Correlators of operators on Wilson loops in N=4 SYM and AdS_2/CFT_1

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• correlation functions of operators on susy and standard WL in $\mathcal{N} = 4$ SYM and dual $AdS_5 \times S^5$ superstring theory: novel examples of 1d defect CFT 's

• non-gravitational example of $\mathrm{AdS}_2/\mathrm{CFT}_1$ defined by world-sheet string action

• WL: $\langle \mathrm{Tr}\, \mathcal{P} e^{i\int A} \rangle$ important observable in any gauge theory
no log div; power div factorize

• WML: $\mathcal{N} = 4$ SYM: special Wilson-Maldacena loop
$iA_\mu \dot{x}^\mu \to iA_\mu \dot{x}^\mu + \Phi_a \dot{y}^a$
if $(\dot{x}^\mu)^2 = (\dot{y}^a)^2$, i.e. $\dot{y}^a(\tau) = |\dot{x}(\tau)|\theta^a, \ \ \theta^2 = 1$:
locally-supersymmetric, better UV properties
straight line: $\frac{1}{2}$ global susy (BPS): $\langle W(\text{line}) \rangle = 1$

• non-susy WL is also of interest in AdS/CFT context:
large N expectation value for circle or cusp $\to$
non-trivial functions of 't Hooft coupling $\lambda = g^2 N$
not fixed by susy but may be by integrability

• $AdS_5 \times S^5$ string side: WML – Dirichlet b.c. in S^5 (susy)
WL – Neumann b.c. in S^5 (non-susy) [Alday, Maldacena]

• corr. functs of local operators inserted on line:
new examples of AdS_2/CFT_1 duality
WML: local ops on $\frac{1}{2}$ -BPS line – define CFT_1
with $OSp(4^*|4)$ 1d superconformal symmetry
WL: different defect CFT_1 with $SO(3) \times SO(6)$ symmetry
[Cooke, Dekel, Drukker; Giombi, Roiban, AT]

• 1-parameter family of Wilson loops:
WL ($\zeta = 0$) and WML ($\zeta = 1$) [Polchinski, Sully]

$$W^{(\zeta)}(C) = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \oint_C d\tau \left[i\, A_\mu(x)\, \dot{x}^\mu + \zeta \Phi_m(x)\, \theta^m\, |\dot{x}|\, \right]$$

θ^m=const: e.g. $\Phi_m\, \theta^m = \Phi_6$

• $\langle W^{(\zeta)} \rangle$ has log divergences for $\zeta \neq 0, 1$
can be absorbed into renormalization of 1d coupling ζ

$$\langle W^{(\zeta)} \rangle \equiv W(\lambda; \zeta(\mu), \mu) \,, \qquad \mu \frac{\partial}{\partial \mu} W + \beta_\zeta \frac{\partial}{\partial \zeta} W = 0$$

at weak coupling $\lambda \ll 1$ (at large N) [PS]

$$\beta_\zeta = \mu \frac{d\zeta}{d\mu} = \frac{\lambda}{8\pi^2} \zeta(\zeta^2 - 1) + \mathcal{O}(\lambda^2)$$

WL $\zeta = 0$ and WML $\zeta = 1$ are UV and IR conformal points
cf. 1d QFT, conformal pert. theory by $O = \zeta \Phi_6$ near $\zeta = 0$

• circular WML ($\zeta = 1$): exact result due to 1/2 susy

[Ericson, Semenoff, Zarembo; Drukker, Gross; Pestun]

$$\langle W^{(1)}(\text{circle})\rangle_{N\to\infty} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

$$\stackrel{\lambda\ll 1}{=} \quad 1 + \frac{1}{8}\lambda + \frac{1}{192}\lambda^2 + \cdots$$

$$\stackrel{\lambda\gg 1}{=} \quad \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{(\sqrt{\lambda})^{3/2}} \left(1 - \frac{3}{8\sqrt{\lambda}} + \cdots\right)$$

$\langle W^{(1)}(\text{line})\rangle = 1$: anomaly in conf map of line to circle [DG]
due to IR behaviour of vector propagator – same for WL ?

• WL case: no log div; if power div factorized $\langle W^{(0)}(\text{line})\rangle = 1$
then $\langle W^{(0)}(\text{circle})\rangle = \langle W^{(1)}(\text{circle})\rangle$?
yes, at leading orders at weak & strong λ but not beyond

• weak coupling: $\langle W^{(\zeta)}\rangle = 1 + \frac{1}{8}\lambda + O(\lambda^2)$
strong coupling: same min surface: AdS_2 with S^1 as bndry
subtracting linear div in $V_{AdS_2} = 2\pi(\frac{1}{a} - 1)$ gives
universal $\langle W^{(\zeta)}\rangle \sim e^{\sqrt{\lambda}}$
• subleading terms at $\lambda \ll 1$: $\langle W^{(\zeta)}(\text{circle})\rangle$ depends on ζ
[Beccaria, Giombi, AT]

$$\langle W^{(\zeta)}\rangle = 1 + \frac{1}{8}\lambda + \Big[\frac{1}{192} + \frac{1}{128\,\pi^2}(1-\zeta^2)^2\Big]\lambda^2 + \mathcal{O}(\lambda^3)$$

interpolates between WML at $\zeta = 1$ and WL at $\zeta = 0$:

$$\langle W^{(0)}\rangle = 1 + \frac{1}{8}\lambda + \Big(\frac{1}{192} + \frac{1}{128\,\pi^2}\Big)\lambda^2 + \mathcal{O}(\lambda^3)$$

• no susy/localization but may be exact expression
from integrability?

Consistency checks:

• UV finiteness of 2-loop λ^2 term: no ζ in 1-loop term
UV log divergences appear first at λ^3 order

• conf points $\zeta = 1$ and $\zeta = 0$ are extrema of $\langle W^{(\zeta)} \rangle$:

$$\frac{\partial}{\partial \zeta} \log \langle W^{(\zeta)} \rangle = \mathcal{C}\, \beta_\zeta \,, \qquad \beta_\zeta = \tfrac{\lambda}{8\pi^2} \zeta(\zeta^2 - 1) + \ldots \,, \;\; \mathcal{C} = \tfrac{1}{4}\lambda + \ldots$$

• may interpret $\langle W^{(\zeta)} \rangle$ as a 1d QFT part funct Z_{S^1} on S^1
computed in pert. theory near $\zeta = 1$ or $\zeta = 0$ conf points:
$d = 1$ case of relation $\frac{\partial F}{\partial g_i} = \mathcal{C}^{ij} \beta_j, \quad F = -\log Z_{S^d}$
cf. F-theorem in odd dimensions [Klebanov, Safdi, Pufu]

• present case: flow driven by $O = \Phi_6$ restricted to the line

$\frac{\partial}{\partial\zeta}\langle W^{(\zeta)}\rangle\big|_{\zeta=0,1} = 0 \quad \to \quad \langle O\rangle\big|_{\zeta=0,1} = 0$

as required by 1d conformal invariance

• ζ: marginally relevant coupling

running from $\zeta = 0$ in UV to $\zeta = 1$ in IR

• 2-loop result implies

$$\langle W^{(0)}\rangle \; > \; \langle W^{(1)}\rangle$$

$\langle W^{(\zeta)}\rangle = Z_{S^1} = e^{-F}$ partition function of defect QFT$_1$ on S^1

consistent with the F-theorem in $d = 1$

$$\tilde{F}_{\rm UV} > \tilde{F}_{\rm IR}\,, \qquad \tilde{F}\Big|_{d=1} = \log Z_{S^1} = -F$$

• $\langle W^{(\zeta)}\rangle$ decreases monotonically with $0 < \zeta < 1$

- 2nd derivative of $\langle W^{(\zeta)} \rangle \propto$ anomalous dimension

$$\frac{\partial^2}{\partial \zeta^2} \log \langle W^{(\zeta)} \rangle \Big|_{\zeta=0,1} = \mathcal{C} \, \frac{\partial \beta_\zeta}{\partial \zeta} \Big|_{\zeta=0,1}$$

$\frac{\partial \beta_\zeta}{\partial \zeta} \Big|_{\zeta=0,1} \to \Delta$ of Φ_6 at $\zeta = 1$ and $\zeta = 0$ conf points

- weak coupling: dim of Φ_6

$$\Delta(\zeta) - 1 = \frac{\partial \beta_\zeta}{\partial \zeta} = \frac{\lambda}{8\pi^2}(3\zeta^2 - 1) + \mathcal{O}(\lambda^2) \, ,$$

$$\Delta(1) = 1 + \frac{\lambda}{4\pi^2} + \dots \, , \qquad \Delta(0) = 1 - \frac{\lambda}{8\pi^2} + \dots \, .$$

Strong coupling

• interpretation of $\langle W^{(\zeta)} \rangle$ as partition function of 1d QFT supported by its strong-coupling representation as $AdS_5 \times S^5$ string partition function on disc with mixed b.c. for S^5 coordinates (D for $\zeta = 1$ and N for $\zeta = 0$) [AM, PS]

• large λ asymptotics:

instead of $\quad \langle W^{(1)} \rangle \sim (\sqrt{\lambda})^{-3/2} e^{\sqrt{\lambda}} + ...$

find $\quad \langle W^{(0)} \rangle \sim \sqrt{\lambda}\, e^{\sqrt{\lambda}} + ...$ [BGT]

i.e. F-theorem $\langle W^{(0)} \rangle > \langle W^{(1)} \rangle$ satisfied also at $\lambda \gg 1$

Map of operators to AdS_2 fields or string coordinates:

• WL: $\zeta = 0 \qquad$ O(6) is unbroken

scalars $\Phi_A \to$ embedding coordinates Y_A of S^5

$$\Phi_A \leftrightarrow Y_A, \qquad A = 1, ..., 6$$

• WML: $\zeta = 1$ O(6) is broken to O(5)
by selection of Φ_6 direction or point of S^5 $(a = 1, ..., 5)$
$\Phi_a \leftrightarrow Y_a = y_a + ...,$ $\Phi_6 \leftrightarrow Y_6 = 1 - \frac{1}{2} y_a y_a + ...$
Φ_a and Φ_6 get different dimensions
• bndry perturbation of string action by $\varkappa \int dt\, Y_6$ near $\zeta = 0$
induces boundary RG flow from N b.c. to D b.c.:
$\varkappa = \mathrm{f}(\zeta; \lambda)$: 0 for $\zeta = 0$ and ∞ for $\zeta = 1$
with RG beta-function $\beta_\varkappa = (-1 + \frac{5}{\sqrt{\lambda}})\varkappa + ...$
• implies that strong-coupling dimensions of Φ_6
near 2 conf points are [AM, GRT]

$$\lambda \gg 1: \qquad \Delta(0) = \frac{5}{\sqrt{\lambda}} + ... , \qquad \Delta(1) = 2 - \frac{5}{\sqrt{\lambda}} + ...$$

consistent with interpolation from $\lambda \ll 1$

$$\lambda \ll 1: \qquad \Delta(0) = 1 - \frac{\lambda}{8\pi^2} + \ldots , \qquad \Delta(1) = 1 + \frac{\lambda}{4\pi^2} + \ldots$$

Correlators on WML at strong coupling: AdS_2/CFT_1

• novel sector of observables in AdS/CFT:
gauge-invariant correlators of operators inserted on Wilson loop
• described by an effective ("defect") CFT_1
"induced" from $\mathcal{N} = 4$ SYM
• $\frac{1}{2}$-BPS line WML: leads to example of AdS_2/CFT_1
quantum theory in AdS_2 defined by superstring action
• in BPS WML "vacuum" have AdS/CFT map:
elementary SYM fields (Φ, F $\perp$ to the line)
$\leftrightarrow$ string coordinates as fields in AdS_2
[cf. $\mathrm{Tr}(\Phi^n ... D^m F^k ...)$ $\leftrightarrow$ closed-string vertex operators]
• 4-point correlators at strong coupling:
Witten diagrams for AdS/CFT correlators, OPE, etc.

• $\frac{1}{2}$ BPS: infinite straight line (or circle), θ^I=const

$x^0 = t \in (-\infty, \infty),\ \theta^I \Phi^I = \Phi_6, \qquad W = \mathrm{tr} P e^{\int dt (i A_t + \Phi_6)}$

• $O_i(x(t_i))$ on WML: gauge inv correlator

$$\langle\langle O_1(t_1) O_2(t_2) \cdots O_n(t_n) \rangle\rangle$$

$$\equiv \langle \mathrm{tr} P \left[O_1(t_1)\, e^{\int dt (i A_t + \Phi_6)}\, O_2(t_2)\ \cdots\ O_n(t_n)\, e^{\int dt (i A_t + \Phi_6)} \right] \rangle$$

$\langle\langle 1 \rangle\rangle = \langle W \rangle = 1$ and similar normalization for circle

• operator insertions are equivalent to deformations of WL

[Drukker, Kawamoto:06; Cooke, Dekel, Drukker:17]

complete knowledge of correlators $\leftrightarrow$ expectation value of general Wilson loop – deformation of line or circle

• symmetries preserved by $\frac{1}{2}$-BPS WL vacuum:

$SO(5) \subset SO(6)$ R-symmetry: 5 scalars Φ^a, $a = 1, \ldots, 5$

$SO(2,1) \times SO(3) \subset SO(2,4)$: $SO(3)$ rotations around line

$SO(2,1)$ – dilations, transl and special conf along line

$d=1$ conformal group + 16 supercharges preserved by line:

$d=1, \mathcal{N}=8$ superconformal group $OSp(4^*|4)$

• operator insertions $O(t)$ classified by $OSp(4^*|4)$ reps

labelled by dim Δ and rep of "internal" $SO(3)\times SO(5)$

• correlators define "defect" CFT_1 on the line

[Drukker et al:06; Sakaguchi, Yoshida:07; Cooke et al:17]

determined by spectrum of dims and OPE coeffs

• $\langle\!\langle ... \rangle\!\rangle$ correlators satisfy all usual properties of CFT:

$O(t)$ = "operators in CFT_1"

without reference to their (non-local) origin in SYM

• "elementary excitations": short rep of $OSp(4^*|4)$

8 bosonic (+ 8 fermionic) ops with protected Δ:

5 scalars: Φ^a $(\Delta=1)$ that do not couple to WL;

3 "displacement operators": $\mathbb{F}_{ti} \equiv iF_{ti} + D_i\Phi_6$ ($i = 1,2,3$) with protected $\Delta = 2$ (WI for breaking of $\perp$ translations)

- protected dims: exact 2-point functions in planar SYM

$$\langle\!\langle \Phi^a(t_1)\Phi^b(t_2)\rangle\!\rangle = \delta^{ab}\frac{C_\Phi(\lambda)}{t_{12}^2}\,, \qquad t_{12} = t_1 - t_2$$

$$\langle\!\langle \mathbb{F}_{ti}(t_1)\mathbb{F}_{tj}(t_2)\rangle\!\rangle = \delta_{ij}\frac{C_{\mathbb{F}}(\lambda)}{t_{12}^4}$$

$$C_\Phi(\lambda) = 2B(\lambda)\,, \quad C_{\mathbb{F}}(\lambda) = 12B(\lambda)\,, \quad B(\lambda) = \frac{\sqrt{\lambda}\, I_2(\sqrt{\lambda})}{4\pi^2\, I_1(\sqrt{\lambda})}$$

$B(\lambda)$ – Bremsstrahlung function [Correa, Henn, Maldacena, Sever:12]

- 3-point functions vanish by $SO(3) \times SO(5)$ symmetry
- 4-point functions: depend on $t_1, ..., t_2$ and λ

String theory side

4-point functions at strong coupling ($N = \infty$, $\lambda \gg 1$)
from string theory in $AdS_5 \times S^5$

• $\frac{1}{2}$-BPS Wilson line (or circle): minimal surface is
AdS_2 embedded in AdS_5

• fundamental open string stretched in AdS_5:
preserves same $OSp(4^*|4)$ as $\frac{1}{2}$-BPS WL
1d conf group $SO(2,1)$ realized as isometry of AdS_2

• expanding string action around AdS_2 surface:
AdS_2 multiplet of fluctuations transverse to string –
5 ($m^2 = 0$) scalars y^a in S^5; 3 ($m^2 = 2$) scalars x^i in AdS_5;
8 ($m^2 = 1$) fermions [Drukker, Gross, AT:00]

• identify 8+8 fields in AdS_2 with elementary CFT_1 insertions
(cf. waves on line $\to$ change of minimal area)

• $m^2 = \Delta(\Delta - d)$ for AdS$_{d+1}$ scalar masses and CFT$_d$ dims:
massless S^5 fields y^a dual to Φ^a in CFT$_1$ with $\Delta = 1$
massive AdS$_5$ fields x^i dual to $\mathbb{F}_{ti}$ with $\Delta = 2$
• AdS/CFT: closed superstring vertex operators $\to$
single-trace gauge inv local operators in SYM;
add open-string sector (strings ending at bndry) $\to$
gauge-inv operators = WL with insertions of local operators

• other gauge-invariant correlators:
(i) WL with single-trace ops e.g. $\langle W \, \mathrm{tr} Z^J \rangle$
point away from line (Tr ^{2}: subleading at large N)
[Berenstein et al:98; Semenoff, Zarembo:01; Pestun, Zarembo:02]
(ii) mixed correlators of ops on line and ops away from line

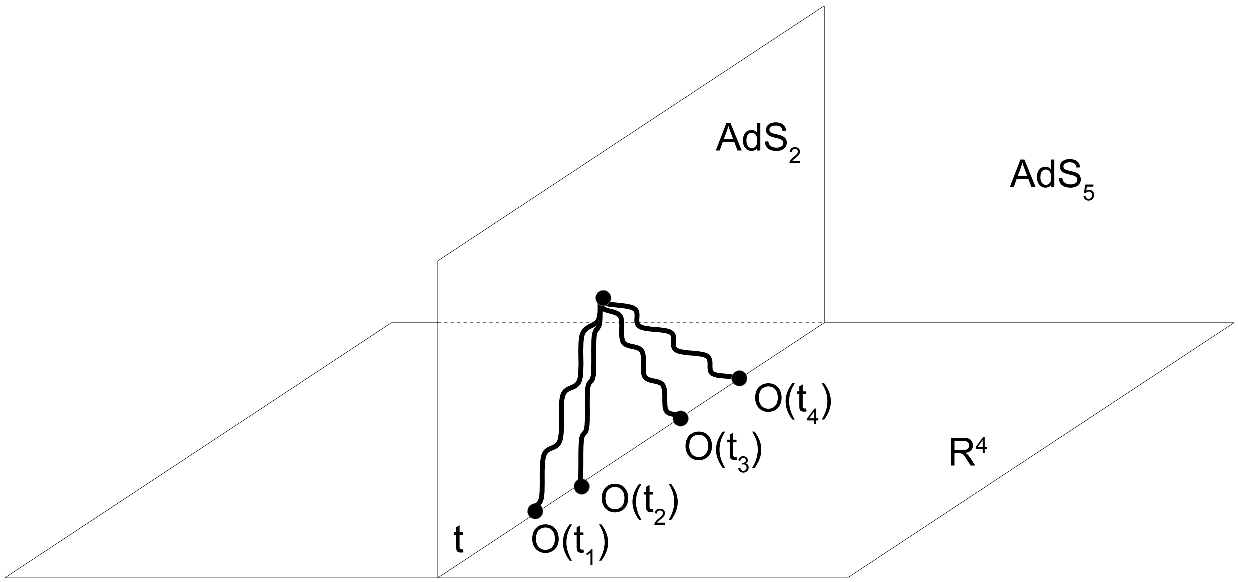
Strategy:

string action $\to$ interaction vertices for "light" AdS_2 fields
$\to$ tree-level Witten diagrams in AdS_2 $\to$ prediction for
4-point functions of protected ops on WL:
expansion parameter $\frac{1}{\sqrt{\lambda}}$ (action $S = \sqrt{\lambda} \int d^2\sigma \sqrt{h} \partial x \partial x + ...$)
(cf. $\frac{1}{N^2}$ in 4-points in AdS_5 sugra: $S = N^2 \int d^5x \sqrt{g} R + ...$)

• AdS_2 QFT: superstring action UV finite
AdS_2/CFT_1 duality should hold for any $T = \frac{\sqrt{\lambda}}{2\pi}$
• AdS_2 Witten diagrams with loops should be well defined
e.g. 1-loop correction to boundary-to-boundary propagator
protected 2-point function: subleading term in

$$C_\Phi = \frac{\sqrt{\lambda}}{2\pi^2} - \frac{3}{4\pi^2} + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

[Buchbinder, AT:13]



(i) compute tree-level 4-point functions

(ii) use OPE to extract strong coupling corrections to dims of "2-particle" ops built of 2 of protected insertions: $\Phi\partial_t^n\Phi$, etc.

$AdS_5 \times S^5$ string in static gauge: AdS_2 bulk theory

bosonic part of superstring action $\quad (T = \frac{\sqrt{\lambda}}{2\pi})$

$$S_B = \tfrac{1}{2} T \int d^2\sigma \sqrt{h}\, h^{\mu\nu} \Big[\frac{1}{z^2} \left(\partial_\mu x^r \partial_\nu x^r + \partial_\mu z \partial_\nu z \right) + \frac{\partial_\mu y^a \partial_\nu y^a}{(1 + \frac{1}{4} y^2)^2} \Big]$$

$\sigma^\mu = (t, s), \;\; r = (0, i) = (0, 1, 2, 3), \;\; a = 1, ..., 5$

minimal surface for straight Wilson line at Euclidean boundary

$$z = s\,, \qquad x^0 = t\,, \qquad\qquad x^i = 0\,, \quad y^a = 0$$

induced metric is AdS_2: $\quad g_{\mu\nu} d\sigma^\mu d\sigma^\nu = \frac{1}{s^2}(dt^2 + ds^2)$.

• compute correlators of small fluctuations of "transverse" coordinates (x^i, y^a) near AdS_2 minimal surface

• global symmetry of action $SO(2,1) \times [SO(3) \times SO(6)]$

• make $SO(2,1)$ manifest: AdS$_2$ adapted coordinates

$$ds^2_{\text{AdS}_5} = \frac{(1+\frac{1}{4}x^2)^2}{(1-\frac{1}{4}x^2)^2} ds^2_{\text{AdS}_2} + \frac{dx^i dx^i}{(1-\frac{1}{4}x^2)^2}, \qquad ds^2_{\text{AdS}_2} = \frac{1}{z^2}(dx_0^2 + dz^2)$$

• action in static gauge: $z = s$ and $x^0 = t$

$$S_B = T \int d^2\sigma \sqrt{h}$$

$$h_{\mu\nu} = \frac{(1+\frac{1}{4}x^2)^2}{(1-\frac{1}{4}x^2)^2} g_{\mu\nu}(\sigma) + \frac{\partial_\mu x^i \partial_\nu x^i}{(1-\frac{1}{4}x^2)^2} + \frac{\partial_\mu y^a \partial_\nu y^a}{(1+\frac{1}{4}y^2)^2}, \qquad g_{\mu\nu} = \frac{1}{s^2}\delta_{\mu\nu}$$

= action of straight fundamental string in AdS$_5 \times S^5$ along z:
2d theory of 3+5 scalars in AdS$_2$ with $SO(2,1) \times [SO(3) \times SO(6)]$

• bulk AdS$_2$ theory $\leftrightarrow$ CFT$_1$ at $z = s = 0$ bndry:
CFT$_1$ defined by insertions on straight WL

$$L_B = L_2 + L_{4x} + L_{2x,2y} + L_{4y} + ...$$

$$L_2 = \frac{1}{2} g^{\mu\nu}\partial_\mu x^i \partial_\nu x^i + x^i x^i + \frac{1}{2} g^{\mu\nu}\partial_\mu y^a \partial_\nu y^a$$

$$L_{4x} = \frac{1}{8}(g^{\mu\nu}\partial_\mu x^i \partial_\nu x^i)^2 - \frac{1}{4}(g^{\mu\nu}\partial_\mu x^i \partial_\nu x^j)\ (g^{\rho\kappa}\partial_\rho x^i \partial_\kappa x^j)$$
$$+\frac{1}{4} x^i x^i (g^{\mu\nu}\partial_\mu x^j \partial_\nu x^j) + \frac{1}{2} x^i x^i\ x^j x^j$$

$$L_{2x,2y} = \frac{1}{4}(g^{\mu\nu}\partial_\mu x^i \partial_\nu x^i)(g^{\rho\kappa}\partial_\rho y^a \partial_\kappa y^a) - \frac{1}{2}(g^{\mu\nu}\partial_\mu x^i \partial_\nu y^a)(g^{\rho\kappa}\partial_\rho x^i \partial_\kappa y^a)$$

$$L_{4y} = -\frac{1}{4}(y^b y^b)(g^{\mu\nu}\partial_\mu y^a \partial_\nu y^a) + \frac{1}{8}(g^{\mu\nu}\partial_\mu y^a \partial_\nu y^a)^2$$
$$-\frac{1}{4}(g^{\mu\nu}\partial_\mu y^a \partial_\nu y^b)\ (g^{\rho\kappa}\partial_\rho y^a \partial_\kappa y^b)$$

• superstring: (3+5) bosons + 8 fermions ($m^2 = 1$)
resulting 2d theory is UV finite and dual to CFT_1
for any coupling $T = \frac{\sqrt{\lambda}}{2\pi}$

• bndry correlators $\langle\!\langle O(t_1)O(t_2)...O(t_n)\rangle\!\rangle$ reproduced
by AdS_2 amplitudes of string sigma model – series in $\frac{1}{\sqrt{\lambda}}$

• operator $O \leftrightarrow$ string coordinates $X = (x, y)$

$$\langle\!\langle O(t_1)O(t_2)...O(t_n)\rangle\!\rangle_{\rm SYM} = \langle X(t_1)X(t_2)....X(t_n)\rangle_{{\rm AdS}_2}$$

• $X \sim y^a \to O \sim \Phi^a$ ($a = 1, ..., 5$) with $\Delta = 1$
$X \sim x^i \to O \sim \mathbb{F}_{it}$ ($i = 1, 2, 3$) with $\Delta = 2$
• $\lambda \gg 1$: $\langle W(C)\rangle$ from $AdS_5 \times S^5$ open str. path int. with Dirichlet b.c. (disc or half-plane w-surface ending at bndry)
$\log\langle W(C)\rangle$ = minimal area = string action on solution
• string action as 2d bulk theory in AdS_2:
same as AdS/CFT procedure for $\langle X(t_1)X(t_2)....X(t_n)\rangle_{{\rm AdS}_2}$
• expanding on-shell string action (gen.f. for tree "S-matrix") in powers of fluctuations $\delta C(t)$ from straight line:
same correlators as from bulk correlators connected to bndry points by bulk-to-bndry propagators

Comments:

• novel example of AdS_2/CFT_1:
critical string – no dynamical 2d gravity: fixed AdS_2 background
defect CFT with no "stress tensor" $\leftrightarrow$ AdS_2 with no gravity
$SO(2,1)$ as isometry of AdS_2 metric, no 1d reparam inv
(cf. dilaton gravity [Ahlmeiri, Polchinski:14; Maldacena, Stanford:16])

• original WL has a reparam inv, fixed by identification $x^0 = t$;
remaining symm $SO(2,1) \subset SO(2,4)$ that preserves the line;
before fixing static gauge string ("bulk") action is reparam inv
but gravity non-dynamical in critical superstring
(no analog of pseudo-Goldstone mode in bndry theory
related to spont. broken reparams)

4-point functions and conformal blocks in CFT_1

• local operators in CFT_1 on line $\mathbb{R} = \{t\}$
covariant under $SO(2,1)$

$$\langle O_\Delta(t_1) O_\Delta(t_2) O_\Delta(t_3) O_\Delta(t_4)\rangle = \frac{1}{(t_{12}t_{34})^{2\Delta}}\, G(\chi)$$

$$\chi = \frac{t_{12}t_{34}}{t_{13}t_{24}}$$

usual cross-ratios u, v not independent in $d = 1$

$$u \equiv \frac{t_{12}^2 t_{34}^2}{t_{13}^2 t_{24}^2} = \chi^2\,, \qquad\qquad v \equiv \frac{t_{14}^2 t_{23}^2}{t_{13}^2 t_{24}^2} = (1-\chi)^2$$

one χ: $SO(2,1)$ allows to fix 3 points on the line

• OPE expansion

$$G(\chi) = \sum_h c_{\Delta,\Delta;h}\, \chi^h\, F_h(\chi)\ , \qquad F_h = {}_2F_1(h,h,2h,\chi)$$

h= dim of exchanged operator; $c_{\Delta,\Delta;h} = \dfrac{C^2_{O_\Delta O_\Delta O_h}}{C^2_{O_\Delta O_\Delta} C_{O_h O_h}}$

$\chi^h\, {}_2F_1(h,h,2h,\chi)$ – conf block in $d=1$ [Dolan, Osborn:11]

"Generalized free fields"

(e.g. $g=0$ large N CFT) [Heemskerk et al:09, Fitzpatrick et al:11]

• case of identical operators of dim Δ:

$G(u,v) = 1 + u^\Delta + (\frac{u}{v})^\Delta$, i.e. in $d=1$

$$\langle O_\Delta(t_1) O_\Delta(t_2) O_\Delta(t_3) O_\Delta(t_4)\rangle = \frac{1}{(t_{12}t_{34})^{2\Delta}}\Big[1 + \chi^{2\Delta} + \frac{\chi^{2\Delta}}{(1-\chi)^{2\Delta}}\Big]$$

ops exchanged in OPE are only 1 and "2-particle" ops

$$\mathcal{O} = \left[O_\Delta O_\Delta\right]_{2n} \sim O_\Delta \partial_t^{2n} O_\Delta \ , \qquad \Delta_{\mathcal{O}} = 2\Delta + 2n, \ \ n = 0, 1, \ldots$$

corresponding OPE coeffs:

$$c_{\Delta,\Delta;2\Delta+2n} = \frac{2\left[\Gamma(2n+2\Delta)\right]^2 \Gamma(2n+4\Delta-1)}{\left[\Gamma(2\Delta)\right]^2 \Gamma(2n+1)\Gamma(4n+4\Delta-1)}$$

4-point function of S^5 fluctuations

tree-level 4-point Witten diagram of S^5 fluctuations y^a

$$\langle\!\langle \Phi^{a_1}(t_1)\Phi^{a_2}(t_2)\Phi^{a_3}(t_3)\Phi^{a_4}(t_4) \rangle\!\rangle$$

$$= \langle y^{a_1}(t_1) y^{a_2}(t_2) y^{a_3}(t_3) y^{a_4}(t_4)\rangle_{\mathrm{AdS}_2} = \frac{\left[C_\Phi(\lambda)\right]^2}{t_{12}^2 t_{34}^2}\, G^{a_1 a_2 a_3 a_4}(\chi;\lambda)$$

Φ^a – protected dimension $\Delta = 1$

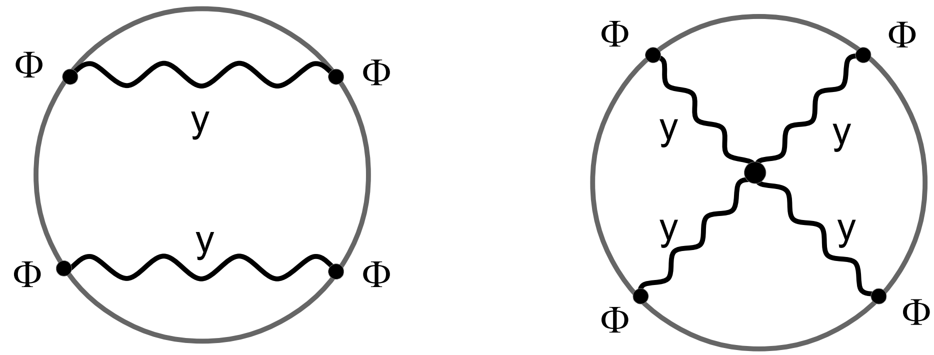
$$\langle y^{a_1}(t_1) y^{a_2}(t_2)\rangle_{\mathrm{AdS}_2} = \langle\!\langle \Phi^{a_1}(t_1)\Phi^{a_2}(t_2)\rangle\!\rangle = \delta^{a_1 a_2}\frac{C_\Phi(\lambda)}{t_{12}^2}$$

• decompose into $SO(5)$ singlet, antisymm and symm traceless

$$G^{a_1a_2a_3a_4}(\chi) = G_S(\chi)\delta^{a_1a_2}\delta^{a_3a_4} + G_A(\chi)\left(\delta^{a_1a_3}\delta^{a_2a_4} - \delta^{a_2a_3}\delta^{a_1a_4}\right)$$
$$+ G_T(\chi)\left(\delta^{a_1a_3}\delta^{a_2a_4} + \delta^{a_2a_3}\delta^{a_1a_4} - \tfrac{2}{5}\delta^{a_1a_2}\delta^{a_3a_4}\right)$$
$$G_{S,T,A}(\chi) = G^{(0)}_{S,T,A}(\chi) + \tfrac{1}{\sqrt{\lambda}}G^{(1)}_{S,T,A}(\chi) + \dots$$

• leading terms $G^{(0)}_{S,T,A}(\chi)$ from disconnected 4-point function
– given by generalized free field result

$$G^{a_1a_2a_3a_4}_{\text{disconn.}} = \frac{\left[C_\Phi(\lambda)\right]^2}{t_{12}^2 t_{34}^2}\left[\delta^{a_1a_2}\delta^{a_3a_4} + \chi^2\delta^{a_1a_3}\delta^{a_2a_4} + \frac{\chi^2}{(1-\chi)^2}\delta^{a_1a_4}\delta^{a_2a_3}\right]$$



$$G_S^{(0)}(\chi) = 1 + \tfrac{2}{5} G_T^{(0)}(\chi)\,, \qquad G_{T,A}^{(0)}(\chi) = \tfrac{1}{2}\Big[\chi^2 \pm \tfrac{\chi^2}{(1-\chi)^2}\Big]$$

• connected part: using 4-vertices in string action and normalized bulk-to-bndry prop. $\langle O_\Delta(x_1) O_\Delta(x_2)\rangle = \frac{\mathcal{C}_\Delta}{x_{12}^{2\Delta}}$

$$K_\Delta(z,x;x') = \mathcal{C}_\Delta \Big[\tfrac{z}{z^2+(x-x')^2}\Big]^\Delta \equiv \mathcal{C}_\Delta\, \mathrm{K}_\Delta(z,x;x')$$

for $d=1, \quad \Delta=1, \quad t \equiv x^0$

$$K_1(z,t;t') = \frac{1}{\pi}\,\frac{z}{z^2+(t-t')^2}\,, \qquad\qquad \mathcal{C}_{\Delta=1} = \frac{1}{\pi}$$

• 4-point in terms of D-functions: in AdS_{d+1} [D'Hoker et al 89]

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1,x_2,x_3,x_4)$$
$$= \int \frac{dz d^d x}{z^{d+1}}\,\mathrm{K}_{\Delta_1}(z,x;x_1)\mathrm{K}_{\Delta_2}(z,x;x_2)\,\mathrm{K}_{\Delta_3}(z,x;x_3)\mathrm{K}_{\Delta_4}(z,x;x_4)$$

• "reduced" $\bar{D}$ $\quad (\Sigma \equiv \frac{1}{2}\sum_i \Delta_i)$

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4} = \frac{\pi^{\frac{d}{2}}\Gamma\left(\Sigma-\frac{d}{2}\right)}{2\,\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)\Gamma(\Delta_4)}\,\frac{x_{14}^{2(\Sigma-\Delta_1-\Delta_4)}\,x_{34}^{2(\Sigma-\Delta_3-\Delta_4)}}{x_{13}^{2(\Sigma-\Delta_4)}\,x_{24}^{2\Delta_2}}\,\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)$$

$$\bar{D} = \int d\alpha d\beta d\gamma\,\delta(\alpha+\beta+\gamma-1)\,\alpha^{\Delta_1-1}\beta^{\Delta_2-1}\gamma^{\Delta_3-1}\,\frac{\Gamma(\Sigma-\Delta_4)\Gamma(\Delta_4)}{(\alpha\gamma+\alpha\beta\,u+\beta\gamma\,v)^{\Sigma-\Delta_4}}$$

• in $d=1$: $\quad u=\chi^2,\ v=(1-\chi)^2$

$$\bar{D}_{1111}(\chi) = -\frac{2}{1-\chi}\log|\chi| - \frac{2}{\chi}\log|1-\chi|$$

$$\langle yyyy\rangle^{a_1a_2a_3a_4}_{\text{conn}} = \frac{(\mathcal{C}_1)^2}{t_{12}^2 t_{34}^2} G^{a_1a_2a_3a_4}(\chi)$$

$$\begin{aligned}
G_S^{(1)}(\chi) &= -\frac{2\left(\chi^4-4\chi^3+9\chi^2-10\chi+5\right)}{5(\chi-1)^2} + \frac{\chi^2\left(2\chi^4-11\chi^3+21\chi^2-20\chi+10\right)}{5(\chi-1)^3}\log|\chi| \\
&\quad - \frac{2\chi^4-5\chi^3-5\chi+10}{5\chi}\log|1-\chi|\,, \\
G_T^{(1)}(\chi) &= -\frac{\chi^2\left(2\chi^2-3\chi+3\right)}{2(\chi-1)^2} + \frac{\chi^4\left(\chi^2-3\chi+3\right)}{(\chi-1)^3}\log|\chi| - \chi^3\log|1-\chi| \\
G_A^{(1)}(\chi) &= \frac{\chi\left(-2\chi^3+5\chi^2-3\chi+2\right)}{2(\chi-1)^2} \\
&\quad + \frac{\chi^3\left(\chi^3-4\chi^2+6\chi-4\right)}{(\chi-1)^3}\log|\chi| - (\chi^3-\chi^2-1)\log|1-\chi|
\end{aligned}$$

• OPE limit $\chi \to 0$

$$G_S^{(1)}(\chi) = \frac{1}{30}\chi^2\big(-60\log|\chi| - 43\big) + \frac{1}{30}\chi^3\big(-60\log|\chi| - 73\big) + \dots$$
$$G_T^{(1)}(\chi) = -\frac{3}{2}\chi^2 - \frac{3}{2}\chi^3 + \frac{1}{12}\chi^4\big(-36\log|\chi| - 18\big) + \dots$$
$$G_A^{(1)}(\chi) = \frac{1}{6}\chi^3\big(24\log|\chi| + 7\big) + \frac{3}{4}\chi^4\big(8\log|\chi| + 5\big) + \dots$$

Dimensions of two-particle operators from OPE

$$G(\chi) = \sum_h c_h\, \chi^h\, F_h(\chi) = G^{(0)}(\chi) + \frac{1}{\sqrt{\lambda}}G^{(1)}(\chi) + \dots$$
$$F_h(\chi) \equiv {}_2F_1(h, h, 2h, \chi)$$

• disconnected part: leading $O(1)$ – gen. free fields – exchanged "2-particle" ops: $[\Phi\Phi]^S_{2n} \sim \Phi^a \partial_t^{2n} \Phi^a$,

$[\Phi\Phi]^T_{2n} \sim \Phi^{(a} \partial_t^{2n} \Phi^{b)}$, $[\Phi\Phi]^A_{2n+1} \sim \Phi^{[a} \partial_t^{2n+1} \Phi^{b]}$

- connected part: $\frac{1}{\sqrt{\lambda}}$ corrections to Δ and OPE coeffs
- complication: ops may mix – degeneracies at leading order

$[\Phi\Phi]^S_{2n}$ with $n \geq 1$ can mix with $\mathbb{F}\partial_t^{2n-2}\mathbb{F}$ and $\psi\partial_t^{2n-1}\psi$;
$[\Phi\Phi]^A_{2n+1}$ can mix with $\psi\partial_t^{2n}\psi$ in $(1, 10)$ of $SU(2) \times Sp(4)$

- $[\Phi\Phi]^T_{2n}$ – no mixing $O(t) = [\Phi\Phi]^T_{2n} \sim \Phi^{(a}\partial_t^{2n}\Phi^{b)}$

$$\Delta_{[\Phi\Phi]^T_{2n}} = 2 + 2n - \frac{2n^2+3n}{\sqrt{\lambda}} + O(\frac{1}{\lambda})$$

$n = 0$: protected $\Phi^{(a}\Phi^{b)}$; $n \geq 1$ – unprotected – long multiplet

- $n = 0$ exception: $\Phi^a\Phi^a$ does not mix

$$\Delta_{\Phi^a\Phi^a} = 2 - \frac{5}{\sqrt{\lambda}} + O(\frac{1}{\lambda}) \ , \qquad c_{\Phi\Phi[\Phi\Phi]^S_0} = \frac{2}{5} - \frac{43}{30\sqrt{\lambda}} + \dots$$

- large n limit of all dims – same asymptotic form

$$\Delta_{n\gg 1} = 2n - \frac{2n^2}{\sqrt{\lambda}} + \dots$$

4-point functions with AdS_5 fluctuations

$$\langle x^{i_1}(t_1)x^{i_2}(t_2)y^{a_1}(t_3)y^{a_2}(t_4)\rangle_{\mathrm{AdS}_2}$$

$$= \langle\!\langle\, \mathbb{F}_t^{i_1}(t_1)\, \mathbb{F}_t^{i_2}(t_2)\, \Phi^{a_1}(t_3)\, \Phi^{a_2}(t_4)\, \rangle\!\rangle = \delta^{i_1 i_2}\delta^{a_1 a_2}\frac{G(\chi)}{t_{12}^4 t_{34}^2}$$

$$\langle x^{i_1}(t_1)x^{i_2}(t_2)x^{i_3}(t_3)x^{i_4}(t_4)\rangle_{\mathrm{AdS}_2}$$

$$= \langle\!\langle\, \mathbb{F}_t^{i_1}(t_1)\, \mathbb{F}_t^{i_2}(t_2)\, \mathbb{F}_t^{i_3}(t_3)\, \mathbb{F}_t^{i_4}(t_4)\, \rangle\!\rangle = \frac{G^{i_1 i_2 i_3 i_4}(\chi)}{t_{12}^4 t_{34}^4}$$

$$G_{\mathrm{conn}}(\chi) = \tfrac{1}{\sqrt{\lambda}}\tfrac{2}{3\pi^2}G^{(1)}\,, \qquad G^{(1)} = -4\left[1 - \left(\tfrac{1}{2} - \tfrac{1}{\chi}\right)\ln|1-\chi|\right]$$

- dimensions of 2-particle ops in OPE

$$[\Phi^a \mathbb{F}_{it}]_n \sim \Phi^a \partial_t^n \mathbb{F}_{it}\,, \qquad \Delta = 3 + n - \tfrac{(n+1)(n+4)}{\sqrt{2\lambda}} + \dots$$

$$G^{i_1 i_2 i_3 i_4}(\chi) = G_S^{(1)} \delta^{i_1 i_2} \delta^{i_3 i_4} + G_A^{(1)} \left(\delta^{i_1 i_3} \delta^{i_2 i_4} - \delta^{i_1 i_4} \delta^{i_2 i_3} \right) + G_T^{(1)} \left(\delta^{i_1 i_3} \delta^{i_2 i_4} + \delta^{i_1 i_4} \delta^{i_2 i_3} - \tfrac{2}{3} \delta^{i_1 i_2} \delta^{i_3 i_4} \right)$$

$$G_S^{(1)}(\chi) = -\frac{\left(24\chi^8 - 90\chi^7 + 125\chi^6 - 76\chi^5 + 125\chi^4 - 306\chi^3 + 438\chi^2 - 288\chi + 72\right)}{9(\chi-1)^4} - \frac{2\left(4\chi^6 - \chi^5 - 6\chi + 12\right)}{3\chi} \log|1-\chi| + \frac{2\chi^4\left(4\chi^6 - 21\chi^5 + 45\chi^4 - 50\chi^3 + 30\chi^2 - 6\chi + 2\right)}{3(\chi-1)^5} \log|\chi| , \quad etc.$$

- ops in OPE: $[\mathbb{F}\mathbb{F}]^S_{2n} \sim \mathbb{F}_{ti} \partial_t^{2n} \mathbb{F}_{it}$, $[\mathbb{F}\mathbb{F}]^T_{2n} \sim \mathbb{F}_{t(i} \partial_t^{2n} \mathbb{F}_{j)t}$, etc.
- symmetric traceless $[\mathbb{F}\mathbb{F}]^T_{2n}$ not expected to mix:

$$\Delta_{[\mathbb{F}\mathbb{F}]^T_{2n}} = 4 + 2n - \frac{2n^2+7n+5}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)$$

- $\Delta_{[\mathbb{F}\mathbb{F}]^T_{2n}} = \Delta_{[\Phi\mathbb{F}]_{2n+1}} = \Delta_{[\Phi\Phi]^T_{2n+2}}$ ops in same long multiplet

Correlators on standard Wilson loop

• no scalar coupling in WL: $SO(6)\ \times SO(2,1) \times SO(3)$, no susy

weak coupling ($A = 1, ..., 6$)

$\langle\langle \Phi_A(t_1)\Phi_B(t_2)\rangle\rangle = \delta_{AB}\frac{C'_\Phi}{(t_{12})^{2\Delta}}\,, \qquad \Delta = 1 - \frac{\lambda}{8\pi^2} + \cdots$

• C'_Φ scheme dependent but $C'_{\rm F}$ is definite function of λ:

displacement op. dual to x^i is $\mathrm{F}_{ti} = iF_{ti}$: $\Delta_F = 2$ (protected)

• leading correction to Δ: [Alday,Maldacena:07]

rederived from integrability of an $SO(6)$ spin chain [Correa:2018]

• aim: CFT_1 correlators at strong coupling using $\mathrm{AdS}_2/\mathrm{CFT}_1$

same AdS_2 minimal surface and same (3+5) + 8 fluctuations

but Dirichlet b.c. for $y^a \to$ Neumann b.c.

- supersymmetric WML expansion is around a point in S^5

$$Y_a = \frac{y_a}{1+\frac{1}{4}y^2}, \qquad Y_6 = \sqrt{1 - Y_a Y_a} = \frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2}, \qquad Y_A Y_A = 1$$

$$ds^2_{S^5} = dY_A dY_A = \frac{dy_a dy_a}{(1+\frac{1}{4}y^2)^2}$$

- Neumann b.c.: integration over point in S^5 restoring $SO(6)$
- massless AdS_2 scalar: $\Delta(\Delta-1)=0 \to$ D: $\Delta=1,$ N: $\Delta=0$

$$N: \qquad \Delta = \frac{5}{\sqrt{\lambda}} + \frac{d_2}{(\sqrt{\lambda})^2} + ...$$

$$\langle\!\langle \Phi_{A_1}(t_1)\cdots\Phi_{A_n}(t_n)\rangle\!\rangle = \langle Y_{A_1}(t_1)\cdots Y_{A_n}(t_n)\rangle_{\text{AdS}_2}$$

$$L_B = \sqrt{\det(g_{\mu\nu} + \partial_\mu Y_A \partial_\nu Y_A)} = \sqrt{g}\,(1 + L_2 + L_4 + \cdots)\,,$$

$$L_2 = \tfrac{1}{2}\partial^\mu Y_A \partial_\mu Y_A, \qquad L_4 = \tfrac{1}{8}\,(\partial^\mu Y_A \partial_\mu Y_A)^2 - \tfrac{1}{4}(\partial^\mu Y_A \partial_\mu Y_B)^2$$

$$Z = \int \mathcal{D}Y\,\delta(Y^2 - 1)\,e^{-T\int d^2\sigma\sqrt{g}[L_2(Y)+L_4(Y)+\ldots]}\,, \qquad T = \tfrac{\sqrt{\lambda}}{2\pi}$$

embedding coordinates:

$$Y^A = n^A + \zeta^A + \ldots\,, \quad n^A = \text{const}, \quad n^A n^A = 1, \quad n^A \zeta^A = 0$$

$$Y^A = \sqrt{1-\zeta^2}\,n^A + \zeta^A = \left[1 - \tfrac{1}{2}\zeta^2 + \ldots\right]n^A + \zeta^A\,, \qquad n^A\zeta^A = 0$$

$$Z = \int [dn] \int \mathcal{D}\zeta\,\delta\left(n_A\zeta_A\right)e^{-T\int d^2\sigma\sqrt{g}[L_2+L_4+\ldots]}$$

$$L_2 = \tfrac{1}{2}\partial^\mu\zeta^A\,\partial_\mu\zeta^A$$

$$L_4 = \tfrac{1}{2}\,\zeta^A\zeta^B\,\partial^\mu\zeta^A\partial_\mu\zeta^B + \tfrac{1}{8}\,(\partial^\mu\zeta^A\,\partial_\mu\zeta^A)^2 - \tfrac{1}{4}(\partial^\mu\zeta^A\,\partial_\mu\zeta^B)^2$$

Neumann propagator in AdS_2 (on half-plane $z > 0$)

$$\langle \zeta^A(\sigma)\zeta^B(\sigma')\rangle = P^{AB}(n)\, G_N(\sigma,\sigma'), \qquad P^{AB} = \delta^{AB} - n^A n^B$$

$$G_N(\sigma,\sigma') = -\tfrac{1}{4\pi}\Big(\log[(t-t')^2 + (z-z')^2] + \log[(t-t')^2 + (z+z')^2]\Big)$$

- bulk-to-boundary propagator

$$G_N(t,z;t') \equiv G_N(t,z;t',0) = -\tfrac{1}{2\pi}\log[(t-t')^2 + z^2]$$

boundary-to-boundary propagator

$$G_N(t_1,t_2) \equiv G_N(t_1,0;t_2,0) = -\tfrac{1}{2\pi} N_{12}\,, \qquad N_{12} \equiv \log(t_{12}^2)$$

- averaging over S^5:

$$\langle n^A n^B\rangle = \tfrac{1}{6}\delta^{AB}\,, \qquad \langle n^A n^B n^C n^D\rangle = \tfrac{1}{48}\big(\delta^{AB}\delta^{CD} + \delta^{AC}\delta^{BD} + \delta^{AD}\delta^{BC}\big)$$

$$\langle P^{AB}\rangle = \tfrac{5}{6}\delta^{AB}\,, \qquad \langle P^{AB}P^{CD}\rangle = \tfrac{33}{48}\delta^{AB}\delta^{CD} + \tfrac{1}{48}\big(\delta^{AC}\delta^{BD} + \delta^{AD}\delta^{BC}\big)$$

Two-point function $\langle Y^A Y^B \rangle$

2-point f. of $Y^A(t) \equiv Y^A(t, z=0)$ by 1d conf invariance

$$\langle Y^A(t_1)\, Y^B(t_2)\rangle = \frac{C_Y\,\delta^{AB}}{|t_{12}|^{2\Delta}} = \delta^{AB} C_Y \Big[1 - \big(\tfrac{d_1}{\sqrt{\lambda}} + \tfrac{d_2}{(\sqrt{\lambda})^2} + ...\big)\ \log(t_{12}^2)$$
$$+ \big(\tfrac{d_1^2}{2\,(\sqrt{\lambda})^2} + ...\big) \log^2(t_{12}^2) + \cdots \Big]$$

$$\Delta = \frac{d_1}{\sqrt{\lambda}} + \frac{d_2}{(\sqrt{\lambda})^2} + \frac{d_3}{(\sqrt{\lambda})^3} + \cdots\ , \qquad\qquad d_1 = 5$$

normalization: $C_Y = \frac{1}{6}$

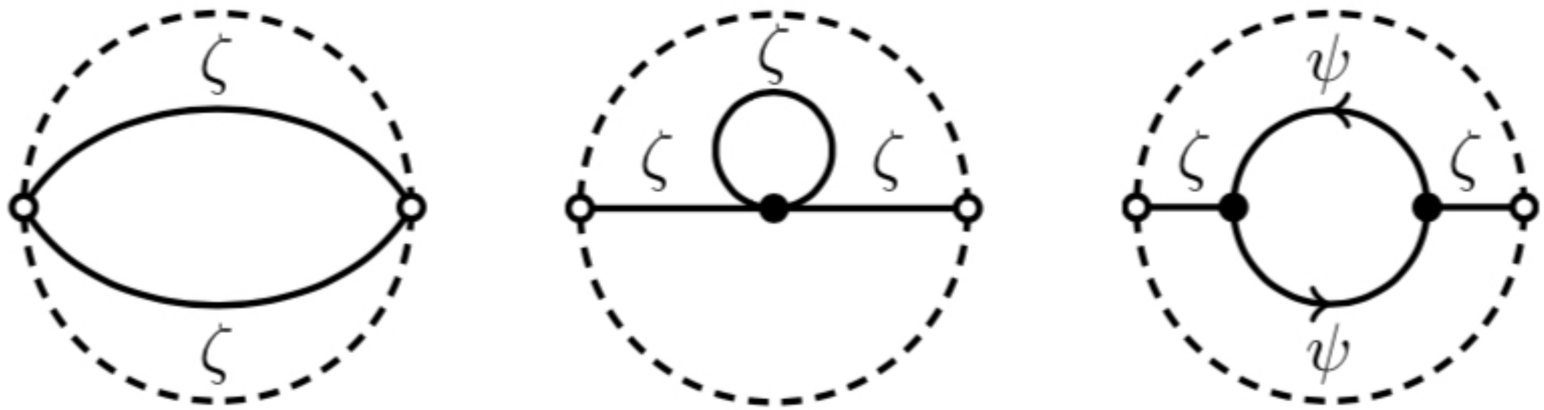
- leading order: $T^{-1} = \frac{2\pi}{\sqrt{\lambda}}$: $\quad J(J+4) = 5$ for $J = 1$

$$\langle Y^A(\sigma_1)\, Y^B(\sigma_2)\rangle = \langle \big[n^A + \zeta^A\big]\,\big[n^B + \zeta^B + \big]\rangle = \tfrac{1}{6}\delta^{AB}\,\big[1 + 5\,T^{-1} \mathrm{G_N}\big]$$

- subleading $\frac{1}{(\sqrt{\lambda})^2}$ order: $d_2 \log + \frac{d_1^2}{2} \log^2$

log from 1-loop graphs

$\log^2$ from tree + 1-loop: should exponentiate: $\frac{d_1^2}{2} = \frac{25}{2}$


ζ
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$\log^2$ terms come only from $\zeta^A\zeta^B\partial^\mu\zeta^A\partial_\mu\zeta^B$ vertex

use particular scheme with $\partial_\mu\partial'_\mu \mathrm{G_N}(\sigma,\sigma')\Big|_{\sigma=\sigma'} = \frac{1}{2\pi z^2}$ and

$\int \frac{dzdt}{z^2}\, \mathrm{G_N}(t,z;t_1)\, \mathrm{G_N}(t,z;t_2) = \frac{1}{4\pi}\, \log^2(t_{12}^2)$

Mixed correlator $\langle x^i x^j Y^A Y^B \rangle$

$\mathrm{F}_t^i \equiv iF_t^i$ dual to x^i: has interpretation of displacement operator

$\Delta = 2$ protected also in non-supersymmetric WL case

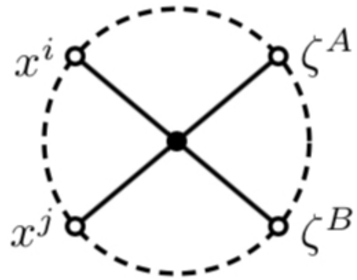
$$\langle\!\langle \mathrm{F}_t^i(t_1)\,\mathrm{F}_t^j(t_2)\rangle\!\rangle = \langle x^i(t_1)\,x^j(t_2)\rangle = \delta^{ij}\frac{C'_x}{(t_{12})^4}$$

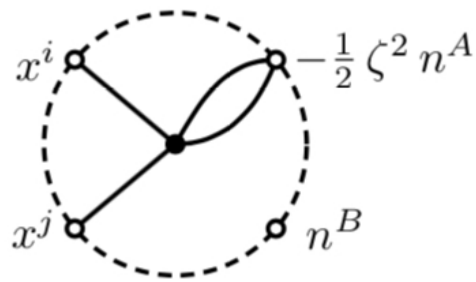
$$\langle\!\langle \mathrm{F}_t^i(t_1)\,\mathrm{F}_t^i(t_2)\Phi_A(t_3)\Phi_B(t_4)\rangle\!\rangle = \langle x^i(t_1)x^j(t_2)Y_A(t_3)Y_B(t_4)\rangle$$

$$= \tfrac{1}{6}\delta^{ij}\delta_{AB}\frac{C'_x}{(t_{12})^4\,(t_{34})^{2\Delta}}\,G(\chi)$$

$$G(\chi) = 1 + \tfrac{1}{\sqrt{\lambda}}G^{(1)} + \tfrac{1}{(\sqrt{\lambda})^2}G^{(2)}\cdots, \qquad \Delta = \frac{5}{\sqrt{\lambda}} + ...$$

connected contribution comes from $\partial x\partial x\partial Y\partial Y$ vertex


x^i
ζ^A
x^j
ζ^B


x^i
$-\frac{1}{2}\,\zeta^2\,n^A$
x^j
n^B

$$K_2(t,z;t') = \mathcal{C}_2\, \mathrm{K}_2(t,z;t')\ , \qquad \mathrm{K}_2(t,z;t') \equiv \Big[\frac{z}{(t-t')^2+z^2}\Big]^2$$

$$G_{\mathrm{N}}(t,z;t') = \mathcal{C}_{\mathrm{N}}\, \mathrm{N}(t,z;t')\ , \qquad \mathrm{N}(t,z;t') \equiv \log[(t-t')^2+z^2]$$

$$\mathcal{C}_2 = \frac{2}{3\pi}\ , \qquad \mathcal{C}_{\mathrm{N}} \equiv -\frac{1}{2\pi}$$

$$\frac{G_{\mathrm{conn}}(\chi)}{t_{12}^4\ t_{34}^{2\Delta}} = -5 \times \big(\tfrac{2\pi}{\sqrt{\lambda}}\big)^2\, \mathcal{C}_2\, (\mathcal{C}_{\mathrm{N}})^2\, \mathsf{Q}_{xy}$$

$$\mathsf{Q}_{xy} \equiv \int \frac{dtdz}{z^2}\Big[\partial\mathrm{K}_2(t_1)\partial\mathrm{K}_2(t_2)\,\partial\mathrm{N}(t_3)\partial\mathrm{N}(t_4)$$

$$-\partial\mathrm{K}_2(t_1)\partial\mathrm{N}(t_3)\,\partial\mathrm{K}_2(t_2)\partial\mathrm{N}(t_4) - \partial\mathrm{K}_2(t_1)\partial\mathrm{N}(t_4)\,\partial\mathrm{K}_2(t_2)\partial\mathrm{N}(t_3)\Big]$$

- doing bulk integral get:

$$G(\chi) = 1 + \frac{1}{(\sqrt{\lambda})^2} G^{(2)} + \ldots$$

$$G^{(2)} = -20\Big[1 - \big(\tfrac{1}{2} - \tfrac{1}{\chi}\big)\ \log(1-\chi)\Big]$$

• related to $\langle x^i x^j y^a y^b \rangle$ in supersymmetric (D) case:

$$G_{\rm N}^{(2)} = 5\, G_{\rm D}^{(1)} \ , \qquad\qquad G_{\rm D}^{(1)} = -4\Big[1 - \left(\tfrac{1}{2} - \tfrac{1}{\chi}\right)\, \log(1-\chi)\Big]$$

• OPE interpretation of $G(\chi)$:
by $t_2 \leftrightarrow t_3$ get

$$\langle\!\langle {\rm F}^i_t(t_1)\, \Phi_A(t_2)\, {\rm F}^i_t(t_3)\, \Phi_B(t_4) \rangle\!\rangle = \tfrac{1}{6}\delta^{ij}\delta_{AB}\, \frac{C'_x}{(t_{12}\, t_{34})^{2+\Delta}}\ \left|\frac{t_{24}}{t_{13}}\right|^{2-\Delta}\ \mathbb{G}(\chi)$$

$$\mathbb{G}(\chi) \equiv \chi^{2+\Delta}\, G(\chi^{-1}) = \chi^{2+\Delta}\left(1 - \tfrac{20}{(\sqrt{\lambda})^2}\left[1 + (\chi - \tfrac{1}{2})\, \log \tfrac{1-\chi}{\chi}\right]\right)$$

$$\mathbb{G}(\chi) = \textstyle\sum_h c_h\, \chi^h\, {}_2F_1(h+2-\Delta, h-2+\Delta, 2\,h, \chi)$$

intermediate operator dimensions and coefficients c_h

$$h_2 = 2 + \tfrac{5}{\sqrt{\lambda}} - \tfrac{10-d_2}{(\sqrt{\lambda})^2} + \cdots\ , \qquad c_{h_2} = 1 - \tfrac{20}{(\sqrt{\lambda})^2} + \cdots$$

$n \geq 3$: operators ${\rm F}\partial^n\Phi$

$$h_n = n - \tfrac{(n+3)(n-4)}{2}\, \tfrac{1}{\sqrt{\lambda}} + ..., \qquad c_{h_n} = \left(-\tfrac{1}{4}\right)^n \tfrac{20}{3}\, \tfrac{n}{n-2}\, \tfrac{\sqrt{\pi}\,(n+1)!}{\Gamma(n-\frac{1}{2})}\, \tfrac{1}{\sqrt{\lambda}} + ...$$

4-point function $\langle Y^A Y^B Y^C Y^D \rangle$

$$\langle Y^A(t_1) Y^B(t_2) Y^C(t_3) Y^D(t_4) \rangle = \frac{1}{|t_{12}\ t_{34}|^{2\Delta}}\, G^{ABCD}(\chi)$$

$$G^{ABCD} = \tfrac{1}{36} G_S\, \delta^{AB}\delta^{CD} + G_T \Big[\delta^{AC}\delta^{BD} + \delta^{BC}\delta^{AD} - \tfrac{1}{3}\, \delta^{AB}\delta^{CD} \Big]$$
$$+ G_A \Big[\delta^{AC}\delta^{BD} - \delta^{BC}\delta^{AD} \Big]$$

$$\langle Y^A(t_1) Y^A(t_2) Y^B(t_3) Y^B(t_4) \rangle = \frac{1}{|t_{12} t_{34}|^{2\Delta}}\, G_S$$

$$Y^A = n^A + \zeta^A - \tfrac{1}{2} n^A \zeta^2 + \mathcal{O}(\zeta^4), \quad n_A \zeta_A = 0, \quad n_A n_A = 1$$

in singlet n^A dependence drops out: S^5 averaging is trivial

Leading-order contributions

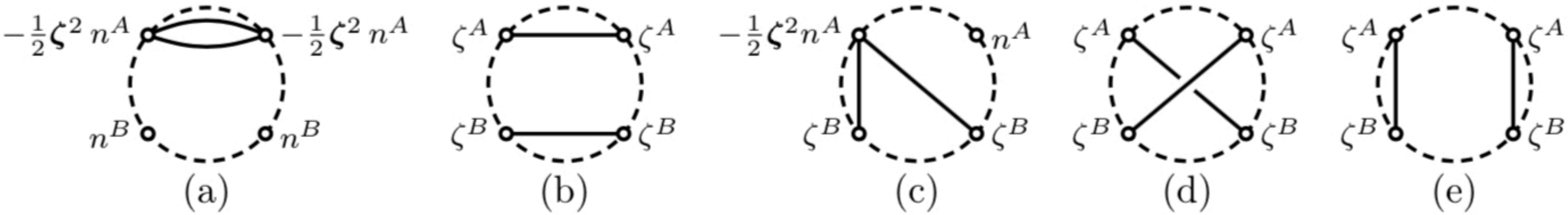
$$\langle Y^A(t_1) Y^A(t_2) Y^B(t_3) Y^B(t_4) \rangle = 1 + \tfrac{1}{\sqrt{\lambda}} Q^{(1)} + \tfrac{1}{(\sqrt{\lambda})^2} Q^{(2)} + \cdots$$

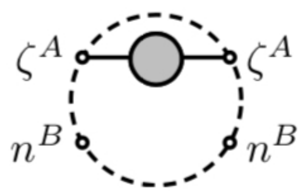
- tree-level terms $\langle \zeta_A \zeta_A n_B n_B \rangle + \langle n_A n_A \zeta_B \zeta_B \rangle$

$$Q^{(1)} = -5\big(\mathrm{N}_{12} + \mathrm{N}_{34}\big)\ , \qquad \mathrm{N}_{12} = \log t_{12}^2$$

correspond to leading term $(t_{12}t_{34})^{-2\Delta}$, $\Delta = \frac{5}{\sqrt{\lambda}} + \ldots$

- $\frac{1}{(\sqrt{\lambda})^2}$ order: tree-level diagrams + 1-loop prop. corrections





separating contributions to prefactor gives:

$$G_S(\chi) = 1 + \frac{1}{(\sqrt{\lambda})^2} G_S^{(2)} + \mathcal{O}(\tfrac{1}{(\sqrt{\lambda})^3}) \,,$$

$$G_S^{(2)} = 10 \log^2(1-\chi)$$

using $SO(6)$ crossing relations:

$$G_T = -\tfrac{3}{20}\, G_S(\chi) + \tfrac{9}{28}\Big[\chi^{2\Delta}\, G_S\big(\tfrac{1}{1-\chi}\big) + \big(\tfrac{\chi}{\chi-1}\big)^{2\Delta} G_S(1-\chi)\Big]$$

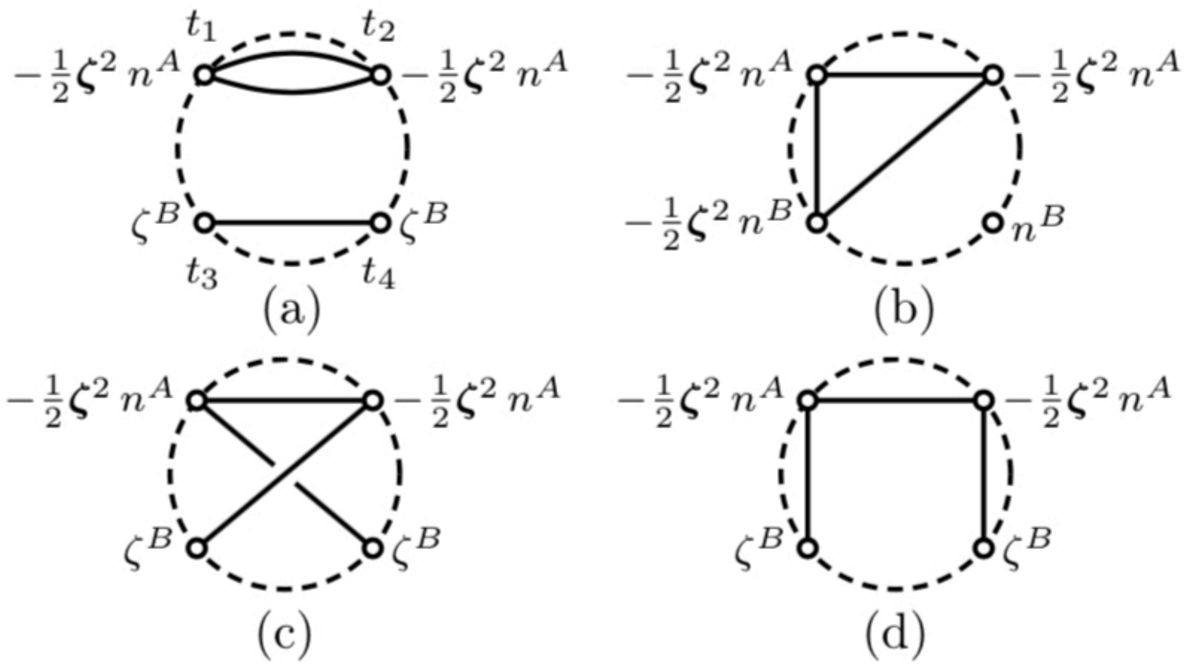
$$G_A = \tfrac{3}{5}\Big[\chi^{2\Delta}\, G_S\big(\tfrac{1}{1-\chi}\big) - \big(\tfrac{\chi}{\chi-1}\big)^{2\Delta} G_S(1-\chi)\Big]$$

$$G_T = \tfrac{3}{4} + \tfrac{9}{2\sqrt{\lambda}} \log \tfrac{\chi^2}{1-\chi} + \tfrac{3}{2(\sqrt{\lambda})^2}\big(9 \log^2 \tfrac{\chi^2}{1-\chi} + ...\big)$$

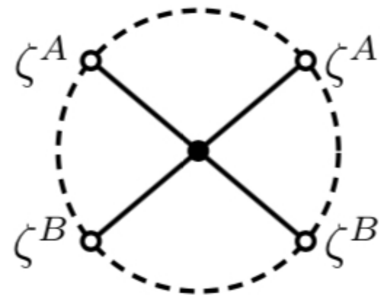
$$G_A = \tfrac{6}{\sqrt{\lambda}}\, \log(1-\chi) + \tfrac{6}{(\sqrt{\lambda})^2}\, \log(1-\chi)\, \big(4\, \log \tfrac{\chi^2}{1-\chi} + \tfrac{1}{5} d_2\big) + ...$$

Order $\frac{1}{(\sqrt{\lambda})^3}$ contributions

(i) "reducible": tree level diagrams (+ with prop. corrections)



(ii) "irreducible" (connected): tree-level with bulk 4-vertices



$$G_S = 1 + \tfrac{1}{(\sqrt{\lambda})^2} G_S^{(2)} + \tfrac{1}{(\sqrt{\lambda})^3} G_S^{(3)} + \mathcal{O}(\tfrac{1}{(\sqrt{\lambda})^4})$$

$$G_S^{(3)} = G_{S,\text{red}}^{(3)} + G_{S,\text{conn}}^{(3)}\,, \qquad G_{S,\text{red}}^{(3)} = G_{S,\log^2}^{(3)} + G_{S,\log^3}^{(3)}$$

$$G_{S,\log^2}^{(3)} = d_2\Big[-4\,(\mathrm{N}_{12}^2 + \mathrm{N}_{34}^2) + 4\log^2(1-\chi)\Big], \qquad \mathrm{N}_{ij} = \log(t_{ij}^2)$$

$$G_{S,\log^3}^{(3)} = \tfrac{25}{2}(\mathrm{N}_{12} + \mathrm{N}_{34})(\mathrm{N}_{13} + \mathrm{N}_{24} - \mathrm{N}_{14} - \mathrm{N}_{23})^2$$

$$+5\big[\mathrm{N}_{12}(\mathrm{N}_{13} - \mathrm{N}_{14})(\mathrm{N}_{23} - \mathrm{N}_{24}) + \mathrm{N}_{34}(\mathrm{N}_{13} - \mathrm{N}_{23})(\mathrm{N}_{14} - \mathrm{N}_{24})\big]$$

total G_S is conf inv: function of χ

connected part: compute bulk integrals with N-propagator applying ∂_{t_k} to reduce to D-propagator integrals

$$\mathrm{N}'(t_a) \equiv \partial_{t_a}\mathrm{N}(t_a) = 2\frac{t_a-t}{(t-t_a)^2+z^2} = \frac{2(t_a-t)}{z}\mathrm{K}_1(t_a)$$

$$\mathrm{N}(t_a) = \log\left[(t-t_a)^2+z^2\right], \qquad \mathrm{K}_1(t_a) = \frac{z}{(t-t_a)^2+z^2} = \tfrac{1}{2}\partial_z\mathrm{N}(t_a)$$

$$\partial_\mu\mathrm{N}'(t_a) = 2\,\epsilon_{\mu\nu}\partial_\nu\mathrm{K}_1(t_a)\,, \qquad \partial_\mu = (\partial_t,\partial_z)$$

$$\langle x^i(t_1)x^j(t_2)Y^A(t_3)Y^B(t_4)\rangle : \quad \partial_{t_3}\partial_{t_4}G_{\mathrm{N}} = -\frac{2}{t_{34}^2}\,G_{\mathrm{D}}(\chi)$$

$$\langle Y^A(t_1)...Y^D(t_4)\rangle : \quad \partial_{t_1}\partial_{t_2}\partial_{t_3}\partial_{t_4}G_{\mathrm{N}} = \frac{4}{t_{12}^2\,t_{34}^2}\,G_{\mathrm{D}}(\chi)+\Omega$$

acting on a function of cross-ratio $\chi = \frac{t_{12}t_{34}}{t_{13}t_{24}}$

$$t_{34}^2\,\partial_{t_3}\partial_{t_4}f(\chi) = -\mathscr{D}f(\chi)\,, \qquad \mathscr{D} \equiv \chi^2\,(1-\chi)\,\partial_\chi^2 - \chi^2\,\partial_\chi$$

$\mathscr{D}$ = conformal Casimir operator for $SO(1,2)$

$$\mathscr{D}\mathrm{F}_h = h(h-1)\mathrm{F}_h\,, \qquad \mathrm{F}_h = \chi^h\,F_h(\chi)\,, \quad F_h \equiv {}_2F_1(h,h,2h,\chi)$$

compute ∂_t derivatives, then integrate

• final result for $\frac{1}{(\sqrt{\lambda})^3}$ term in total G_S:

$$G_S = 80\left[\mathrm{Li}_3(\chi) + \mathrm{Li}_3\left(\tfrac{\chi}{\chi-1}\right) - \mathrm{Li}_2(\chi)\,\log(1-\chi)\right]$$

$$+\,40\,\log\tfrac{\chi}{1-\chi}\,\log^2(1-\chi) - 10\,\tfrac{\chi^2}{1-\chi}\,\log\chi$$

$$+\,5\left(5 - \tfrac{10}{\chi} - 2\,\chi\right)\,\log(1-\chi) - 50 = (30\log\chi + \tfrac{205}{6})\chi^2 + \ldots$$

similar results for G_T and G_A

• more complicated than in susy (D) case: polylogs
relation to 1d bootsrtrap?   interpolation to weak coupling?

Anomalous dimensions from OPE

$\frac{1}{(\sqrt{\lambda})^2}$ terms in $G_{S,T,A}$: OPE – extract anom dims

$$G_{S,T} = c_0\,\chi^{h_0}\,F_{h_0} + c_2\,\chi^{h_2}\,F_{h_2} + \ldots$$

$$G_A = c_1\,\chi^{h_1}\,F_{h_1} + c_3\,\chi^{h_3}\,F_{h_3} + \ldots$$

$$c_n = c_{n,0} + c_{n,1}\,\tfrac{1}{\sqrt{\lambda}} + c_{n,2}\,\tfrac{1}{(\sqrt{\lambda})^2} + \ldots$$

$$h_n = n + d_{n,1}\,\tfrac{1}{\sqrt{\lambda}} + d_{n,2}\,\tfrac{1}{(\sqrt{\lambda})^2} + \ldots$$

S-channel: $Y^A Y^A = 1$ identity operator: $h_{0,S} = 0$, $c_{0,S} = 1 + \ldots$

$Y\partial^2 Y:\qquad h_{2,S} = 2 + \ldots\,,\qquad c_{2,S} = \frac{10}{(\sqrt{\lambda})^2} + \ldots$

$T:\qquad h_{0,T} = \frac{12}{\sqrt{\lambda}} + \frac{12\,d_2}{5}\,\frac{1}{(\sqrt{\lambda})^2} + \ldots\,,\qquad c_{0,T} = \frac{3}{4} + \ldots$

$h_{2,T} = 2 + \ldots\,,\qquad c_{2,T} = \frac{5}{24\,(\sqrt{\lambda})^2} + \ldots,$

$A:\qquad h_{1,A} = 1 + \frac{8}{\sqrt{\lambda}} + \ldots\,,\qquad c_{1,A} = -\frac{6}{\sqrt{\lambda}} - \frac{6d_2}{5}\,\frac{1}{(\sqrt{\lambda})^2} + \ldots$

$h_{3,A} = 3 + \ldots\,,\qquad c_{3,A} = -\frac{8}{3}\,\frac{1}{(\sqrt{\lambda})^2} + \ldots$

Remarks and open questions:

• AdS_2 loop corrections including fermions? compute d_2

• intrinsic description of "induced" CFT_1 ?

"$\mathcal{N} = 8$ conformal QM" in WML case ?

non-local? (cf. SYK-like models [Gross,Rosenhaus:17])

possible derivation: 1d fermion rep for WL and integrate out A

• 1d analog of large spin expansion? semiclassical approxim

to explain universal large n behaviour of Δ of $\Phi\partial_t^n\Phi$, etc.?

• relation to integrability? how integrability of $\mathrm{AdS}_5 \times S^5$ string

is encoded in correlators in AdS_2 in static gauge?

connection to factorization of 2d S-matrix in l.c. gauge?

• extension to all orders in $\frac{1}{\sqrt{\lambda}}$?

relation to conformal bootstrap in $d = 1$?

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More on strong coupling expansion

string description: $AdS_5 \times S^5$ path integral on a disc

- WML: D b.c. for S^5 (fixed scalar position – point in S^5)
  WL: N b.c. for S^5 (no scalar coupling)
- leading term: minimal surface ending on line or circle – AdS_2

AdS_2 as homogeneous space: $\log Z \sim$ volume

line: $V = \frac{L}{\epsilon} \to 0$ after factorizing linear div: $\langle W^{(0)} \rangle = 1$

circle ($R = 1$): $V_{\mathrm{AdS}_2} = 2\pi\big(\frac{1}{\epsilon} - 1\big) \to -2\pi$

$\langle W^{(0)} \rangle$ non-trivial function of string tension $\frac{\sqrt{\lambda}}{2\pi}$

- leading $\sqrt{\lambda}$ term is universal

$$\langle W^{(\zeta)} \rangle \equiv e^{-F^{(\zeta)}(\lambda)}\,, \qquad F^{(\zeta)} = -\sqrt{\lambda} + F_1^{(\zeta)} + \mathcal{O}(\frac{1}{\sqrt{\lambda}})$$

1-loop term

• $\zeta = 1$: [Drukker, Gross, AT; Buchbinder, AT;...]

spectrum of fluctuations: 3 AdS_5 modes $m^2 = 2$;
5 S^5 modes $m^2 = 0$; 8 fermions $m^2 = 1$

• $\zeta = 0$: same spectrum, except for b.c. of S^5 modes
ratio of D/N massless scalars

$$\frac{\langle W^{(1)}\rangle}{\langle W^{(0)}\rangle} = \frac{e^{-F^{(1)}}}{e^{-F^{(0)}}} = \mathcal{N}_0^{-1} \Big[\frac{\det(-\nabla^2)_{\rm D}}{\det'(-\nabla^2)_{\rm N}}\Big]^{-5/2} \Big[1 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})\Big]$$

$\mathcal{N}_0$ is S^5 zero mode factor in N case: $\quad \mathcal{N}_0 = c_0\, (\sqrt{\lambda})^{5/2}$

$$F_1^{(0)} = F_1^{(1)} - 5\delta\Gamma = F_1^{(1)} + \tfrac{5}{2}\log(2\pi) - \tfrac{5}{2}\log\sqrt{\lambda} + \tfrac{5}{2}\log k$$

while exact gauge-theory prediction expanded at $\lambda \gg 1$

$$F_{1\,\rm tot}^{(1)} = \tfrac{1}{2}\log(2\pi) - \log 2 + \tfrac{3}{2}\log\sqrt{\lambda}$$

$\frac{3}{2}\log\sqrt{\lambda}$ is normalization of Möbius symmetry

3 zero modes on disc [Drukker, Gross]

$\log 2$ difference understood recently [Medina-Rincon, AT, Zarembo]

• for standard WL at strong coupling

$$F^{(0)}_{1\,\rm tot} = F^{(1)}_{1\,\rm tot} + \tfrac{5}{2}\log(2\pi) + \log\mathcal{N}_0 = -\log\sqrt{\lambda} + \log(4\pi^3 k^{5/2})$$

• thus $\tilde{F}^{(\zeta)} \equiv \log\langle W^{(\zeta)}\rangle = -F^{(\zeta)}_{\rm tot}$= same $\sqrt{\lambda}$ at leading order

but subleading $\tilde{F}^{(0)}_1 > \tilde{F}^{(1)}$

in agreement with 1d analog of F-theorem

General ζ case

$\langle W^{(\zeta)}(\lambda)\rangle$ expanded at $\lambda \gg 1$

should interpolate between $\zeta = 1$ and $\zeta = 0$ results

• proposal for string description of non-conformal case: [PS]

start with WL case in static gauge $x^0 = \tau,\ z = \sigma$

induced AdS$_2$ metric $ds^2 = \frac{1}{\sigma^2}(d\tau^2 + d\sigma^2),\ \ \partial_z Y^a\big|_{z\to 0} = 0$

perturb string action $I_0 = T\int d\tau d\sigma\big(\frac{1}{2}\sqrt{h}h^{mn}\partial_m Y^a \partial_n Y^a + ...\big)$

$$I_{\varkappa} = I_0 - \varkappa T\int d\tau\, Y_6$$

$$Y_6 = \sqrt{1 - Y_a Y_a} = 1 - \tfrac{1}{2}Y_a Y_a + ...\,, \qquad T = \frac{\sqrt{\lambda}}{2\pi}$$

$\varkappa = 0 \to \zeta = 0$ and $\varkappa = \infty \to \zeta = 1$

like ζ here $\varkappa$ will run with 2d scale

• variation of $I_{\varkappa}$: $\nabla^2 Y_a = 0$, i.e. near AdS$_2$ boundary

$Y^a = z^{\Delta_+}u^a + z^{\Delta_-}v^a + \mathcal{O}(z^2) = zu^a + v^a + \mathcal{O}(z^2)$

with the mixed (Robin) boundary condition

$$(-\partial_z + \varkappa) y^a\Big|_{z=0} = 0\,, \quad \text{i.e.} \qquad - u^a + \varkappa v^a = 0$$

• special case of "open-string tachyon" coupling:

$\delta I_{\rm b} = \Lambda \int d\tau\, \mathcal{T}_{\rm b}(Y)\,, \qquad \Lambda \mathcal{T}_{\rm b} = \mu\big[\mathcal{T} - \log \frac{\Lambda}{\mu} (\alpha' D^2 + ...)\mathcal{T} + ...\big]$

$\beta_{\mathcal{T}} = \mu \frac{d\mathcal{T}}{d\mu} = -\mathcal{T} - \alpha' D^2 \mathcal{T} + ...\,, \qquad \alpha' = \frac{R^2}{\sqrt{\lambda}}$

D^2 = Laplacian on S^5: for $\mathcal{T} = \varkappa Y_6$ and small Y_a

$D^2 Y_6 = (\partial_Y^2 + ...)(-\frac{1}{2} Y_a Y_a + ...) = -5 + ...$

$$\beta_\varkappa = \mu \frac{d\varkappa}{d\mu} = \big(-1 + \frac{5}{\sqrt{\lambda}} + ...\big) \varkappa + ...$$

• Φ_6 perturbation near $\zeta = 0$ corresponds to $Y_6 = 1 - \frac{1}{2} Y_a^2 + ..$

dimension $\Delta - 1 = \frac{d\beta_\varkappa}{d\varkappa}$ then gives $\Delta(0) = \frac{5}{\sqrt{\lambda}} + ...$

near $\zeta = 1$: $\beta_\varkappa \to -\beta_\varkappa$, i.e. $\Delta - 1 = -(-1 + \frac{5}{\sqrt{\lambda}} + ...)$