

Brief introduction

The genus two four graviton amplitude in type II string theory

Modular graph functions for the $D^8 \mathcal{R}^4$ term

Varying the Beltrami differentials

The eigenvalue equation for some modular graphs

Eigenvalue equation for genus two modular graphs

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Outline of the talk

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- Multiloop string amplitudes provide useful insight into the structure of terms in the effective action of string theory, which encodes the dynamics of the massless modes of the theory.
- It yields S–matrix elements which contain terms both analytic as well as non–analytic in the external momenta of the particles.
- Terms analytic in the external momenta arise from the integration over the interior of the moduli space of the Riemann surface.
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- Calculating amplitudes becomes progressively difficult as one considers higher genus string amplitudes.
- Beyond tree level, one has to integrate over the geometric moduli of the Riemann surface which is non-trivial.
- At genus one, in order to calculate the analytic terms in the low momentum expansion, it is very useful to obtain eigenvalue equations which the modular invariant integrand satisfies.
- This helps us not only to have an understanding of the detailed structure of the integrand, but also to calculate the integral over moduli space.

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- At every genus if one considers the analytic terms, the integrand at a fixed order in the derivative expansion can be described diagrammatically by graphs, referred to as modular graph functions.
- Roughly, the vertices of the graphs are the positions of insertions of the vertex operators on the worldsheet, while the links are given by the scalar Green function connecting the vertices.
- These graphs depend on the moduli of the worldsheet and transform with fixed weights under $Sp(2g, \mathbb{Z})$ transformations for the genus g Riemann surface, such that the integrand is $Sp(2g, \mathbb{Z})$ invariant.

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- Our aim is to understand certain properties of some graphs at genus two.
- We shall consider the low momentum expansion of the genus two four graviton amplitude in type II superstring theory. The integrand is simpler than other string theories, thanks to the maximal supersymmetry the type II theory enjoys.

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- The genus two four graviton amplitude is the same in type IIA and IIB string theory ([Green, Kwon, Vanhove](#)).
- It is given by ([D'Hoker, Phong](#); [Berkovits](#); [Berkovits, Mafra](#))

$$\mathcal{A} = \frac{\pi}{64} \kappa_{10}^2 e^{2\phi} \mathcal{R}^4 \int_{\mathcal{M}_2} \frac{|d^3\Omega|^2}{(\det Y)^3} \mathcal{B}(s, t, u; \Omega, \bar{\Omega}),$$

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- $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$.
- The period matrix is given by $\Omega = X + iY$, where X, Y are matrices with real entries.
- The measure is

$$|d^3\Omega|^2 = \prod_{I \leq J} id\Omega_{IJ} \wedge d\bar{\Omega}_{IJ}.$$

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- The dynamics is contained in

$$\mathcal{B}(\mathbf{s}, t, \mathbf{u}; \Omega, \bar{\Omega}) = \int_{\Sigma^4} \frac{|\mathcal{Y}|^2}{(\det Y)^2} e^{-\alpha' \sum_{i < j} k_i \cdot k_j G(z_i, z_j)/2},$$

where each factor of Σ represents an integral over the genus two worldsheet.

- The string Green function is given by

$$G(z, w) = -\ln|E(z, w)|^2 + 2\pi Y_{IJ}^{-1} \left(\text{Im} \int_z^w \omega_I \right) \left(\text{Im} \int_z^w \omega_J \right),$$

where $Y_{IJ}^{-1} = (Y^{-1})_{IJ}$, $E(z, w)$ is the prime form and ω_I ($I = 1, 2$) are the abelian differential one forms.

- Finally,

$$3\mathcal{Y} = (t - u)\Delta(1, 2) \wedge \Delta(3, 4) + (s - t)\Delta(1, 3) \wedge \Delta(4, 2) \\ + (u - s)\Delta(1, 4) \wedge \Delta(2, 3),$$

where the bi-holomorphic form is given by

$$\Delta(i, j) \equiv \Delta(z_i, z_j) dz_i \wedge dz_j = \epsilon_{IJ} \omega_I(z_i) \wedge \omega_J(z_j).$$

- The Mandelstam variables are given by $s = -\alpha'(k_1 + k_2)^2/4$, $t = -\alpha'(k_1 + k_4)^2/4$, $u = -\alpha'(k_1 + k_3)^2/4$, where $\sum_i k_i = 0$ and $k_i^2 = 0$.

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- The amplitude is conformally invariant, as it is invariant under

$$G(z, w) \rightarrow G(z, w) + c(z) + c(w)$$

even though the string Green function $G(z, w)$ is not.

- To consider the analytic terms in the low momentum expansion, define

$$\mathcal{B}(s, t, u; \Omega, \bar{\Omega}) = \sum_{p, q=0}^{\infty} \mathcal{B}^{(p, q)}(\Omega, \bar{\Omega}) \frac{\sigma_2^p \sigma_3^q}{p! q!}$$

where

$$\sigma_n = s^n + t^n + u^n.$$

- Thus $\mathcal{B}^{(p, q)}(\Omega, \bar{\Omega})$ is a sum of various graphs with distinct topologies. Each of them involves factors of $G(z, w)$ in the integrand and hence is not generically conformally invariant, even though it is modular invariant.

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- Of course, the total contribution from all the graphs is conformally invariant.
- It is natural to consider contributions coming from graphs each of which is conformally as well as modular invariant.

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- This is obtained by considering

$$\mathcal{B}(s, t, u; \Omega, \bar{\Omega}) = \int_{\Sigma^4} \frac{|\mathcal{Y}|^2}{(\det Y)^2} e^{-\alpha' \sum_{i < j} k_i \cdot k_j \mathcal{G}(z_i, z_j)/2}$$

and performing the low energy expansion, where $\mathcal{G}(z, w)$ is the conformally invariant Arakelov Green function.

- To define the Arakelov Green function, consider the Kahler form

$$\kappa = \frac{1}{4} Y_{IJ}^{-1} \omega_I \wedge \bar{\omega}_J,$$

which satisfies

$$\int_{\Sigma} \kappa = 1,$$

on using the Riemann bilinear relation

$$\int_{\Sigma} \omega_I \wedge \bar{\omega}_J = 2Y_{IJ}.$$

- The Arakelov Green function is defined by

$$\mathcal{G}(z, w) = G(z, w) - \gamma(z) - \gamma(w) + \gamma_1,$$

where

$$\gamma(z) = \int_{\Sigma_w} \kappa(w) G(z, w),$$

and

$$\gamma_1 = \int_{\Sigma} \kappa(z) \gamma(z).$$

- Defining the dressing factor

$$(z_1, \bar{z}_2) = Y_{IJ}^{-1} \omega_I(z_1) \overline{\omega_J(z_2)},$$

we obtain the useful relation

$$\int_{\Sigma_z} \mu(z) \mathcal{G}(z, w) = 0,$$

where $\mu(z) = (z, \bar{z})$.

- Let us consider the modular graphs that arise at low orders in the momentum expansion.

- The $D^4\mathcal{R}^4$ term is given by (D'Hoker,Gutperle,Phong)

$$\mathcal{B}^{(1,0)}(\Omega, \bar{\Omega}) = \frac{1}{2} \int_{\Sigma^4} \frac{|\Delta(1,2) \wedge \Delta(3,4)|^2}{(\det Y)^2} = 32.$$

- The $D^6 \mathcal{R}^4$ term is given by

$$\begin{aligned}
 \mathcal{B}^{(0,1)}(\Omega, \bar{\Omega}) &= -\frac{1}{3} \int_{\Sigma^4} \frac{|\Delta(1,2) \wedge \Delta(3,4) - \Delta(1,4) \wedge \Delta(2,3)|^2}{(\det Y)^2} \\
 &\quad \times \left(\mathcal{G}(z_1, z_2) + \mathcal{G}(z_3, z_4) - \mathcal{G}(z_1, z_3) - \mathcal{G}(z_2, z_4) \right) \\
 &= 16 \int_{\Sigma^2} \prod_{i=1}^2 d^2 z_i \mathcal{G}(z_1, z_2) P(z_1, z_2),
 \end{aligned}$$

where

$$P(z_1, z_2) = (z_1, \bar{z}_2)(z_2, \bar{z}_1).$$

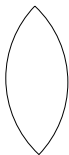
- This graph is given by the Kawazumi–Zhang invariant and satisfies an eigenvalue equation.
(D'Hoker, Green, Pioline, Russo)
- All modular graphs are given by skeleton graphs with links given by Arakelov Green function, along with dressing factors involving the integrated vertices.

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- All modular graphs are given by skeleton graphs with links given by Arakelov Green function, along with dressing factors involving the integrated vertices.

- The $D^8\mathcal{R}^4$ term is given by

$$\mathcal{B}^{(2,0)}(\Omega, \bar{\Omega}) = \frac{1}{4} \int_{\Sigma^4} \frac{|\Delta(1, 2) \wedge \Delta(3, 4)|^2}{(\det Y)^2} \\ \times \left(\mathcal{G}(z_1, z_4) + \mathcal{G}(z_2, z_3) - \mathcal{G}(z_1, z_3) - \mathcal{G}(z_2, z_4) \right)^2.$$

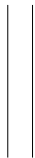
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(i)



(ii)



(iii)

- We denote

$$\mathcal{B}^{(2,0)}(\Omega, \bar{\Omega}) = \sum_{i=1}^3 \mathcal{B}_i^{(2,0)}(\Omega, \bar{\Omega}),$$

where we define $\mathcal{B}_i^{(2,0)}(\Omega, \bar{\Omega})$ next.

- We have that

$$\begin{aligned} \mathcal{B}_1^{(2,0)}(\Omega, \bar{\Omega}) &= \int_{\Sigma^4} \frac{|\Delta(1,2) \wedge \Delta(3,4)|^2}{(\det Y)^2} \mathcal{G}(z_1, z_4)^2 \\ &= 4 \int_{\Sigma^2} \prod_{i=1}^2 d^2 z_i \mathcal{G}(z_1, z_2)^2 Q_1(z_1, z_2), \end{aligned}$$

where

$$Q_1(z_1, z_2) = \mu(z_1)\mu(z_2).$$

- We have that

$$\begin{aligned}
 \mathcal{B}_2^{(2,0)}(\Omega, \bar{\Omega}) &= -2 \int_{\Sigma^4} \frac{|\Delta(1,2) \wedge \Delta(3,4)|^2}{(\det Y)^2} \mathcal{G}(z_1, z_4) \mathcal{G}(z_1, z_3) \\
 &= 4 \int_{\Sigma^3} \prod_{i=1}^3 d^2 z_i \mathcal{G}(z_1, z_2) \mathcal{G}(z_1, z_3) \mu(z_1) P(z_2, z_3).
 \end{aligned}$$

- We have that

$$\begin{aligned}
 \mathcal{B}_3^{(2,0)}(\Omega, \bar{\Omega}) &= \int_{\Sigma^4} \frac{|\Delta(1, 2) \wedge \Delta(3, 4)|^2}{(\det Y)^2} \mathcal{G}(z_1, z_4) \mathcal{G}(z_2, z_3) \\
 &= \int_{\Sigma^4} \prod_{i=1}^4 d^2 z_i \mathcal{G}(z_1, z_4) \mathcal{G}(z_2, z_3) P(z_1, z_2) P(z_3, z_4).
 \end{aligned}$$

- Our aim is to obtain eigenvalue equation(s) satisfied by genus two modular graphs on moduli space.
- Variations of the moduli are captured by variations of the Beltrami differentials.
- The holomorphic deformation with respect to the Beltrami differential μ is given by

$$\delta_\mu \phi = \frac{1}{2\pi} \int_\Sigma d^2 w \mu_{\bar{w}}^w \delta_{w\bar{w}} \phi.$$

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- The relevant formulae can be derived using the known relations for the variations of the abelian differentials, period matrix and the prime form.

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- The relevant formulae can be derived using the known relations for the variations of the abelian differentials, period matrix and the prime form.

- A useful formula for both variations is

$$\delta_{ww} \left(Y_{IJ}^{-1} \omega_J(z) \right) = -Y_{IJ}^{-1} \omega_J(w) \partial_z \partial_w \mathcal{G}(w, z).$$

- We also use the formulae

$$\begin{aligned}\bar{\partial}_w \partial_z \mathcal{G}(z, w) &= 2\pi \delta^2(z - w) - \pi(z, \bar{w}), \\ \bar{\partial}_z \partial_z \mathcal{G}(z, w) &= -2\pi \delta^2(z - w) + \frac{\pi}{2} \mu(z)\end{aligned}$$

very often.

- For the holomorphic variations, we use

$$\begin{aligned} \delta_{ww} \mathcal{G}(z_1, z_2) &= -\partial_w \mathcal{G}(w, z_1) \partial_w \mathcal{G}(w, z_2) \\ &\quad - \frac{1}{4} \int_{\Sigma} d^2 u(w, \bar{u}) \partial_w \mathcal{G}(w, u) \partial_u \left(\mathcal{G}(u, z_1) + \mathcal{G}(u, z_2) \right). \end{aligned}$$

- For the anti-holomorphic variations, we also use

$$\begin{aligned} \bar{\delta}_{uu} \partial_w \mathcal{G}(w, z) &= \pi(w, \bar{u}) \left(\bar{\partial}_u \mathcal{G}(u, z) - \frac{1}{2} \bar{\partial}_u \mathcal{G}(u, w) \right) \\ &\quad + \frac{\pi}{4} \int_{\Sigma} d^2 x(x, \bar{u})(w, \bar{x}) \bar{\partial}_u \mathcal{G}(u, x). \end{aligned}$$

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- This leads to manifestly conformally covariant expressions.

- By varying the Beltrami differentials, we perform the mixed variation for each of the three modular graphs.
- For each graph $\mathcal{B}_i^{(2,0)}$, we obtain contributions involving four, two and zero derivatives ($\mathcal{B}_1^{(2,0)}$ has no contribution involving zero derivatives).
- They act on the Arakelov Green functions.

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- Schematically, the contributions with four derivatives are of the form $\partial_w^2 \bar{\partial}_u^2$, $\partial_w^2 \bar{\partial}_u \bar{\partial}_z + h.c.$, and $\partial_w \bar{\partial}_u \partial_{z_i} \bar{\partial}_{z_j} + h.c.$.
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- First let us consider the terms involving four derivatives that arise from the mixed variations of the graphs $\mathcal{B}_i^{(2,0)}$.
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- Varying $\mathcal{B}_1^{(2,0)}$, we get that

$$\frac{1}{8} \bar{\delta}_{uu} \delta_{ww} \mathcal{B}_1^{(2,0)} = \sum_{\alpha=A}^E \Phi_{1,\alpha} + \dots$$

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- Varying $\mathcal{B}_3^{(2,0)}$, we get that

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$$-\frac{1}{8} \bar{\delta}_{uu} \delta_{ww} \mathcal{B}_2^{(2,0)} = \sum_{\alpha=A}^E \Phi_{2,\alpha} + \dots$$

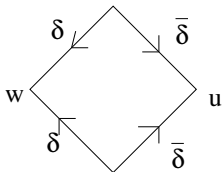
- Varying $\mathcal{B}_3^{(2,0)}$, we get that

$$\frac{1}{2} \bar{\delta}_{uu} \delta_{ww} \mathcal{B}_3^{(2,0)} = \sum_{\alpha=A}^E \Phi_{3,\alpha} + \dots$$

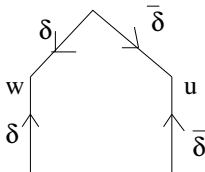
- Here $\Phi_{i,\alpha}$ ($\alpha = A, B, C, D, E$) involves the various contributions with four derivatives, and we have ignored other contributions.
- It is very useful to denote the various contributions by skeleton graphs. We do not include the dressing factors for the sake of brevity.

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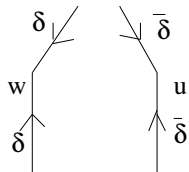
- The skeleton graphs for (i) $\Phi_{1,A}$, (ii) $\Phi_{2,A}$ and (iii) $\Phi_{3,A}$ are given by



(i)



(ii)



(iii)

- For example,

$$\begin{aligned} \Phi_{1,A} = & \int_{\Sigma^2} \prod_{i=1,2} d^2 z_i Q_1(z_1, z_2) \partial_w \mathcal{G}(w, z_1) \partial_w \mathcal{G}(w, z_2) \\ & \times \bar{\partial}_u \mathcal{G}(u, z_1) \bar{\partial}_u \mathcal{G}(u, z_2) \end{aligned}$$

on including the dressing factor.

- These graphs are of the form $\partial_w^2 \bar{\partial}_u^2$.

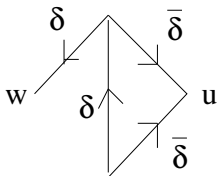
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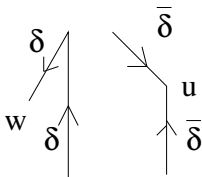
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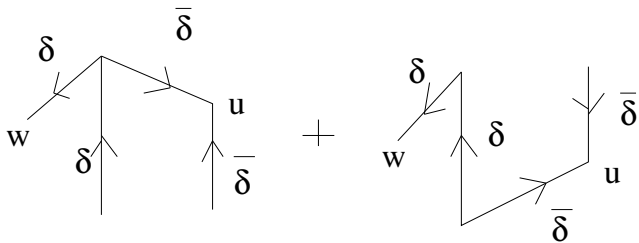
(i)



(ii)

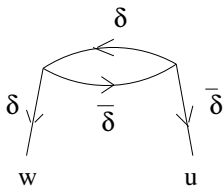
along with their hermitian conjugates.

- The skeleton graphs for $\Phi_{2,B}$ are given by

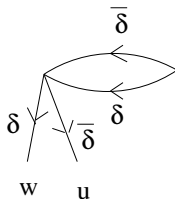


- These graphs are of the form $\partial_w^2 \bar{\partial}_u \bar{\partial}_z + h.c..$

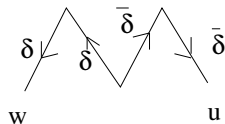
- The skeleton graphs for (i) $\Phi_{1,C}$, (ii) $\Phi_{1,D}$ and (iii) $\Phi_{1,E}$ are given by



(i)

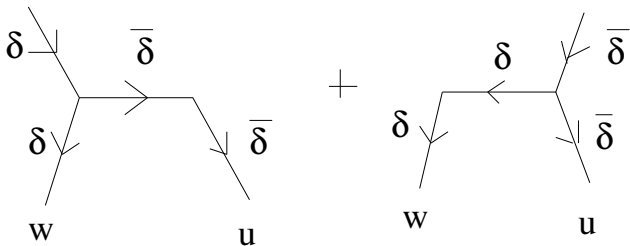


(ii)

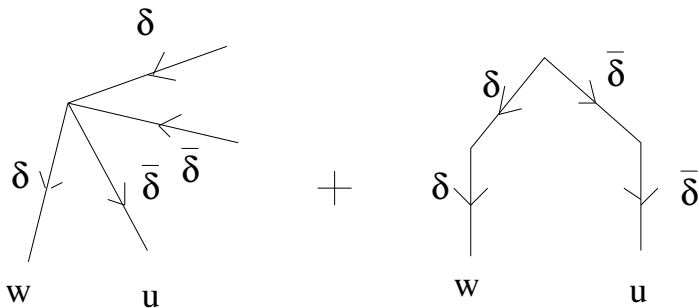


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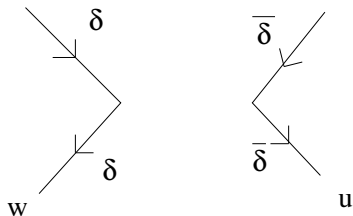
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- These graphs are of the form $\partial_w \bar{\partial}_u \partial_{z_i} \bar{\partial}_{z_j} + h.c. .$

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- However it is expected that certain linear combinations of these terms involving different $\mathcal{B}_i^{(2,0)}$ can potentially simplify, much like the analysis for genus one graphs.

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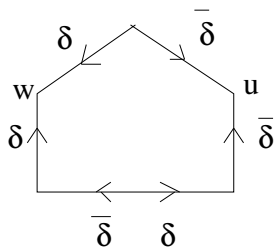
- Let us first consider the contributions that arise from varying $\mathcal{B}_1^{(2,0)}$ and $\mathcal{B}_2^{(2,0)}$.
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- Consider the auxiliary graph given by

$$\begin{aligned} \Phi_{12,A} = & \int_{\Sigma^3} \prod_{i=1,2,3} d^2 z_i \partial_w \mathcal{G}(w, z_1) \partial_w \mathcal{G}(w, z_2) \bar{\partial}_u \mathcal{G}(u, z_1) \\ & \times \bar{\partial}_u \mathcal{G}(u, z_3) \mu(z_1)(z_2, \bar{z}_3) \bar{\partial}_{z_2} \partial_{z_3} \mathcal{G}(z_2, z_3). \end{aligned}$$

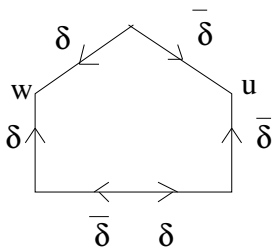
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- We get that

$$\Phi_{12,A} = \pi(2\Phi_{1,A} + \Phi_{2,A}).$$

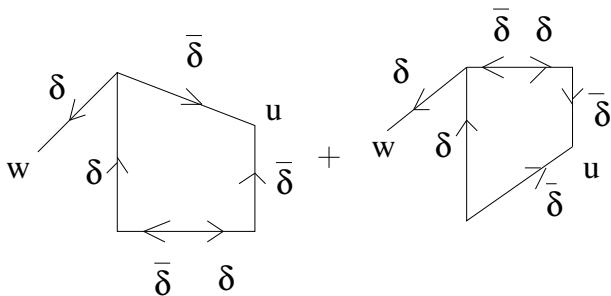
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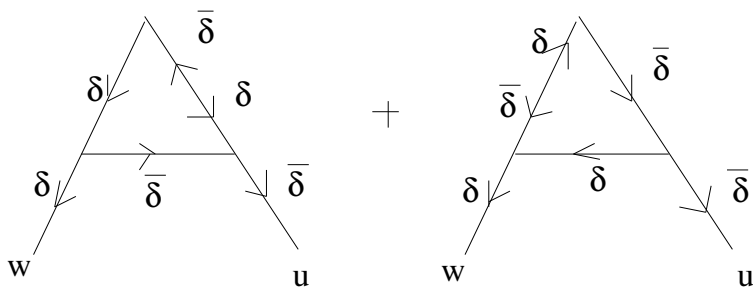
- From the auxiliary skeleton graph $\Phi_{12,B}$ given by (along with its hermitian conjugate)



we get that

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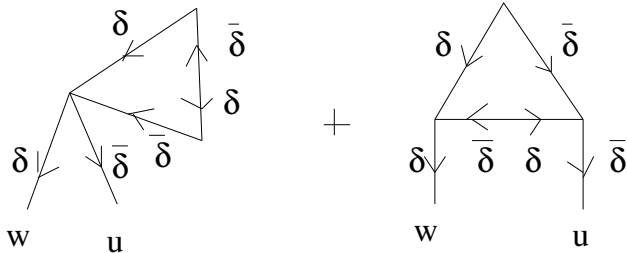
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$$\Phi_{12,C} = \pi(2\Phi_{1,C} + \Phi_{2,C}).$$

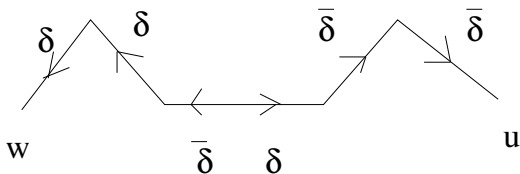
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$$\Phi_{12,D} = \pi(2\Phi_{1,D} + \Phi_{2,D}).$$

- From the auxiliary skeleton graph $\Phi_{12,E}$ given by



we get that

$$\Phi_{12,E} = -4\pi(2\Phi_{1,E} + \Phi_{2,E}).$$

- Crucially, we always end up with the expression proportional to

$$2\Phi_{1,\alpha} + \Phi_{2,\alpha}.$$

- Thus the mixed variation

$$\bar{\delta}_{uu}\delta_{ww}\left(\mathcal{B}_1^{(2,0)} - \frac{1}{2}\mathcal{B}_2^{(2,0)}\right)$$

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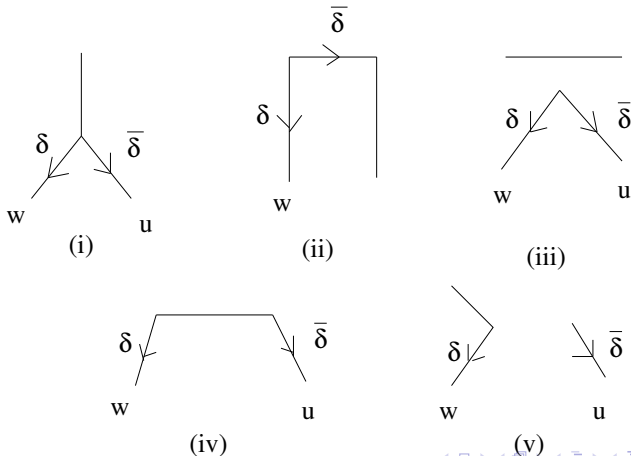
can be expressed in terms of these auxiliary graphs, as well as other contributions involving two or no derivatives.

- What is special about these auxiliary graphs?
- We can integrate a ∂ and a $\bar{\partial}$ by parts in each such graph to reduce it to contributions having only two derivatives.
- Thus the mixed variation of $\mathcal{B}_1^{(2,0)} - \mathcal{B}_2^{(2,0)}/2$ involves only contributions having at most two derivatives.

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- The contributions with derivatives are essentially given by the skeleton graphs



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- This is given by $\mathcal{B}_1^{(2,0)} - \mathcal{B}_2^{(2,0)} + \mathcal{B}_3^{(2,0)}$.

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- We have that

$$\frac{1}{4} \bar{\delta}_{uu} \delta_{ww} \left(\mathcal{B}_1^{(2,0)} - \mathcal{B}_2^{(2,0)} + \mathcal{B}_3^{(2,0)} \right) = \left(\Psi_1 - \Psi_2 - \Psi_3 \right) + \Phi_0.$$

- Ψ_1 , Ψ_2 and Ψ_3 involve two derivatives while Φ_0 has no derivatives.

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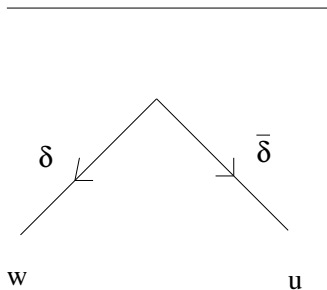
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- Including dressing factors, we have that

$$\begin{aligned} \psi_1 = & \frac{\pi}{\det Y} \int_{\Sigma^3} \prod_{i=1}^3 d^2 z_i P(z_2, z_3) \mathcal{G}(z_2, z_3) \partial_w \mathcal{G}(w, z_1) \bar{\partial}_u \mathcal{G}(u, z_1) \\ & \times \Delta(w, z_1) \overline{\Delta(u, z_1)}. \end{aligned}$$

- This is given by the skeleton graph

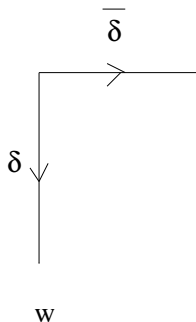


- Also, we have that

$$\begin{aligned} \psi_2 = & -\frac{\pi}{\det Y} \int_{\Sigma^3} \prod_{i=1}^3 d^2 z_i(z_2, \bar{z}_1)(z_3, \bar{u}) \mathcal{G}(z_2, z_3) \bar{\partial}_{z_3} \mathcal{G}(z_1, z_3) \\ & \times \partial_w \mathcal{G}(w, z_1) \overline{\Delta(u, z_2)} \Delta(w, z_1), \end{aligned}$$

along with its hermitian conjugate.

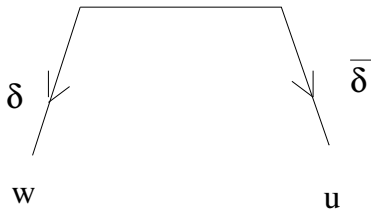
- This is given by the skeleton graph (along with its hermitian conjugate)



- Finally, we have that

$$\begin{aligned} \Psi_3 &= \frac{4\pi}{\det Y} \int_{\Sigma^2} \prod_{i=1}^2 d^2 z_i \mathcal{G}(z_1, z_2) \partial_w \mathcal{G}(w, z_1) \bar{\partial}_u \mathcal{G}(u, z_2) (z_2, \bar{z}_1) \\ &\quad \times \Delta(w, z_1) \overline{\Delta(u, z_2)}. \end{aligned}$$

- This is given by the skeleton graph



- Let us consider the equation

$$\frac{1}{4} \bar{\delta}_{uu} \delta_{ww} \left(\mathcal{B}_1^{(2,0)} - \mathcal{B}_2^{(2,0)} + \mathcal{B}_3^{(2,0)} \right) = \left(\Psi_1 - \Psi_2 - \Psi_3 \right) + \Phi_0$$

once again.

- The right hand side has some terms which involve two derivatives, which we would like to get rid of to get a simple eigenvalue equation.
- We shall explain later why we want this.

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- How to proceed to obtain such an equation? One possibility is to consider more modular graphs which might do the trick.
- Hence consider graphs with the same skeleton graphs as before, but with different dressing factors.

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- Hence consider graphs with the same skeleton graphs as before, but with different dressing factors.

- Recall that



(i)



(ii)



(iii)

- We consider

$$\mathcal{B}_4^{(2,0)} = 4 \int_{\Sigma^2} \prod_{i=1}^2 d^2 z_i \mathcal{G}(z_1, z_2)^2 P(z_1, z_2)$$

which has the same skeleton graph as figure (i), but different dressing factors compared to $\mathcal{B}_1^{(2,0)}$, and

$$\mathcal{B}_5^{(2,0)} = 4 \int_{\Sigma^3} \prod_{i=1}^3 d^2 z_i \mathcal{G}(z_1, z_2) \mathcal{G}(z_1, z_3) (z_1, \bar{z}_2) (z_2, \bar{z}_3) (z_3, \bar{z}_1)$$

which has the same skeleton graph as figure (ii), but different dressing factors compared to $\mathcal{B}_2^{(2,0)}$.

- Also consider

$$\mathcal{B}_6^{(2,0)} = \int_{\Sigma^4} \prod_{i=1}^4 d^2 z_i \mathcal{G}(z_1, z_4) \mathcal{G}(z_2, z_3) (z_1, \bar{z}_4) (z_4, \bar{z}_3) (z_3, \bar{z}_2) (z_2, \bar{z}_1)$$

and

$$\mathcal{B}_7^{(2,0)} = \int_{\Sigma^4} \prod_{i=1}^4 d^2 z_i \mathcal{G}(z_1, z_4) \mathcal{G}(z_2, z_3) (z_1, \bar{z}_2) (z_2, \bar{z}_4) (z_4, \bar{z}_3) (z_3, \bar{z}_1)$$

which have the same skeleton graph as figure (iii), but different dressing factors compared to $\mathcal{B}_3^{(2,0)}$.

- Including the dressing factors, we can denote them graphically.

- The graphs (i) $B_1^{(2,0)}$, (ii) $B_4^{(2,0)}$, (iii) $B_2^{(2,0)}$, (iv) $B_5^{(2,0)}$, (v) $B_3^{(2,0)}$, (vi) $B_6^{(2,0)}$, and (vii) $B_7^{(2,0)}$



(i)



(ii)



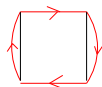
(iii)



(iv)



(v)



(vi)



(vii)

- For the new graphs, we proceed as we had before for the three graphs.
- We get that

$$\frac{1}{4} \bar{\delta}_{uu} \delta_{ww} \left(\mathcal{B}_4^{(2,0)} - \mathcal{B}_5^{(2,0)} + \mathcal{B}_6^{(2,0)} \right) = \left(\Psi_1 - \Psi_2 - \Psi_3 \right) + \Phi_1,$$

where Φ_1 has no derivatives.

- Strikingly, exactly the same set of terms with derivatives arise in both the equations.

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- Strikingly, exactly the same set of terms with derivatives arise in both the equations.

- Defining

$$\mathcal{B} = \left(\mathcal{B}_1^{(2,0)} - \mathcal{B}_4^{(2,0)} \right) - \left(\mathcal{B}_2^{(2,0)} - \mathcal{B}_5^{(2,0)} \right) + \left(\mathcal{B}_3^{(2,0)} - \mathcal{B}_6^{(2,0)} \right),$$

we get that

$$\frac{1}{4} \bar{\delta}_{uu} \delta_{ww} \mathcal{B} = \Phi_0 + \Phi_1 \equiv \Theta.$$

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- Thus there are no terms involving derivatives on the right hand side of the equation.

- We are now in a position to obtain the eigenvalue equation involving these modular graph functions.

- The left hand side of the equation is given by

$$\frac{1}{4} \bar{\delta}_{uu} \delta_{ww} \mathcal{B} = \pi^2 \omega_I(w) \omega_J(w) \overline{\omega_K(u) \omega_L(u)} \partial_{IJ} \bar{\partial}_{KL} \mathcal{B},$$

on using the expression for the partial derivative

$$\partial_{IJ} = \frac{1}{2} (1 + \delta_{IJ}) \frac{\partial}{\partial \Omega_{IJ}}$$

in the composite index notation. This follows from the fact that the holomorphic quadratic differential $\delta_{ww} \Phi$ for arbitrary Φ can be expanded in a basis of $\omega_I(w) \omega_J(w)$ for $I \leq J$, and similarly for the anti-holomorphic variation.

- Since there are no derivatives on the right hand side of the equation, we can trivially pull out a factor of $\omega_I(w)\omega_J(w)\omega_K(u)\omega_L(u)$ with coefficients that are independent of w and u which follows from the structure of the various terms.

- The terms involving derivatives have factors of $\partial_w \mathcal{G}(w, z)$ and/or $\bar{\partial}_u \mathcal{G}(u, z')$ in the integrand, and hence this simplification does not work.

- Thus expressing Θ as

$$\Theta = 4\pi^2 \omega_I(\mathbf{w}) \omega_J(\mathbf{w}) \overline{\omega_K(\mathbf{u}) \omega_L(\mathbf{u})} \Theta_{IJ;KL},$$

we have that

$$\partial_{IJ} \bar{\partial}_{KL} \mathcal{B} = \Theta_{IJ;KL} + \Theta_{IJ;LK} + \Theta_{JI;KL} + \Theta_{JI;LK}$$

on symmetrizing in IJ and KL separately.

- Using the expressions for the Laplacian on moduli space

$$\Delta = 2 \left(Y_{IK} Y_{JL} + Y_{IL} Y_{JK} \right) \partial_{IJ} \bar{\partial}_{KL},$$

we get the equation

$$\frac{1}{8} \Delta \mathcal{B} = \left(Y_{IK} Y_{JL} + Y_{IL} Y_{JK} \right) \Theta_{IJ;KL}.$$

- This gives us the desired eigenvalue equation

$$\Delta \mathcal{B} = 3 \left(\mathcal{B}_1^{(2,0)} - \mathcal{B}_4^{(2,0)} \right) - \frac{7}{2} \mathcal{B}_2^{(2,0)} + 4 \mathcal{B}_5^{(2,0)} + 4 \left(\mathcal{B}_3^{(2,0)} - \mathcal{B}_6^{(2,0)} \right) - \mathcal{B}_7^{(2,0)}$$

involving seven modular graph functions.

- Graphically

$$\begin{aligned}
 & \Delta \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right] \\
 &= 3 \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] - \frac{7}{2} \left[\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right] + 4 \left[\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right] \\
 &+ 4 \left[\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right] - \left[\begin{array}{c} \text{Diagram 9} \end{array} \right]
 \end{aligned}$$

The diagrams are modular graphs representing various topologies and internal structures. Diagram 1 is a genus-2 surface with two external legs. Diagram 2 is a genus-2 surface with two internal lines. Diagram 3 is a genus-2 surface with one external leg and one internal line. Diagram 4 is a genus-2 surface with two internal lines. Diagram 5 is a genus-2 surface with two internal lines. Diagram 6 is a genus-2 surface with two internal lines. Diagram 7 is a genus-2 surface with two internal lines. Diagram 8 is a genus-2 surface with two internal lines. Diagram 9 is a square with two diagonal lines.

- Similar manipulation of the $D^6\mathcal{R}^4$ term yields

$$\Delta\mathcal{B}^{(0,1)} = 5\mathcal{B}^{(0,1)}.$$

(D'Hoker, Green, Pioline, Russo)

- This involves a modular graph which has only one factor of the Green function.
- Our analysis involves graphs with two factors of the Arakelov Green function, with the extra factor of the Green function in the integrand essentially leading to the need for the involved analysis.

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- Our analysis involves graphs beyond those that arise in the four graviton amplitude upto this order in the derivative expansion. We would like to know which amplitudes yield them. Perhaps they arise in the low momentum expansion of higher point amplitudes in the same theory.
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- This procedure is general enough to be used at all orders in the derivative expansion.

- We have obtained only one eigenvalue equation involving several graphs. In order to integrate over moduli space, we would like to obtain more differential equations involving them. This, in general, would be quite interesting.

- Also interesting to analyse modular graphs in theories with lesser supersymmetry, and in compactifications to lower dimensions.