Pure Spinor Action in Hamiltonian Form

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Previous results

- Berkovits and Howe (2001)

$$
\begin{aligned}
S_{PS} = \int d^2z \, \Big[& \frac{1}{2}\bar{\partial} Z^N \partial Z^M (G_{MN} + B_{MN}) + \bar{\partial} Z^M E_M^{\alpha} d_{\alpha} + \partial Z^M E_M^{\widehat{\alpha}} \widehat{d}_{\widehat{\alpha}} \\
& + \bar{\partial} Z^M \Omega_{M\alpha}{}^{\beta} \lambda^{\alpha} \omega_{\beta} + \partial Z^M \widehat{\Omega}_{M\widehat{\alpha}}{}^{\widehat{\beta}} \widehat{\lambda}^{\alpha} \widehat{\omega}_{\widehat{\beta}} + \omega_{\alpha} \bar{\partial} \lambda^{\alpha} + \widehat{\omega}_{\widehat{\alpha}} \partial \widehat{\lambda}^{\widehat{\alpha}} \\
& + C_{\alpha}^{\beta\widehat{\beta}} \lambda^{\alpha} \omega_{\beta} \widehat{d}_{\widehat{\beta}} + \widehat{C}_{\widehat{\alpha}}^{\widehat{\beta}\beta} \widehat{\lambda}^{\widehat{\alpha}} \widehat{\omega}_{\widehat{\beta}} d_{\beta} + S_{\alpha\widehat{\alpha}}^{\beta\widehat{\beta}} \lambda^{\alpha} \omega_{\beta} \widehat{\lambda}^{\widehat{\alpha}} \widehat{\omega}_{\widehat{\beta}} + P^{\alpha\widehat{\alpha}} d_{\alpha} \widehat{d}_{\widehat{\alpha}} \Big]
\end{aligned}
$$

BRST currents $j_B = \lambda^{\alpha} d_{\alpha}$ and $\widetilde{j}_B = \widehat{\lambda}^{\widehat{\alpha}} \widehat{d}_{\widehat{\alpha}}$

$$
\begin{aligned}
j_B &= \lambda^{\alpha} E_{\alpha}{}^{M} \Big(P_M - (\lambda \Omega_M \omega) - (\widehat{\lambda} \widehat{\Omega}_M \widehat{\omega}) - \partial_{\sigma} Z^N B_{NM} \Big) \\
\widetilde{j}_B &= \widehat{\lambda}^{\widehat{\alpha}} E_{\widehat{\alpha}}{}^{M} \Big(P_M - (\lambda \Omega_M \omega) - (\widehat{\lambda} \widehat{\Omega}_M \widehat{\omega}) - \partial_{\sigma} Z^N B_{NM} \Big)
\end{aligned}
$$

- Nilpotency and holomorphicity implies Type II SUGRA constraints

What are we trying to solve?

- Straightforward computation, no clear geometrical meaning
- Goal: Look for more symmetrical formulation between matter and ghosts
- In some backgrounds (e.g. AdS) there are transformations between matter and ghosts
- How to proceed? Second order formalism? Too restrictive
- Approach: Hamiltonian formalism (or first order formulation)

Pure Spinor action

- Assume the RR background $P^{\alpha\widehat{\alpha}}$ is invertible and solve for d_α and $\widehat{d}_{\widehat{\alpha}}$

$$d_\alpha = \left(\partial Z^M E_M{}^{\widehat{\alpha}} + (\lambda C \omega)^{\widehat{\alpha}}\right) P^{-1}_{\widehat{\alpha}\alpha}$$
$$\hat{d}_{\widehat{\alpha}} = -P^{-1}_{\widehat{\alpha}\alpha}\left(\bar{\partial} Z^M E_M{}^{\alpha} + (\widehat{\lambda}\widehat{C}\omega)^{\alpha}\right)$$

- Action and BRST currents become dependent on $P^{-1}_{\widehat{\alpha}\alpha}$
- When S_{PS} is expressed in Hamiltonian form

$$P_M = \frac{\delta S_{PS}}{\delta \partial_t Z^M}$$

  all these terms depending on $P^{-1}_{\widehat{\alpha}\alpha}$ cancel!
- Action is describing the general case again

Pure Spinor action

$$
\begin{aligned}
S_{PS} = \int & \partial_t Z^M P_M + \omega_\alpha \bar{\partial} \lambda^\alpha + \widehat{\omega}_{\widehat{\alpha}} \partial \widehat{\lambda}^{\widehat{\alpha}} \\
& - \frac{1}{2} \partial_\sigma Z^N \partial_\sigma Z^M \Big(G_{MN} - B_{ML} A^{LP} B_{PN} - E_{(M|}{}^{\widehat{\alpha}} E_{\widehat{\alpha}}{}^P B_{P|N)} \\
& \qquad\qquad + E_{(M|}{}^{\alpha} E_{\alpha}{}^P B_{P|N)} \Big) \\
& - \frac{1}{2} (P_M - \lambda \Omega_M \omega - \widehat{\lambda} \widehat{\Omega}_M \widehat{\omega}) A^{MN} (P_N - \lambda \Omega_N \omega - \widehat{\lambda} \widehat{\Omega}_N \widehat{\omega}) \\
& + (\lambda C \omega^{\widehat{\alpha}} E_{\widehat{\alpha}}{}^M + \widehat{\lambda} \widehat{C} \widehat{\omega}^\alpha E_\alpha{}^M)(P_M - \lambda \Omega_M \omega - \widehat{\lambda} \widehat{\Omega}_M \widehat{\omega}) \\
& + \partial_\sigma Z^N \Big(B_{NP} A^{PM} + E_N{}^{\widehat{\alpha}} E_{\widehat{\alpha}}{}^M - E_N{}^\alpha E_\alpha{}^M \Big) \\
& \qquad\qquad \times (P_M - \lambda \Omega_M \omega - \widehat{\lambda} \widehat{\Omega}_M \widehat{\omega}) \\
& + \partial_\sigma Z^N (\lambda C \omega)^{\widehat{\alpha}} E_{\widehat{\alpha}}{}^M B_{MN} + \partial_\sigma Z^N (\widehat{\lambda} \widehat{C} \widehat{\omega})^\alpha E_\alpha{}^M B_{MN} \\
& + \partial_\sigma Z^M (\widehat{\lambda} \widehat{\Omega}_M \widehat{\omega}) - \partial_\sigma Z^M (\lambda \Omega_M \omega) + (\lambda \widehat{\lambda} S)^{\alpha \widehat{\alpha}} \omega_\alpha \widehat{\omega}_{\widehat{\alpha}}
\end{aligned}
$$

Pure Spinor action

Consider $(\mathbb{Z}^i) = (Z^M, \lambda^\alpha, \widehat{\lambda}^{\widehat{\alpha}})$ and $(\mathbb{P}_i) = (P_M, \omega_\alpha, \widehat{\omega}_{\widehat{\alpha}})$

$$S_{PS} = \int \left\{ \partial_t \mathbb{Z}^i \mathbb{P}_i - \frac{1}{2}(\partial_\sigma \mathbb{Z}^i \ \mathbb{P}_i) \begin{pmatrix} \mathbb{A}_{ij} & \mathbb{A}_i{}^j \\ \mathbb{A}^i{}_j & \mathbb{A}^{ij} \end{pmatrix} \begin{pmatrix} \partial_\sigma \mathbb{Z}^j (-)^j \\ \mathbb{P}_j \end{pmatrix} \right\}$$

where the matrix $\mathbb{A}$ is

$$\mathbb{A}_{PS} = \left(\begin{array}{ccc|ccc} \mathbb{A}_{MN} & 0 & 0 & \mathbb{A}_M{}^N & \mathbb{A}_M{}^\beta & \mathbb{A}_M{}^{\widehat{\beta}} \\ 0 & 0 & 0 & 0 & \delta_\alpha{}^\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta_{\widehat{\alpha}}{}^{\widehat{\beta}} \\ \hline \mathbb{A}^M{}_N & 0 & 0 & \mathbb{A}^{MN} & \mathbb{A}^{M\beta} & \mathbb{A}^{M\widehat{\beta}} \\ \mathbb{A}^\alpha{}_N & \delta^\alpha{}_\beta & 0 & \mathbb{A}^{\alpha N} & \mathbb{A}^{\alpha\beta} & \mathbb{A}^{\alpha\widehat{\beta}} \\ \mathbb{A}^{\widehat{\alpha}}{}_N & 0 & -\delta^{\widehat{\alpha}}{}_{\widehat{\beta}} & \mathbb{A}^{\widehat{\alpha}N} & \mathbb{A}^{\widehat{\alpha}\beta} & \mathbb{A}^{\widehat{\alpha}\widehat{\beta}} \end{array} \right)$$

Simple example: bosonic string

- Sigma model with target space M

$$S = \frac{1}{2} \int \partial x^m \bar{\partial} x^n (G_{mn} + B_{mn})$$

- Transformation $\delta x^m = V^m(x)$ is symmetry if

$$\mathcal{L}_V G = 0 \quad \text{and} \quad \mathcal{L}_V B = dF$$

- Noether current $(j_z, j_{\bar{z}})$

$$\begin{aligned} j_z &= \frac{1}{2} \partial x^n (G_{nm} + B_{nm}) V^m + \frac{1}{2} \partial x^n F_n \\ j_{\bar{z}} &= \frac{1}{2} \bar{\partial} x^n (G_{mn} + B_{mn}) V^m - \frac{1}{2} \bar{\partial} x^n F_n \end{aligned}$$

for some one-form F_m

Simple example: bosonic string

- In Hamiltonian form

$$S = \int \partial_t x^m p_m - \frac{1}{2}(\partial_\sigma x^m \ p_m) \begin{pmatrix} A_{mn} & A_m{}^n \\ A^m{}_n & A^{mn} \end{pmatrix} \begin{pmatrix} \partial_\sigma x^n \\ p_n \end{pmatrix}$$

where

$$A = \begin{pmatrix} (G - BG^{-1}B)_{mn} & -(BG^{-1})_m{}^n \\ (G^{-1}B)^m{}_n & (G^{-1})^{mn} \end{pmatrix}$$

- The components of the current

$$j_z = \frac{1}{2}(V \ F)\left[\mathbb{I} + U_A\right] \begin{pmatrix} p \\ \partial_\sigma x \end{pmatrix}$$

$$j_{\bar{z}} = \frac{1}{2}(V \ F)\left[\mathbb{I} - U_A\right] \begin{pmatrix} p \\ \partial_\sigma x \end{pmatrix}$$

where

$$U_A := A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -BG^{-1} & G - BG^{-1}B \\ G^{-1} & G^{-1}B \end{pmatrix}$$

Generalized metric

- (Fiber-wise) symmetric bilinear form

$$A : (TM \oplus T^*M) \times (TM \oplus T^*M) \longrightarrow C^\infty(M)$$

- Satisfies $O(d,d)$ condition

$$A^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which implies that

$$(U_A)^2 = 1 \qquad \text{for} \qquad U_A = A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

i.e. $P_\pm = \frac{1}{2}(\mathbb{I} \pm U_A)$ are projectors onto (± 1)-eigenspaces of U_A

Pure Spinor action

Target space: supermanifold M parametrized $(\mathbb{Z}^i) = (Z^M, \lambda^\alpha, \widehat{\lambda}^{\widehat{\alpha}})$

$$S_{PS} = \int \left\{ \partial_t \mathbb{Z}^i \mathbb{P}_i - \frac{1}{2} (\partial_\sigma \mathbb{Z}^i \ \mathbb{P}_i) \begin{pmatrix} \mathbb{A}_{ij} & \mathbb{A}_i{}^j \\ \mathbb{A}^i{}_j & \mathbb{A}^{ij} \end{pmatrix} \begin{pmatrix} \partial_\sigma \mathbb{Z}^j (-)^j \\ \mathbb{P}_j \end{pmatrix} \right\}$$

where $\mathbb{A}_{PS}$ satisfies

- $\mathbb{A}_{PS}$ is a bilinear (super)symmetric
- Satisfies the $OSp(d, d|2s)$ condition

$$\begin{pmatrix} \mathbb{A}_{ik} & \mathbb{A}_i{}^k \\ \mathbb{A}^i{}_k & \mathbb{A}^{ik} \end{pmatrix} \begin{pmatrix} 0 & \delta^k{}_l \\ \delta_k{}^l & 0 \end{pmatrix} \begin{pmatrix} \mathbb{A}_{lj} & \mathbb{A}_l{}^j \\ \mathbb{A}^l{}_j & \mathbb{A}^{lj} \end{pmatrix} = \begin{pmatrix} 0 & \delta_i{}^j \\ \delta^i{}_j & 0 \end{pmatrix}$$

- The $OSp(d, d|2s)$ condition guarantees that the action still posseses the $SO(1, 1)$ worldsheet symmetry (not manifestly)

Current

- Given a section $(\mathbb{V},\ \mathbb{F})$, define current $(j_z, j_{\bar{z}})$

$$j_z := \left(\mathbb{V}^i\ \mathbb{F}_i\right)\left[\frac{1}{2}\left(\mathbb{I}+\mathbb{U}_{\mathbb{A}}\right)\right]\begin{pmatrix} \mathbb{P}_j \\ \partial_\sigma \mathbb{Z}^j(-)^j \end{pmatrix}$$
$$j_{\bar{z}} := \left(\mathbb{V}^i\ \mathbb{F}_i\right)\left[\frac{1}{2}\left(\mathbb{I}-\mathbb{U}_{\mathbb{A}}\right)\right]\begin{pmatrix} \mathbb{P}_j \\ \partial_\sigma \mathbb{Z}^j(-)^j \end{pmatrix}$$

- Current conservation $\bar{\partial} j_z + \partial j_{\bar{z}} = 0$ implies

$$\mathcal{L}_{\mathbb{V}}\mathbb{A} = (d\mathbb{F})\cdot\mathbb{A} \quad \Longrightarrow \quad \begin{aligned} (\mathcal{L}_{\mathbb{V}}\mathbb{A})_{ij} &= -(d\mathbb{F})_{(i|k}\mathbb{A}^{k}{}_{|j)} \\ (\mathcal{L}_{\mathbb{V}}\mathbb{A})_i{}^j &= -(d\mathbb{F})_{ik}\mathbb{A}^{kj} \\ (\mathcal{L}_{\mathbb{V}}\mathbb{A})^{ij} &= 0 \end{aligned} \tag{1}$$

Holomorphicity

- If the section $(\mathbb{V},\ \mathbb{F})$ is a $(+1)$-eigenvector of $\mathbb{U}_{\mathbb{A}}$

$$j_z = \mathbb{V}^i \mathbb{P}_i + \mathbb{F}_i \partial_\sigma \mathbb{Z}^i (-)^i \ , \qquad j_{\bar{z}} = 0$$

  If the section $(\mathbb{V},\ \mathbb{F})$ is a (-1)-eigenvector of $\mathbb{U}_{\mathbb{A}}$

$$j_z = 0 \ , \qquad j_{\bar{z}} = \mathbb{V}^i \mathbb{P}_i + \mathbb{F}_i \partial_\sigma \mathbb{Z}^i (-)^i$$

- In any case, the current conservation conditions become the holomorphicity $\bar{\partial} j_z = 0$ or antiholomorphicity $\partial j_{\bar{z}} = 0$ conditions

Nilpotency

- Given a current $j_{(\mathbb{V},\mathbb{F})} := \mathbb{V}^i\mathbb{P}_i + \mathbb{F}_i\partial_\sigma\mathbb{Z}^i(-)^i$

$$\left\{j_{(\mathbb{V}_1,\mathbb{F}_1)}(\sigma_1), j_{(\mathbb{V}_2,\mathbb{F}_2)}(\sigma_2)\right\}_{PB} = j_{[\![(\mathbb{V}_1,\mathbb{F}_1),(\mathbb{V}_2,\mathbb{F}_2)]\!]}\delta(\sigma_1-\sigma_2) - \langle(\mathbb{V}_1,\mathbb{F}_1),(\mathbb{V}_2,\mathbb{F}_2)\rangle\delta'(\sigma_1-\sigma_2)$$

- For a fermionic current, nilpotency $\{j_{(\mathbb{Q},\mathbb{F})}, j_{(\mathbb{Q},\mathbb{F})}\}_{PB} = 0$ implies

$$[\![(\mathbb{Q},\mathbb{F}),(\mathbb{Q},\mathbb{F})]\!] = 0 \qquad (2)$$

or in components $[\mathbb{Q},\mathbb{Q}] = 0$, $\quad \mathcal{L}_\mathbb{Q}\mathbb{F} + \iota_\mathbb{Q} d\mathbb{F} = 0$

BRST currents

- Exist two sections $(\mathbb{Q},\ \mathbb{F})$ and $(\widetilde{\mathbb{Q}},\ \widetilde{\mathbb{F}})$

$$(\mathbb{Q}^i) = \begin{pmatrix} \lambda^\alpha E_\alpha{}^M \\ -\lambda^\beta E_\beta{}^M (\lambda\Omega_M)^\alpha \\ -\lambda^\beta E_\beta{}^M (\widehat{\lambda}\widehat{\Omega}_M)^{\widehat{\alpha}} \end{pmatrix} \ , \quad (\mathbb{F}_i) = \begin{pmatrix} Q^N B_{NM} \\ 0 \\ 0 \end{pmatrix}$$

  Same for $(\widetilde{\mathbb{Q}},\ \widetilde{\mathbb{F}})$ with hatted expressions.
- These are $(+1)$ and (-1) eigenvectors of $\mathbb{U}_{(\mathbb{A}_{PS})}$

$$\begin{aligned} (\mathbb{Q},\ \mathbb{F}) : &\qquad j_z = j_B \ , &\qquad j_{\bar{z}} = 0 \\ (\widetilde{\mathbb{Q}},\ \widetilde{\mathbb{F}}) : &\qquad j_z = 0 \ , &\qquad j_{\bar{z}} = \widetilde{j}_B \end{aligned}$$

Type II SUGRA constraints

- Nilpotency

$$\lambda^\alpha \lambda^\beta T_{\alpha\beta}{}^C = \lambda^\alpha \lambda^\beta \lambda^\gamma R_{\alpha\beta\gamma}{}^\sigma = \lambda^\alpha \lambda^\beta \hat{R}_{\alpha\beta\widehat{\gamma}}{}^{\widehat{\sigma}} = \lambda^\alpha \lambda^\beta H_{\alpha\beta C} = 0$$

- Holomorphicity

$$T_{\alpha(bc)} = H_{\alpha bc} = H_{\alpha\widehat{\beta}\gamma} = T_{\alpha\beta c} + H_{\alpha\beta c} = T_{\alpha\widehat{\beta}c} - H_{\alpha\widehat{\beta}c} = 0$$

$$T_{\alpha b}{}^\gamma + T_{\alpha\widehat{\gamma}b} P^{\gamma\widehat{\gamma}} = \ldots = T_{\alpha\beta}{}^{\widehat{\gamma}} - \frac{1}{2} H_{\alpha\beta\gamma} P^{\gamma\widehat{\gamma}} = T_{\alpha\widehat{\beta}}{}^\gamma = 0$$

$$C_\alpha^{\beta\widehat{\beta}} + \nabla_\alpha P^{\beta\widehat{\beta}} - T_{\alpha\rho}{}^\beta P^{\rho\widehat{\beta}} = \widehat{R}_{a\beta\widehat{\gamma}}{}^{\widehat{\sigma}} + T_{\beta\rho a} \widehat{C}_{\widehat{\gamma}}^{\widehat{\sigma}\rho} = \ldots = 0$$

$$S_{\alpha\widehat{\alpha}}^{\beta\widehat{\beta}} + \widehat{R}_{\alpha\widehat{\gamma}\widehat{\alpha}}{}^{\widehat{\beta}} P^{\beta\widehat{\gamma}} + \nabla_\alpha \widehat{C}_{\widehat{\alpha}}^{\widehat{\beta}\beta} - T_{\alpha\gamma}{}^\beta \widehat{C}_{\widehat{\alpha}}^{\widehat{\beta}\gamma} = 0$$

$$\lambda^\alpha \lambda^\beta \left(R_{a\alpha\beta}{}^\gamma + T_{\alpha\widehat{\beta}a} C_\beta^{\gamma\widehat{\beta}} \right) = \lambda^\alpha \lambda^\beta R_{\widehat{\beta}\alpha\beta}{}^\gamma = 0$$

$$\lambda^\alpha \lambda^\beta \left(\nabla_\alpha C_\beta^{\gamma\widehat{\gamma}} - R_{\alpha\sigma\beta}{}^\gamma P^{\sigma\widehat{\gamma}} \right) = \lambda^\alpha \lambda^\beta \left(\nabla_\alpha S_{\beta\widehat{\beta}}^{\gamma\widehat{\gamma}} - \ldots \right) = 0$$

Conclusions

- Obtained an action that treats matter and ghosts on the same footing
- Translation of holomorphicity and nilpotency of BRST currents into context of generalized geometry

$$\mathcal{L}_{\mathbb{Q}}\mathbb{A} = (d\mathbb{F}) \cdot \mathbb{A} \qquad \text{(holomorph.)}$$

$$[\![(\mathbb{Q},\ \mathbb{F}), (\mathbb{Q},\ \mathbb{F})]\!] = 0 \qquad \text{(nilpotency)}$$

- Rederived Type II SUGRA constraints

What is it good for?

- Special cases where there is an honest first order formalism

$$S = \int p_M \bar{\partial} z^M + \bar{p}_{\bar{M}} \partial \bar{z}^{\bar{M}} + \omega_\alpha \bar{\partial} \lambda^\alpha + \widehat{\omega}_{\widehat{\alpha}} \partial \widehat{\lambda}^{\widehat{\alpha}}$$

$$- \frac{1}{2} (p_M \ \omega_\alpha) \begin{pmatrix} A^{M\bar{M}} & A^{M\widehat{\alpha}} \\ A^{\alpha\bar{M}} & A^{\alpha\widehat{\alpha}} \end{pmatrix} \begin{pmatrix} \bar{p}_{\bar{M}} \\ \widehat{\omega}_{\widehat{\alpha}} \end{pmatrix}$$

- Pure Spinor on a CY background

$$S = \int \frac{1}{2} (g + b)_{mn} \partial x^m \bar{\partial} x^n + \omega_\alpha \bar{\partial} \lambda^\alpha + \widehat{\omega}_{\widehat{\alpha}} \partial \widehat{\lambda}^{\widehat{\alpha}}$$

$$+ R(\theta^\alpha p_\alpha + \lambda^\alpha \omega_\alpha)(\widehat{\theta}^{\widehat{\alpha}} \widehat{p}_{\widehat{\alpha}} + \widehat{\omega}_{\widehat{\alpha}} \widehat{\lambda}^{\widehat{\alpha}})$$

Thanks!