln|\tau|$$

Due to the $\ln|\tau|$ term, the soft factors do not have well defined $|\tau| \to \infty$ limit

Next guess: The soft expansion has a $\ln\omega^{-1}$ term at the subleading order, given by $S^{(1)}$ with $\ln|\tau|$ replaced by $\ln\omega^{-1}$.

$\omega \equiv k_0$

$$
\begin{aligned}
S^{(1)} &= i\sum_{a=1}^{N} (p_a\cdot k)^{-1}\,\varepsilon_{\mu\nu}\,p_a^\mu\,k_\rho\,J_a^{\rho\nu} \\
&= i\sum_{i=1}^{N} (p_a\cdot k)^{-1}\,\varepsilon_{\mu\nu}\,p_a^\mu\,k_\rho\,(b_a^\rho p_a^\nu - b_a^\nu p_a^\rho)\ln\omega^{-1} \\
&\quad + \text{finite}
\end{aligned}
$$

This has been tested by studying explicit examples of gravitational radiation during scattering in D=4

– perfect agreement with all the cases studied.

Assuming the validity of the $\ln|\tau| \Rightarrow \ln\omega^{-1}$ rule, we can write down the metric deformation up to gauge transformation:

$$\tilde{h}^{\mu\nu} = \frac{2G}{iR}\sum_a \eta_a \frac{p_a^\mu p_a^\nu}{p_a.k}\left\{1 + 2iG\ln(\omega^{-1}R^{-1}) \sum_{b,\eta_b=-1} k.p_b\right\}$$

$$+2\frac{G^2}{R}\ln\omega^{-1}\sum_a \sum_{\substack{b\neq i \\ \eta_a\eta_b=1}} \frac{k_\rho p_a^{(\nu}}{p_a.k}(p_a^{\mu)}p_b^\rho - p_b^{\mu)}p_a^\rho)$$

$$\times \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\left\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\right\} + \text{finite}\,.$$

η_a**: +1 if a is incoming, -1 if a is outgoing.**

$\mathbf{k} = -\omega\,(1, \vec{x}/|\vec{x}|)$

Sahoo, A.S.

In the special case when there is only one object in the initial state, Fourier transform of this gives us back the result described at the beginning of the talk.

For a core collapse supernova explosion in our galaxy, the magnitudes of these terms are near the edge of LIGO detection limits.

Summary

1. Up to subleading order we have universal soft graviton theorem in all dimensions > 4, for all mass and spin of external states.

2. Classical limit of soft theorem determines the low frequency radiative part of the gravitational field during classical scattering

3. The 'classical soft theorem' is valid also in D=4, but at the subleading order there is a term $\propto$ the log of the soft energy, determined from soft theorem

– produces a tail term to the memory effect