Soft Theorems for Massless Particles from Gauge Invariance

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- This talk is based on a paper together with

  Z. Bern, S. Davies, J. Nosh

  and on many papers with

  R. Marotta and M. Mojaza

Plan of the talk

Introduction

- In particle physics we deal with many kinds of symmetries.
- They all have the property of leaving the action invariant, but have very different physical consequences.
- **GLOBAL INTERNAL UNBROKEN SYMMETRIES**
- Unique vacuum annihilated by the symmetry gener.: $Q_a|0\rangle = 0$
- Particles are classified according to multiplets of this symmetry and all particles of a multiplet have the same mass.
- If $m_u = m_d$, QCD would be invariant under an $SU(2)_V$ flavor symmetry.
- and the proton and the neutron would have the same mass (neglecting the electromagnetic interactions).
- Since $\frac{m_u}{m_d} = \frac{2m^2_{\pi^0} - m^2_{\pi^+} + m^2_{k^+} - m^2_{k^0}}{m^2_{k^0} - m^2_{k^+} + m^2_{\pi^+}} = 0.56 \neq 1$, $SU(2)_V$ is only an approximate symmetry.

- **GLOBAL INTERNAL SPONT. BROKEN SYMMETRIES.**
- Degenerate vacua: $Q_a|0\rangle = |0'\rangle$.
- Not realized in the spectrum, but presence in the spectrum of massless particles, called Nambu-Goldstone bosons.
- For zero quark mass, QCD with two flavors is invariant under $SU(2)_L \times SU(2)_R$.
- It is broken to $SU(2)_V \Longrightarrow$ 3 broken generators.
- The NG bosons are the three pions in QCD with 2 flavors.
- This is one physical consequence of the spontaneous breaking.
- Another one is the existence of low-energy theorems.
- The scattering amplitude for a soft pion is zero at low energy: (Adler zeroes).
- If one introduces a mass term, breaking explicitly chiral symmetry and giving a small mass to the pion, one gets the two Weinberg scattering lengths ($\pi\pi \to \pi\pi$):

$$a_0 = \frac{7m_\pi}{32\pi F_\pi^2} \quad ; \quad a_2 = -\frac{m_\pi}{16\pi F_\pi^2}$$

- **SPACE-TIME GLOBAL SYMMETRIES**
- Conformal invariance is the most notable example.
- It is a classical symmetry if the action does not contain any dimensional quantity $\Longrightarrow$ Symmetry of the tree diagrams.
- In general, it is broken by anomalies in the quantum theory.
- Introduction of the dimensional quantity μ in the ren. process.
- It can be also explicitly broken by, for instance, mass terms.
- It can be spontaneously broken by, for instance, non-zero vacuum expectation value of a scalar field.
- As a consequence, one gets a NG boson, called the dilaton that has a universal low-energy behavior: only one NG boson.
- One can derive the universal soft dilaton behavior from the conformal WT identities.
- In general, in the full quantum theory, the dilaton gets a mass proportional to the β-function of the theory.
- In $\mathcal{N} = 4$ super Yang-Mills, it stays massless in the quantum th..

- $\mathcal{N} = 4$ super Yang-Mills contains six scalars.
- When one of them gets a vev, then conformal invariance and $SO(6)$ R-symmetry are spontaneously broken.
- The one with vev is a dilaton (NG boson of broken conformal invariance).
- The other 5 belong to the coset $\frac{SO(6)}{SO(5)}$ and are NG bosons of broken R-symmetry.
- While the dilaton satisfies the soft theorems derived from the Ward identities, the other 5 NG bosons do not have Adler zeroes [M. Bianchi, A. Guerrieri, Y.t. Huang, C.J. Lee and C. Wen (2016)]

- **LOCAL INTERNAL AND SPACE-TIME SYMMETRIES**
- A local internal (space-time) symmetry requires the introduction of massless gauge bosons with spin 1 (spin 2).
- In both cases, local gauge invariance is necessary to reconcile the theory of relativity with quantum mechanics.
- It allows a fully relativistic description, eliminating, at the same time, the presence of negative norm states in the spectrum of physical states.
- Although described by A_μ and $G_{\mu\nu}$, both gluons and gravitons have only two physical degrees of freedom in D=4.
- Another consequence of gauge invariance for photons is charge conservation, while for gravitons is momentum conservation.
- Yet another physical consequence of local gauge invariance is the existence of low-energy theorems for photons and gravitons [F. Low, 1958; S. Weinberg, 1964]

- The interest on the soft theorems was revived few years ago by [Cachazo and Strominger, arXiv:1404.1491[hep-th]].
- They study the behavior of the n-graviton amplitude when the momentum q of one graviton becomes soft ($q \sim 0$).
- The leading term $O(q^{-1})$ was shown to be universal by Weinberg in the sixties,
- They suggest a universal formula for the subleading term $O(q^0)$.
- They speculate that, as the leading term, it may be a consequence of BMS symmetry of asymptotically flat space-times.
- This has been claimed later, in four space-time dimensions, to be a consequence of the BMS Ward-Takahashi identities.

- In this talk we show that, once the structure of the three-point amplitude is given, the conditions:

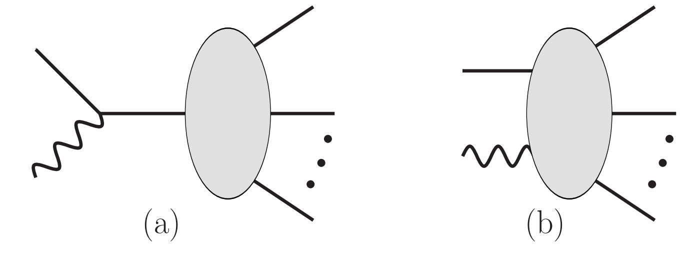
$$q_\mu M^{\mu\nu}_{n+1}(q;k_i) = q_\nu M^{\mu\nu}_{n+1}(q;k_i) = 0$$

completely determine the terms of $\mathcal{O}(q^{-1}), \mathcal{O}(q^0)$ and $\mathcal{O}(q^1)$ of the symmetric part of $M^{\mu\nu}_{n+1}(q;k_i)$ in terms of the amplitude without the soft particle.
- q is the momentum and μ, ν are indices of the soft particle.
- They also determine the terms of $\mathcal{O}(q^0)$ of the antisymmetric part of $M^{\mu\nu}_{n+1}(q;k_i)$ in terms of the amplitude without the soft particle.
- For $B_{\mu\nu}$ the term of $\mathcal{O}(q^{-1})$ is zero and that of $\mathcal{O}(q)$ is not fixed.
- The soft behavior for graviton, dilaton and $B_{\mu\nu}$ obtained above is confirmed by explicit calculations in string amplitudes where the other hard particles are massive scalars or massless gravitons, dilatons and $B_{\mu\nu}$.
- This procedure can be extended to massive particles with any spin in a straightforward way.

- The soft behavior obtained in this way is valid not only for the tree diagrams but also for loop diagrams when the theory containing these soft particles is free from UV and IR divergences.
- The soft theorem is kept also at the loop level if there are no UV divergences (as in string theory) and the IR divergences do not depend on the number of external legs. See below.
- This is the case if, in the bosonic string, we compactify $26 - D$ directions and we keep $D > 4$.
- The soft behavior obtained in this way is confirmed by explicit calculations in string theory (with possibly the addition of string corrections).
- The string corrections are naturally explained by the structure of the three-point amplitudes.

- It turns out that the soft behavior of the gravity dilaton includes the generators of scale and special conformal transformations.
- Then, we consider a conformal theory in D dimensions where the generators of scale and special conformal transformations are spontaneously broken.
- A massless Nambu-Goldstone boson appears, also called dilaton.
- This dilaton has, in principle, nothing to do with the previous dilaton.
- They have the common property of being coupled to the trace of the energy-momentum tensor: that is the origin of the same name.
- In fact, the string dilaton is coupled to the trace of the target space energy-momentum tensor.
- We then show that the Ward-Takahashi identities of scale and special conformal transformations imply a low-energy behavior (at the tree level) that is very similar to that of the gravity dilaton.
- We finally discuss similarities and differences.

One photon and n scalar particles



- The scattering amplitude $M^{\mu}_{n+1}(q; k_1 \ldots k_n)$, involving one photon and n scalar particles, consists of two pieces:

$$
\begin{aligned}
M^{\mu}_{n+1}(q; k_1, \ldots, k_n) &= \sum_{i=1}^{n} e_i \frac{k_i^{\mu}}{k_i \cdot q} M_n(k_1, \ldots, k_i + q, \ldots, k_n) \\
&+ N^{\mu}_{n+1}(q; k_1, \ldots, k_n)\,.
\end{aligned}
$$

- and must be gauge invariant for any value of q:

$$q_\mu M_{n+1}^\mu = \sum_{i=1}^{n} e_i M_n(k_1, \ldots, k_i + q, \ldots, k_n)$$
$$+ q_\mu N_{n+1}^\mu(q; k_1, \ldots, k_n) = 0$$

- Expanding around $q = 0$, we have

$$0 = \sum_{i=1}^{n} e_i \left[M_n(k_1, \ldots, k_i, \ldots, k_n) + q_\mu \frac{\partial}{\partial k_{i\mu}} M_n(k_1, \ldots, k_i, \ldots, k_n) \right]$$
$$+ q_\mu N_{n+1}^\mu(q = 0; k_1, \ldots, k_n) + \mathcal{O}(q^2)\,.$$

- At leading order, this equation reduces to

$$\sum_{i=1}^{n} e_i = 0\,,$$

which is simply a statement of charge conservation [Weinberg, 1964]

- At the next order, we have

$$q_\mu N^\mu_{n+1}(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i q_\mu \frac{\partial}{\partial k_{i\mu}} M_n(k_1, \ldots, k_n) \,.$$

- This equation tells us that $N^\mu_{n+1}(0; k_1, \ldots, k_n)$ is entirely determined in terms of M_n up to potential pieces that are separately gauge invariant.
- It is easy to see that the only expressions local in q that vanish under the gauge-invariance condition $q_\mu E^\mu = 0$ are of the form,

$$E^\mu = q_\nu A^{\mu\nu} \quad ; \quad A^{\mu\nu} = -A^{\nu\mu}$$

  where $A^{\mu\nu}$ is an antisymmetric tensor (local in q) constructed with the momenta of the scalar particles.
- The explicit factor of the soft momentum q in each term means that they are suppressed in the soft limit and do not contribute to $N^\mu_{n+1}(0; k_1, \ldots, k_n)$.

- We can therefore remove q_μ getting

$$N^{\mu}_{n+1}(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i \frac{\partial}{\partial k_{i\mu}} M_n(k_1, \ldots, k_n)\,,$$

thereby determining $N^{\mu}_{n+1}(0; k_1, \ldots, k_n)$ as a function of the amplitude without the photon.

- Inserting this into the original expression yields

$$M^{\mu}_{n+1}(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[k_i^{\mu} - i q_{\nu} L_i^{\mu\nu} \right] M_n(k_1, \ldots, k_n) + \mathcal{O}(q) \,,$$

where

$$L_i^{\mu\nu} \equiv i \left(k_i^{\mu} \frac{\partial}{\partial k_{i\nu}} - k_i^{\nu} \frac{\partial}{\partial k_{i\mu}} \right)$$

is the orbital angular-momentum operator.

- The amplitude with a soft photon with momentum q is entirely determined, up to $\mathcal{O}(q^0)$, in terms of the amplitude $M_n(k_1, \ldots, k_n)$, involving n scalar particles and no photon.
- This goes under the name of F. Low's low-energy theorem.
- Low's theorem for photons is unchanged at loop level.
- Even at loop level, all diagrams containing a pole in the soft momentum are of the form shown, with loops appearing only in the blob and not correcting the external vertex.

- Get an amplitude by contracting $M^{\mu}_{n+1}(q; k_1, \ldots, k_n)$ with the photon polarization $\varepsilon_{q\mu}$.
- Soft-photon limit:

$$M_{n+1}(q; k_1, \ldots, k_n) \to \left[S^{(0)} + S^{(1)}\right] M_n(k_1, \ldots, k_n) + \mathcal{O}(q)\,,$$

where

$$S^{(0)} \equiv \sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q}\,,$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} e_i \frac{\varepsilon_{q\mu} q_\nu L_i^{\mu\nu}}{k_i \cdot q}\,,$$

where $L_i^{\mu\nu}$ is the orbital angular momentum.

- Actually from the gauge invariance condition:

$$\sum_{i=1}^{n} e_i M_n(k_1 \ldots k_i + q \ldots k_n) + q_\mu N^\mu_{n+1}(q; k_1 \ldots k_n) = 0$$

one can get a more general soft theorem.
[Z.Z. Li, H.H. Lin and S.Q. Zhang, arXiv:1802.03148 [hep-th]]

- Defining

$$N^\mu_{n+1}(q; k_1 \ldots k_n) = \sum_{\ell=0}^{\infty} q_{\mu_1} \ldots q_{\mu_\ell} N^{\mu;\mu_1\cdots\mu_\ell}(k_1 \ldots k_n)$$

the gauge inv. condition fixes only the symmetric part of $N^{\mu;\mu_1\cdots\mu_\ell}$.

- One gets

$$N^{\mu;\mu_1\cdots\mu_\ell}(k_1 \ldots k_n) = -\sum_{i=0}^{n} \frac{e_i}{(\ell+1)!} \frac{\partial}{\partial k_{i\mu}} \frac{\partial}{\partial k_{i\mu_1}} \cdots \frac{\partial}{\partial k_{i\mu_\ell}}$$
$$\times M_n(k_1 \ldots k_n) + A^{\mu;\mu_1\cdots\mu_\ell}(k_1 \ldots k_n)$$

where $A^{\mu;\mu_1\cdots\mu_\ell}(k_1 \ldots k_n)$ is antisymmetric under the exchange of μ with μ_i.

- This fixes M_{n+1} to be

$$M^{\mu}_{n+1}(q;k_1\dots k_n)=\sum_{i=1}^{n}e_i\left(\frac{k_i^{\mu}-i\sum_{\ell=0}^{\infty}\frac{1}{(\ell+1)!}q_\nu J_i^{\mu\nu}(q\frac{\partial}{\partial k_i})^{\ell}}{k_i q}\right)$$
$$\times M_n(k_1\dots k_n)+A^{\mu}(q;k_1\dots k_n)$$

where

$$A^{\mu}(q;k_1\dots k_n)=\sum_{\ell=1}^{\infty}q_{\mu_1}\dots q_{\mu_\ell}A^{\mu;\mu_1\dots\mu_\ell}\ \ ;\ \ A^{\mu}(q=0;k_1\dots k_n)=0$$

- $A^{\mu;\mu_1\dots\mu_\ell}$ is antisymmetric exchanging μ with any μ_i.

- An infinite order soft theorem is then obtained:

$$\Omega_{\mu\mu_1\ldots\mu_\ell}\frac{\partial}{\partial q_{\mu_1}}\cdots\frac{\partial}{\partial q_{\mu_\ell}}M^{\mu}_{n+1}(q;k_1\ldots k_n)|_{q=0}=$$

$$=\Omega_{\mu\mu_1\ldots\mu_\ell}\left[\frac{\partial}{\partial q_{\mu_1}}\cdots\frac{\partial}{\partial q_{\mu_\ell}}\sum_{i=1}^{n}\frac{e_i(-i)}{(qk_i)(\ell+1)!}q_\nu J_i^{\mu\nu}\left(q\frac{\partial}{\partial k_i}\right)^{\ell}\right.$$

$$\left.\times M_n(k_1\ldots k_n)\right]_{q=0}$$

  where $\Omega_{\mu\mu_1\ldots\mu_\ell}$ is a symmetric matrix.
- This is in agreement with the result of [Y. Hamada and G. Shiu, arXiv:1801.05528] obtained with other methods.

One graviton/dilaton and n scalar particles

- In the case of a graviton scattering on n scalar particles, one can write

$$M^{\mu\nu}_{n+1}(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{k_i^\mu k_i^\nu}{k_i \cdot q} M_n(k_1, \ldots, k_i + q, \ldots, k_n) + N^{\mu\nu}_{n+1}(q; k_1, \ldots, k_n),$$

- $N^{\mu\nu}_{n+1}(q; k_1, \ldots, k_n)$ is symmetric under the exchange of μ and ν.
- On-shell gauge invariance implies

$$0 = q_\mu M^{\mu\nu}_{n+1}(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} k_i^\nu M_n(k_1, \ldots, k_i + q, \ldots, k_n) + q_\mu N^{\mu\nu}_{n+1}(q; k_1, \ldots, k_n).$$

- and also

$$0 = q_\nu M^{\mu\nu}_{n+1}(q; k_1, \ldots, k_n) = \sum k_i^\mu M_n(k_1, \ldots, k_i + q, \ldots, k_n) + q_\nu N^{\mu\nu}_{n+1}(q; k_1, \ldots, k_n).$$

- Before proceeding further let us be more precise.
- In general, gauge invariance implies the more general conditions:

$$q_\mu \left(M_{n+1}^{\mu\nu}(q; k_1 \ldots k_n) - f(q; k_1 \ldots k_n)\eta^{\mu\nu} \right) = 0$$
$$q_\nu \left(M_{n+1}^{\mu\nu}(q; k_1 \ldots k_n) - f(q; k_1 \ldots k_n)\eta^{\mu\nu} \right) = 0$$

  where f is an arbitrary function of the momenta.
- Such extra function is irrelevant for a soft graviton because its polarization is traceless: $\epsilon_q^{\mu\nu} \eta_{\mu\nu} = 0$.
- But, it would be relevant for the dilaton!
- It turns out, however, that explicit string calculations (at the tree level) show that this term is present only if the amplitude also includes open strings.
- We will see that this term has an important physical interpretation.
- Explicit string calculations (at the multi-loop level) in the bosonic string also show that this term is present and has again a very simple interpretation.

- At leading order in q, we then have

$$\sum_{i=1}^{n} k_i^{\mu} = 0\,,$$

- It is satisfied due to momentum conservation.
- Different couplings to different particles would have prevented the leading term to vanish: Gravitons have universal coupling [Weinberg, 1964].
- At first order in q, one gets

$$\sum_{i=1}^{n} k_i^{\nu} \frac{\partial}{\partial k_{i\mu}} M_n(k_1, \ldots, k_n) + N_{n+1}^{\mu\nu}(0; k_1, \ldots, k_n) = 0\,,$$

- while at second order in q, it gives

$$\sum_{i=1}^{n} k_i^{\nu} \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\rho}} M_n(k_1, \ldots, k_n) + \left[\frac{\partial N_{n+1}^{\mu\nu}}{\partial q_\rho} + \frac{\partial N_{n+1}^{\rho\nu}}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) = 0$$

- Using the previous conditions + the corresponding ones ($\mu \leftrightarrow \nu$) (the other particles are scalar particles) one gets

$$M_{n+1}^{\mu\nu}(q; k_i) = \kappa_D \sum_{i=1}^{n} \left[\frac{k_i^\mu k_i^\nu}{k_i \cdot q} - i \frac{k_i^\mu q_\rho L_i^{\nu\rho}}{2k_i \cdot q} - i \frac{k_i^\nu q_\rho L_i^{\mu\rho}}{2k_i \cdot q} \right.$$
$$\left. - \frac{1}{2} \frac{q_\rho L_i^{\mu\rho} q_\sigma L_i^{\nu\sigma}}{k_i \cdot q} + \frac{1}{2} \left(\eta^{\mu\nu} q^\sigma - q^\mu \eta^{\nu\sigma} - \frac{k_i^\mu q^\nu q^\sigma}{k_i \cdot q} \right) \frac{\partial}{\partial k_i^\sigma} \right]$$
$$\times M_n(k_1, \ldots, k_n) + O(q^2) ,$$

where $\kappa_D^2 = 8\pi G_N^{(D)}$.

- The previous expression is gauge invariant by construction:

$$q_\mu M_{n+1}^{\mu\nu} = q_\nu M_{n+1}^{\mu\nu} = 0$$

- This is precisely the soft-dilaton(graviton) behavior of an amplitude with n tachyons (at the tree level) in the bosonic string ($m_i^2 = -\frac{4}{\alpha'}$) with no α' corrections ! $B_{\mu\nu}$ is not coupled to only tachyons.

- We see that the physical-state conditions for the graviton

$$q^\mu \epsilon_{\mu\nu} = q^\nu \epsilon_{\mu\nu} = 0 \quad ; \quad \eta^{\mu\nu}\epsilon_{\mu\nu} = 0$$

set to zero the terms that are proportional to $\eta^{\mu\nu}$, q^μ and q^ν.
- We are then left with the following expression for the graviton soft limit of a single-graviton, n-scalar amplitude:

$$M_{n+1}(q; k_1, \ldots, k_n) \to \left[S^{(0)} + S^{(1)} + S^{(2)}\right] M_n(k_1, \ldots, k_n) + \mathcal{O}(q^2)\,,$$

- where

$$S^{(0)} \equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q}\,,$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho L_i^{\nu\rho}}{k_i \cdot q}\,,$$

$$S^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} q_\rho L_i^{\mu\rho} q_\sigma L_i^{\nu\sigma}}{k_i \cdot q}\,.$$

- These soft factors follow entirely from gauge invariance.

- On the other hand, if we saturate it with the dilaton polarization:

$$\epsilon^d_{\mu\nu} = \frac{1}{\sqrt{D-2}} \left(\eta_{\mu\nu} - q_\mu \bar{q}_\nu - q_\nu \bar{q}_\mu\right) \;\; ; \;\; q^2 = \bar{q}^2 = 0 \;\; , \;\; q\bar{q} = 1$$

we get

$$\epsilon^d_{\mu\nu} M^{\mu\nu}_{n+1}(q; k_1 \ldots k_n = \frac{\kappa_D}{\sqrt{D-2}} \Bigg\{ - \sum_{i=1}^{n} \frac{m_i^2}{k_i \cdot q} \Bigg(1 + q^\rho \frac{\partial}{\partial k_i^\rho}$$

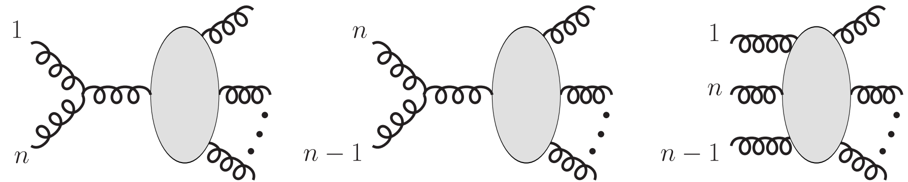
$$+ \frac{1}{2} q^\rho q^\sigma \frac{\partial^2}{\partial k_i^\rho \partial k_i^\sigma} \Bigg) + 2 - \sum_{i=1}^{n} \Bigg[k_i^\mu \frac{\partial}{\partial k_i^\mu}$$

$$+ \frac{q^\rho}{2} \Bigg(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho} \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} \Bigg) \Bigg] \Bigg\} M_n + \mathcal{O}(q^2) \,,$$

where m_i is the mass of the i^{th} scalar particle.

- The soft factors follow again entirely from gauge invariance.

Soft limit of n-gluon amplitude



(a) (b) (c)

- We consider a tree-level color-ordered amplitude where gluon $(n+1)$ becomes soft with $q \equiv k_{n+1}$.
- Being the amplitude color-ordered, we have to consider only two poles.

- By introducing the spin contribution to the angular-momentum operator,

$$(\Sigma_i^{\mu\sigma})^{\mu_i \rho} \equiv i\left(\eta^{\mu\mu_i}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\mu_i\sigma}\right) ,$$

we can write the total amplitude as

$$\begin{aligned} M_{n+1}^{\mu;\mu_1\cdots\mu_n}(q;k_1,\ldots,k_n) \\ = \frac{\delta_\rho^{\mu_1} k_1^\mu - iq_\sigma (\Sigma_1^{\mu\sigma})^{\mu_1}{}_\rho}{\sqrt{2}(k_1\cdot q)} M_n^{\rho\mu_2\cdots\mu_n}(k_1+q,k_2,\ldots,k_n) \\ - \frac{\delta_\rho^{\mu_{n-1}} k_n^\mu - iq_\sigma (\Sigma_{n-1}^{\mu\sigma})^{\mu_n}{}_\rho}{\sqrt{2}(k_{n-1}\cdot q)} M_n^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-2},k_n+q) \\ + N_{n+1}^{\mu;\mu_1\cdots\mu_n}(q;k_1,\ldots,k_n) . \end{aligned}$$

- Notice that the spin terms independently vanish when contracted with q_μ.

- On-shell gauge invariance requires

$$0 = q_\mu M_{n+1}^{\mu;\mu_1\cdots\mu_n}(q; k_1, \ldots, k_n)$$
$$= \frac{1}{\sqrt{2}} M_n^{\mu_1\mu_2\cdots\mu_n}(k_1 + q, k_2, \ldots, k_n)$$
$$- \frac{1}{\sqrt{2}} M_n^{\mu_1\cdots\mu_{n-1}\mu_n}(k_1, \ldots, k_{n-1}, k_n + q)$$
$$+ q_\mu N_{n+1}^{\mu;\mu_1\cdots\mu_n}(q; k_1, \ldots, k_n)\,.$$

- For $q = 0$, this is automatically satisfied.
- At the next order in q, we obtain

$$-\frac{1}{\sqrt{2}}\left[\frac{\partial}{\partial k_{1\mu}} - \frac{\partial}{\partial k_{n\mu}}\right] M_n^{\mu_1\cdots\mu_n}(k_1, k_2 \ldots k_n)$$
$$= N_{n+1}^{\mu;\mu_1\cdots\mu_n}(0; k_1, \ldots, k_n)\,.$$

- Thus, $N_{n+1}^{\mu;\mu_1\cdots\mu_n}(0; k_1, \ldots, k_n)$ is determined in terms of the amplitude without the soft gluon.

- With this, the total expression becomes

$$
\begin{aligned}
& M_{n+1}^{\mu;\mu_1\cdots\mu_n}(q;k_1\ldots k_n) \\
& = \left(\frac{k_1^\mu}{\sqrt{2}(k_1\cdot q)} - \frac{k_n^\mu}{\sqrt{2}(k_n\cdot q)}\right) M_n^{\mu_1\cdots\mu_n}(k_1,\ldots,k_n) \\
& \quad - i\frac{q_\sigma (J_1^{\mu\sigma})^{\mu_1}{}_\rho}{\sqrt{2}(k_1\cdot q)} M_n^{\rho\mu_2\cdots\mu_n}(k_1,\ldots,k_{n-1}) \\
& \quad + i\frac{q_\sigma (J_n^{\mu\sigma})^{\mu_n}{}_\rho}{\sqrt{2}(k_{n-1}\cdot q)} M_n^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_n) + \mathcal{O}(q),
\end{aligned}
$$

where

$$
(J_i^{\mu\sigma})^{\mu_i\rho} \equiv L_i^{\mu\sigma}\eta^{\mu_i\rho} + (\Sigma_i^{\mu\sigma})^{\mu_i\rho},
$$

with

$$
L_i^{\mu\sigma} \equiv i\left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}}\right) \quad ; \quad (\Sigma_i^{\mu\sigma})^{\mu_i\rho} \equiv i\left(\eta^{\mu\mu_i}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\mu_i\sigma}\right)
$$

- In order to write the final result in terms of full amplitudes, we contract with external polarization vectors.
- We must pass polarization vectors $\varepsilon_{1\mu_1}$ and $\varepsilon_{n\mu_n}$ through the spin-one angular-momentum operator such that they will contract with the ρ index of, respectively, $M_n^{\rho\mu_2\cdots\mu_n}(k_1, \ldots, k_n)$ and $M_n^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \ldots, k_n)$.
- It is convenient to write the spin angular-momentum operator as

$$\varepsilon_{i\mu_i}(\Sigma_i^{\mu\sigma})^{\mu_i}{}_\rho M^\rho = i\left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}}\right)\varepsilon_{i\rho}M^\rho \,.$$

- We may therefore write

$$M_{n+1}(q;k_1,\ldots,k_n) \to \left[S_n^{(0)} + S_n^{(1)}\right] M_n(k_1,\ldots,k_n) + \mathcal{O}(q)\,,$$

where

$$S_n^{(0)} \equiv \frac{k_1\cdot\varepsilon}{\sqrt{2}\,(k_1\cdot q)} - \frac{k_n\cdot\varepsilon}{\sqrt{2}\,(k_n\cdot q)}\,,$$
$$S_n^{(1)} \equiv -i\varepsilon_\mu q_\sigma\left(\frac{J_1^{\mu\sigma}}{\sqrt{2}\,(k_1\cdot q)} - \frac{J_n^{\mu\sigma}}{\sqrt{2}\,(k_n\cdot q)}\right).$$

- Here

$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + S_i^{\mu\sigma}\,,$$

where

$$L_i^{\mu\nu} \equiv i\left(k_i^\mu\frac{\partial}{\partial k_{i\nu}} - k_i^\nu\frac{\partial}{\partial k_{i\mu}}\right) \quad , \quad S_i^{\mu\sigma} \equiv i\left(\varepsilon_i^\mu\frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma\frac{\partial}{\partial \varepsilon_{i\mu}}\right).$$

Soft limit of $(n+1)$-graviton/dilaton amplitude

- As before the amplitude is the sum of two pieces:

$$
\begin{aligned}
& M_{n+1}^{\mu\nu;\mu_1\nu_1\cdots\mu_n\nu_n}(q;k_1,\ldots,k_n) \\
&= \sum_{i=1}^{n} \frac{1}{k_i\cdot q}\left[k_i^\mu \eta^{\mu_i\alpha} - iq_\rho(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\left[k_i^\nu \eta^{\nu_i\beta} - iq_\sigma(\Sigma_i^{\nu\sigma})^{\nu_i\beta}\right] \\
& \times M_n^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_n\nu_n}(k_1,\ldots,k_i+q,\ldots,k_n) \\
& \qquad + N_{n+1}^{\mu\nu;\mu_1\nu_1\cdots\mu_n\nu_n}(q;k_1,\ldots,k_n)\,,
\end{aligned}
$$

where

$$
(\Sigma_i^{\mu\rho})^{\mu_i\alpha} \equiv i\left(\eta^{\mu\mu_i}\eta^{\alpha\rho} - \eta^{\mu\alpha}\eta^{\mu_i\rho}\right)\,.
$$

- On-shell gauge invariance implies

$$0 = q_\mu M_{n+1}^{\mu\nu;\mu_1\nu_1\cdots\mu_n\nu_n}(q;k_1,\ldots,k_n)$$
$$= \sum_{i=1}^{n}\left[k_i^\nu \eta^{\nu_i\beta} - iq_\rho(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right] M_n^{\mu_1\nu_1\cdots\mu_i\ \ \cdots\mu_{n-1}\nu_{n-1}}{}_{\beta}(k_1,\ldots,k_i+q,\ldots,k_n)$$
$$+ q_\mu N_{n+1}^{\mu\nu;\mu_1\nu_1\cdots\mu_n\nu_n}(q;k_1,\ldots,k_n)\,.$$

- Proceeding as before we end up getting

$$M_{n+1}^{\mu\nu}(q;k_i) = \kappa_D \sum_{i=1}^{n}\left[\frac{k_i^\mu k_i^\nu}{k_i\cdot q} - i\frac{k_i^\mu q_\rho J_i^{\nu\rho}}{2k_i\cdot q} - i\frac{k_i^\nu q_\rho J_i^{\mu\rho}}{2k_i\cdot q}\right.$$
$$- \frac{1}{2}\frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i\cdot q} + \frac{1}{2}\left(\eta^{\mu\nu}q^\sigma - q^\mu\eta^{\nu\sigma} - \frac{k_i^\mu q^\nu q^\sigma}{k_i\cdot q}\right)\frac{\partial}{\partial k_i^\sigma}$$
$$\left.+\frac{1}{2}\frac{q_\rho q_\sigma \eta_{\mu\nu} - q_\sigma q_\nu \eta_{\rho\mu} - q_\rho q_\mu \eta_{\sigma\nu}}{k_i\cdot q}\epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\sigma}}\right\}$$
$$\times M_n(k_1,\ldots,k_n) + \mathcal{O}(q^2)\,,$$

where $M_n(k_1,\ldots,k_n)$ is the n-point scattering amplitude involving gravitons and/or dilatons

- The previous expression can be more compactly written as follows:

$$M_{n+1}^{\mu\nu}(q;k_i) = \kappa_D$$
$$\times \sum_{i=1}^{n} \left[\frac{k_i^\mu k_i^\nu}{k_i \cdot q} - i\frac{k_i^\mu q_\rho J_i^{\nu\rho}}{2k_i \cdot q} - i\frac{k_i^\nu q_\rho J_i^{\mu\rho}}{2k_i \cdot q} - \frac{1}{2}\frac{: q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma} :}{k_i \cdot q} \right]$$
$$\times M_n(k_1, \ldots, k_n)$$

where $:\;\;:$ means that the first operator does not act on the second and

$$J = L + S$$

- In order to write our expression in terms of amplitudes, we saturate with the soft graviton polarization tensor $\epsilon_{\mu\nu}$

$$q^\mu \epsilon_{\mu\nu} = q^\nu \epsilon_{\mu\nu} = \eta^{\mu\nu} \epsilon_{\mu\nu} = 0$$

- As for gluons, passing the polarization vectors through the spin operators, we get

$$M_{n+1}(q; k_1, \ldots, k_n) = \left[S_n^{(0)} + S_n^{(1)} + S_n^{(2)} \right] M_n(k_1, \ldots, k_n) + \mathcal{O}(q^2) \,,$$

where

$$S_n^{(0)} \equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q} \,,$$

$$S_n^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q} \,,$$

$$S_n^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q} \,.$$

Same result as from string theory apart from α' correct. of O(q).

- These soft factors follow entirely from gauge invariance and agree with those computed by Cachazo and Strominger for $D = 4$.
- Remember that

$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + S_i^{\mu\sigma} \, ,$$

with

$$L_i^{\mu\sigma} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}} \right) \, , \qquad S_i^{\mu\sigma} \equiv i \left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}} \right) \, .$$

- Projecting along the dilaton, we get

$$\epsilon^d_{\mu\nu} M^{\mu\nu}_{n+1}(q; k_1 \dots k_n) = \frac{\kappa_D}{\sqrt{D-2}} \Bigg\{ 2 - \sum_{i=1}^n \Bigg[k_i^\mu \frac{\partial}{\partial k_i^\mu}$$

$$- \frac{q^\rho}{2} \left(-2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} + k_{i\rho} \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} + 2 i S^{(i)}_{\mu\rho} \frac{\partial}{\partial k_{i\mu}} \right) \Bigg]$$

$$+ \sum_{i=1}^n \frac{q^\rho q^\sigma}{2 k_i \cdot q} \left((S^{(i)}_{\rho\mu}) \eta^{\mu\nu} (S^{(i)}_{\nu\sigma}) + D \epsilon_{i\rho} \frac{\partial}{\partial \epsilon_i^\sigma} \right) \Bigg\} M_n + \mathcal{O}(q^2)$$

Same result as from string theory (no α' corrections!).

Soft behavior of the Kalb-Ramond $B_{\mu\nu}$

- We start again with the pole term:

$$M^{pole}_{(n+1)\mu\nu} = \kappa_D \sum_{i=1}^{n} \frac{[k_{i\mu} - iq^\rho S_{\mu\rho}][k_{i\nu} - iq^\sigma \bar{S}_{\nu\sigma}]}{k_i \cdot q} M_n(k_i + q)\,,$$

where

$$S_{\mu\rho} = i\left(\epsilon_{i\mu}\frac{\partial}{\partial\epsilon_i^\rho} - \epsilon_{i\rho}\frac{\partial}{\partial\epsilon_i^\mu}\right)\ ;\ \bar{S}_{\nu\sigma} = i\left(\bar{\epsilon}_{i\nu}\frac{\partial}{\partial\bar{\epsilon}_i^\sigma} - \bar{\epsilon}_{i\nu}\frac{\partial}{\partial\bar{\epsilon}_i^\mu}\right)$$

- For the antisymmetric part one gets

$$M^{pole}_{(n+1)\mu\nu} = \kappa_D \sum_{i=1}^{n} \frac{-ik_{i[\mu}q^\sigma \bar{S}_{i\,\nu]\sigma} - ik_{i[\nu}q^\rho S_{i\,\mu]\rho} - q^\rho S_{i\,[\mu\rho}q^\sigma \bar{S}_{i\,\nu]\sigma}}{k_i \cdot q}$$
$$\times M_n(k_i + q)\,,$$

where $A_{[\mu}B_{\nu]} = \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu)$.

- No leading term (q^{-1}) appears (Weinberg, 1965)

- The previous expression is not gauge invariant
- We can add to it a term that will make it gauge invariant obtaining:

$$M_{\mu\nu} = \kappa_D \sum_{i=1}^{n} \left[\frac{-ik_{i[\mu}q^\sigma \bar{S}_{i\,\nu]\sigma} - ik_{i[\nu}q^\rho S_{i\,\mu]\rho} - q^\rho S_{i\,[\mu\rho}q^\sigma \bar{S}_{i\,\nu]\sigma}}{k_i \cdot q} + \frac{i}{2}\left(S_{i\,\mu\nu} - \bar{S}_{i\,\mu\nu}\right) \right] M_n(k_i + q) + N_{\mu\nu}(q; k_i)\,.$$

- In this case gauge invariance imposes:

$$q^\mu N_{\mu\nu}(q; k_i) = q^\nu N_{\mu\nu}(q; k_i) = 0\,.$$

- They are satisfied if we impose

$$N_{\mu\nu}(q = 0; k_i) = 0 \quad ; \quad \frac{\partial}{\partial q^\rho} N_{\mu\nu}(q; k_i) = A_{\rho\mu\nu}$$

$A_{\rho\mu\nu}$ is a completely antisymmetric tensor.

- In conclusion, one gets

$$M_{(n+1)\mu\nu} = \kappa_D \sum_{i=1}^{n} \left[\frac{-ik_{i[\mu} q^\sigma \bar{S}_{i\,\nu]\sigma} - ik_{i[\nu} q^\rho S_{i\,\mu]\rho} - q^\rho S_{i\,[\mu\rho} q^\sigma \bar{S}_{i\,\nu]\sigma}}{k_i \cdot q} \right.$$
$$\left. + \frac{i}{2} \left(S_{i\,\mu\nu} - \bar{S}_{i\,\mu\nu} \right) \right] M_n(k_i + q) + q^\rho A_{\rho\mu\nu}(q, k_i) \,.$$

[R. Marotta, M. Mojaza and PDV, 1708.02961]
Result confirmed by [Higuchi and Kawai, 1805.11079] using OPE.
- Obviously $A_{\rho\mu\nu}$, being gauge inv., cannot be fixed by gauge inv..
- Subsubleading term is not fixed by gauge invariance.
- In string theory the subsubleading term cannot be written in a factorized form and contains a term with the Bloch-Wigner Dilog that is gauge invariant by itself.

$$2iD_2(z) = Li_2(z) - Li_2(\bar{z}) - \log|z| \log \frac{1-z}{1-\bar{z}} \;\; ; \;\; Li_2 = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

The Unified Massless Closed String Soft Theorem

- Graviton, Dilaton, $B_{\mu\nu}$ Soft Behavior

$$M_{n+1} = \kappa_D \epsilon_\mu \bar{\epsilon}_\nu \left[\left(S^{\mu\nu}_{-1} + S^{\mu\nu}_{0} + S^{\mu\nu}_{1} \right) M_n + q^\rho A_{[\rho\mu\nu]} \right] + \mathcal{O}(q^2)$$

where

$$S^{\mu\nu}_{-1} = \sum_{i=1}^{n} \frac{k_i^\mu k_i^\nu}{k_i q}$$

$$S^{\mu\nu}_{0} = -\frac{i}{2} \sum_{i=1}^{n} \left(\frac{k_i^\nu q_\rho (J_{iL} + S_i)^{\mu\rho}}{k_i q} + \frac{k_i^\mu q_\rho (J_{iR} + \bar{S}_i)^{\nu\rho}}{k_i q} - (S_i^{\mu\nu} - \bar{S}_i^{\mu\nu}) \right)$$

$$S^{\mu\nu}_{1} = \sum_{i=1}^{n} \frac{q_\rho q_\sigma}{k_i q} \left(: J_i^{\rho(\mu} J_i^{\nu)\sigma} : + : J_{iL}^{\rho[\mu} J_{iR}^{\nu]\sigma} : \right) + \alpha' S_{string}$$

with

$$J_{iL} = L_i + S_i \ ; \ \ J_{iR} = L_i + \bar{S}_i \ ; \ \ J_i = L_i + S_i + \bar{S}_i$$

$$A^{(\mu} B^{\nu)} = \frac{1}{2} (A^\mu B^\nu + A^\nu B^\mu) \ ; \ \ A^{[\mu} B^{\nu]} = \frac{1}{2} (A^\mu B^\nu - A^\nu B^\mu)$$

Soft theorems in string theory

- In string theory the soft theorems have been investigated by Ademollo et al (1975) and J. Shapiro (1975)
  B.U.W. Schwab, arXiv:1406.4172 and arXiv:1411.6661
  M. Bianchi, Song He, Yu-tin Huang and Congkao Wen, arXiv:1406.5155
  M. Bianchi and A. Guerrieri, arXiv:1505.05854 and arXiv:1512.00803
  Several papers by A. Sen and Laddha and Sen, 1706.00754
  Several papers with R. Marotta and M. Mojaza
- One can compute the amplitude with a massless closed string state and other particles and derive the soft behavior.
- The leading and the two sub-leading terms are exactly those derived by imposing gauge invariance apart from string corrections in the subsubleading term.

- If the soft particle is a dilaton and the other particles are tachyons (at the tree level) one gets:

$$\epsilon^{d}_{\mu\nu} M^{\mu\nu}_{n+1}(q; k_1 \ldots k_n) = \frac{\kappa_D}{\sqrt{D-2}} \Bigg\{ -\sum_{i=1}^{n} \frac{m_i^2}{k_i \cdot q} \left(1 + q^\rho \frac{\partial}{\partial k_i^\rho} \right.$$

$$\left. + \frac{1}{2} q^\rho q^\sigma \frac{\partial^2}{\partial k_i^\rho \partial k_i^\sigma} \right) + 2 - \sum_{i=1}^{n} \Bigg[k_i^\mu \frac{\partial}{\partial k_i^\mu}$$

$$+ \frac{q^\rho}{2} \left(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho} \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} \right) \Bigg] \Bigg\} M_n + \mathcal{O}(q^2)\,,$$

where $m_i^2 = -\frac{4}{\alpha'}$ and

$$M_n = \frac{8\pi}{\alpha'} \left(\frac{\kappa_D}{2\pi} \right)^{n-2} \int \frac{\prod_{i=1}^{n} d^2 z_i}{dV_{abc}} \prod_{i \neq j} |z_i - z_j|^{\frac{\alpha'}{2} k_i k_j}$$

is the correctly normalized n-tachyon amplitude.

- When the other particles are massless, the soft behavior of a graviton or a dilaton (at the tree level) is given by:

$$M_{(n+1)\mu\nu}(q;k_i) = \kappa_D \sum_{i=1}^{n} \Bigg\{ \frac{1}{k_i q} \Bigg[k_{i\mu}k_{i\nu} - \frac{i}{2} q^\rho \left(k_{i\mu}(J_i)_{\nu\rho} + k_{i\nu}(J_i)_{\mu\rho} \right.$$

$$- \frac{1}{2} \frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i q} \Bigg] - \left(\frac{k_i^\mu q^\nu}{k_i \cdot q} q^\sigma + q^\mu \eta^{\nu\sigma} - \eta^{\mu\nu} q^\sigma \right) \frac{\partial}{\partial k_i^\sigma}$$

$$+ \frac{1}{2} \frac{q_\rho q_\sigma \eta_{\mu\nu} - q_\sigma q_\nu \eta_{\rho\mu} - q_\rho q_\mu \eta_{\sigma\nu}}{k_i \cdot q} \epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\sigma}}$$

$$+ \alpha' \left(q_\sigma k_{i\nu} \eta_{\rho\mu} + q_\rho k_{i\mu} \eta_{\sigma\nu} - \eta_{\rho\mu}\eta_{\sigma\nu}(k_i \cdot q) - q_\rho q_\sigma \frac{k_{i\mu}k_{i\nu}}{k_i \cdot q} \right)$$

$$\times \epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\sigma}} \Bigg\} M_n(k_1 \ldots k_n) + \mathcal{O}(q^2)$$

  obtained through explicit calculations on the full bosonic string amplitude and containing α' correction to the order q.
- String corrections in the heterotic string, but not in superstring.

- Projecting along the dilaton we get

$$\epsilon_d^{\mu\nu} M_{\mu\nu}(q;k_i) = \frac{\kappa_D}{\sqrt{D-2}} \Bigg\{ 2 - \sum_{i=1}^{n} \Bigg[k_i^\mu \frac{\partial}{\partial k_i^\mu} + \frac{q^\rho}{2} \left(2\, k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_i^\rho} - k_{i\rho} \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} \right) - i q^\rho S_{\mu\rho}^{(i)} \frac{\partial}{\partial k_{i\mu}} \Bigg] + \sum_{i=1}^{n} \frac{q^\rho q^\sigma}{2 k_i \cdot q} \left((S_{\rho\mu}^{(i)}) \eta^{\mu\nu} (S_{\nu\sigma}^{(i)}) + D \epsilon_{i\rho} \frac{\partial}{\partial \epsilon_i^\sigma} \right) \Bigg\} M_n + \mathcal{O}(q^2)$$

- No string corrections (in the soft operator) for the soft dilaton.
- Generators of space-time scale and spec. conf. transf.

$$\hat{\mathcal{D}} = x_\mu \hat{\mathcal{P}}^\mu \;; \quad \hat{\mathcal{K}}_\mu = (2 x_\mu x_\lambda - x^2 \eta_{\mu\lambda}) \hat{\mathcal{P}}^\lambda \,.$$

  where $\hat{\mathcal{P}}^\mu$ is the generator of space-time translations.
- Going to momentum space they become:

$$\hat{\mathcal{D}} = -i k_\mu \frac{\partial}{\partial k_\mu} \;; \quad \hat{\mathcal{K}}_\mu = -\left(2 k^\nu \frac{\partial^2}{\partial k^\nu \partial k^\mu} - k_\mu \frac{\partial^2}{\partial k^\nu \partial k_\nu} \right) ,$$

  What is the reason for their presence? See below.

- If we restrict us to (the regular) terms of $\mathcal{O}(q^0)$, for both the bosonic string and superstring, we get:

$$M_{n+1}|_{q^0} = \frac{\kappa_D}{\sqrt{D-2}} \left[2 - \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} \right] M_n \; .$$

  This is the old result of Ademollo et al (1975).
- It can be written in a more suggestive way by observing that, in general, M_n has the following form:

$$M_n = \frac{8\pi}{\alpha'} \left(\frac{\kappa_D}{2\pi} \right)^{n-2} F_n(\sqrt{\alpha'} k_i) \, , \; \kappa_D = \frac{1}{2^{\frac{D-10}{4}}} \frac{g_s}{\sqrt{2}} (2\pi)^{\frac{D-3}{2}} (\sqrt{\alpha'})^{\frac{D-2}{2}} \, ,$$

  where F_n is dimensionless and M_n trivially satisfies the condition:

$$\left(-\sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} + \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} - 2 + (n-2) \frac{D-2}{2} \right) M_n = 0$$

- This equation and the soft behavior imply:

$$M_{n+1} = \frac{\kappa_D}{\sqrt{D-2}} \left[-\sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} + \frac{D-2}{2} g_s \frac{\partial}{\partial g_s} \right] M_n + \mathcal{O}(q) \; .$$

- Same result if we include massless open strings (on a Dp-brane), due to the presence of the term proportional to $\eta^{\mu\nu}$ in the gauge invariance condition.
- The amplitude of a soft dilaton is obtained from the amplitude without a dilaton by a simultaneous rescaling of the Regge slope α' and the string coupling constant g_s.
- Same rescaling that leaves Newton's constant invariant:

$$\left[-\sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} + \frac{D-2}{2} g_s \frac{\partial}{\partial g_s} \right] \kappa_D = 0$$

- No fundamental dimensionless constant in string theory.

- Apply to the case $n = 5$ with 5 dilatons:

$$M_5 = \frac{\kappa_D}{\sqrt{D-2}} \left(2 - \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} \right) M_4 + \mathcal{O}(q)$$

where

$$M_4 = \kappa_D^2 \left(\frac{tu}{s} + \frac{su}{t} + \frac{st}{u} \right) \frac{\Gamma(1 - \frac{\alpha' s}{4})\Gamma(1 - \frac{\alpha' u}{4})\Gamma(1 - \frac{\alpha' t}{4})}{\Gamma(1 + \frac{\alpha' s}{4})\Gamma(1 + \frac{\alpha' u}{4})\Gamma(1 + \frac{\alpha' t}{4})}$$

- In the field theory limit ($\alpha' \to 0$) (supergravity), one gets zero because one has a homogenous function of degree 2.
- This is consistent with the fact that M_5 is vanishing in field theory, due to the Z_2 symmetry.
- In string theory one gets a non-trivial right-hand-side.

- What about the field theory limit of the $\mathcal{O}(q)$ term?
- The Z_2 symmetry of the dilaton in field theory makes all amplitudes with an odd number of dilatons and any number of gravitons vanish (if no B-fields are involved)
- Thus, more generally for an amplitude, M_n involving any number of gravitons and an *even* number of dilatons

$$\lim_{\alpha' \to 0} \left(2 - \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} \right) M_n = 0$$

- As shown in [Loebbert, Mojaza, Plefka, arXiv:1802.05999], the soft theorem also implies invariance under special conformal transformation

$$\lim_{\alpha' \to 0} \left(k_i^{\mu} \frac{\partial^2}{\partial k_i^2} - 2k_i^{\rho} \frac{\partial^2}{\partial k_i^{\rho} k_{i\mu}} - 2iS_i^{\mu\rho} \frac{\partial}{\partial k_i^{\rho}} \right) M_n = 0$$

- Full conformal invariance can be established by introducing a multiplicity dependent scaling dimension $\Delta = \frac{d-2}{n}$.

Origin of string corrections

- In the closed bosonic string the polarization stripped three-point on-shell amplitude for massless states reads:

$$M_3^{\mu\nu;\,\mu_i\nu_i;\,\alpha\beta} = 2\kappa_D \left(\eta^{\mu\mu_i} q^\alpha - \eta^{\mu\alpha} q^{\mu_i} + \eta^{\mu_i\alpha} k_i^\mu - \frac{\alpha'}{2} k_i^\mu q^{\mu_i} q^\mu\right)$$

$$\times \left(\eta^{\nu\nu_i} q^\beta - \eta^{\nu\beta} q^{\nu_i} + \eta^{\nu_i\beta} k_i^\nu - \frac{\alpha'}{2} k_i^\nu q^{\nu_i} q^\nu\right)$$

- It has string corrections with respect to the field theory three-point amplitude.
- They come from the Gauss-Bonnet (α') and R^3 ($(\alpha')^2$) terms [Zwiebach (1985), Matseev+Tseytlin (1987)].
- If, in the pole terms, one keeps also the string corrections in the three-point amplitude, then, from gauge invariance, one obtains precisely the string corrections in the soft behavior.
- String corrections also in the heterotic string but not in superstring.
- Universal behavior of a soft dilaton: never string corrections.

Soft behavior of graviton and dilaton at loop level

With R. Marotta and M. Mojaza, arXiv:1808.04845

- The h-loop amplitude involving one graviton/dilaton and n tachyons is equal

$$M_{n+1}^{(h)} = C_h N_0^{n+1} \int dM \prod_{i<j} \mathrm{e}^{\frac{\alpha'}{2} k_i k_j \mathcal{G}_h(z_i,z_j)} \int d^2 z \prod_{\ell=1}^{n} \mathrm{e}^{\frac{\alpha'}{2} k_\ell q \mathcal{G}_h(z,z_\ell)} \epsilon_q^{\mu} \bar{\epsilon}_q^{\nu}$$

$$\times \left[\frac{\alpha'}{2} \sum_{i,j=1}^{n} k_{i\mu} k_{j\nu} \partial_z \mathcal{G}_h(z, z_i) \partial_{\bar{z}} \mathcal{G}_h(z, z_j) + \frac{1}{2} \eta_{\mu\nu} \omega^I(z) (2\pi Im\tau)^{-1}_{IJ} \bar{\omega}^J(z) \right]$$

[Frau, Lerda, Sciuto, DV, 1987] [Petersen and Sidenius, 1987] [Mandelstam, 1985 and 1992]

- $\mathcal{G}_h$ is the h-loop Green function, ω_I are the h abelian differentials and τ_{IJ} is the period matrix.

- If we compactify $26 - D$ directions on circles with the same radius R, the measure, in the Schottky parametrization , is given by

$$dM = \frac{\prod_{i=1}^{n} d^2 z_i}{dV_{abc}} \prod_{a=1}^{h} \left[\frac{d^2 k_a d^2 \xi_a d^2 \eta_a}{|k_a|^4 |\xi_a - \eta_a|^4} |1 - k_a|^4 \right]$$

$$\prod_{\alpha}{}' \left[\prod_{n=1}^{\infty} |\frac{1}{1 - k_\alpha^n}|^{52} \prod_{n=2}^{\infty} |1 - k_\alpha^n|^4 \right] \frac{(F(\tau, \bar{\tau}))^{26-D}}{(\det Im\tau)^{D/2}}$$

In the Schottky parametrization of a Riemann surface the moduli are k_a (multiplier) and ξ_a and η_a (two fixed points) of Sch. gen. S_a.

- The Schottky generator S_a is a projective transformation that depends on the three parameters k_a, ξ_a, η_a defined by:

$$\frac{S_a(z) - \eta_a}{S_a(z) - \xi_a} = k \frac{z - \eta_a}{z - \xi_a} \; ; \; S_a(z) = \frac{az + b}{cz + d} \; , \; ad - bc = 1$$

- and

$$F(\tau, \bar{\tau}) = \sum_{(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^{2h}} e^{i\pi(\mathbf{p}_R \tau \mathbf{p_R} - \mathbf{p}_L \bar{\tau} \mathbf{p_L})} \; ; \; \mathbf{p}_{R;L} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{\alpha'}}{R} \mathbf{n} \pm \frac{R}{\sqrt{\alpha'}} \mathbf{m} \right)$$

$$\mathcal{V}_{N;h} = \quad C_h(N_0)^N \int dM \langle\Omega| \exp\left[\frac{1}{2}\sum_{i=1}^{N}\sum_{n=0}^{\infty}\frac{\alpha_n^{(i)}}{n!}\alpha_0^{(i)}\frac{\partial^n}{\partial z^n}\log V_i'(z)\Big|_{z=0}\right]$$

$$\times \exp\left[\frac{1}{2}\sum_{i=1}^{N}\sum_{n=0}^{\infty}\frac{\bar{\alpha}_n^{(i)}}{n!}\alpha_0^{(i)}\frac{\partial^n}{\partial \bar{z}^n}\log \bar{V}_i'(\bar{z})\Big|_{\bar{z}=0}\right]$$

$$\times \exp\left[\frac{1}{2}\sum_{i\neq j}\sum_{n,m=0}^{\infty}\frac{\alpha_n^{(i)}}{n!}\partial_z^n\partial_y^m \log\frac{E(V_i(z),V_j(y))}{\sqrt{V_i'(0)V_j'(0)}}\Big|_{z=y=0}\frac{\alpha_m^{(j)}}{m!}\right]$$

$$\times \exp\left[\frac{1}{2}\sum_{i\neq j}\sum_{n,m=0}^{\infty}\frac{\bar{\alpha}_n^{(i)}}{n!}\partial_{\bar{z}}^n\partial_{\bar{y}}^m \log\frac{E(\bar{V}_i(\bar{z}),\bar{V}_j(\bar{y}))}{\sqrt{\bar{V}_i'(0)\bar{V}_j'(0)}}\Big|_{z=y=0}\frac{\bar{\alpha}_m^{(j)}}{m!}\right]$$

$$\times \exp\left[\frac{1}{2}\sum_{i=1}^{N}\sum_{n,m=0}^{\infty}\frac{\alpha_n^{(i)}}{n!}\partial_z^n\partial_y^m \log\frac{E(V_i(z),V_i(y))}{V_i(z)-V_i(y)}\Big|_{z=y=0}\frac{\alpha_m^{(i)}}{m!}\right]$$

$$
\times \exp\left[\frac{1}{2}\sum_{i=1}^{N}\sum_{n,m=0}^{\infty}\frac{\bar{\alpha}_n^{(i)}}{n!}\partial_{\bar{z}}^{n}\partial_{\bar{y}}^{m}\log\frac{E(\bar{V}_i(\bar{z}),\bar{V}_i(\bar{y}))}{\bar{V}_i(\bar{z})-\bar{V}_i(\bar{y})}\Big|_{\bar{z}=\bar{y}=0}\frac{\bar{\alpha}_m^{(i)}}{m!}\right]
$$

$$
\times \exp\left[\sum_{i,j=1}^{N}\sum_{n=0}^{\infty}\left(\frac{\alpha_n^{(i)}}{n!}\partial_z^n+\frac{\bar{\alpha}_n^{(i)}}{n!}\partial_{\bar{z}}^n\right)\mathrm{Re}\left(\int_{z_0}^{V_i(z)}\omega_I\right)(2\pi\mathrm{Im}\tau)^{-1}_{IJ}\right.
$$

$$
\left.\times\sum_{m=0}^{\infty}\left(\frac{\alpha_m^{(j)}}{n!}\partial_z^m+\frac{\bar{\alpha}_m^{(i)}}{m!}\partial_{\bar{z}}^m\right)\mathrm{Re}\left(\int_{z_0}^{V_j(y)}\omega_J\right)\right],
$$

- It is obtained starting from a BRST invariant V_{N+2h} string vertex, that is the BRST invariant version of the one originally proposed by [Lovelace (1970)], and sewing $2h$ legs with a BRST invariant string propagator.
- The sewing procedure has been performed with the functions $V_i(z)$ introduced by Lovelace:

$$V_i(0) = z_i \;\; ; \;\; V_i(1) = z_{i+1} \;\; ; \;\; V_i(\infty) = z_{i-1}$$

- When saturated with N physical states, it gives the corresponding h-loop amplitude.
  [Frau, Hornfeck, Lerda, Sciuto, DV (1988)]
  [Pezzella, Frau, Hornfeck, Lerda, Sciuto, DV (1989)]

- The same amplitude at the tree level is equal to

$$M_0 = C_0 N_0^{n+1} \int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}} \prod_{i<j} \mathrm{e}^{\frac{\alpha'}{2} k_i k_j \mathcal{G}_0(z_i, z_j)} \epsilon_q^{\mu} \bar{\epsilon}_q^{\nu} \frac{\alpha'}{2}$$

$$\times \int d^2 z \prod_{\ell=1}^{n} \left[\mathrm{e}^{\frac{\alpha'}{2} k_\ell q \mathcal{G}_0(z, z_\ell)} \right] \sum_{i,j=1}^{n} k_{i\mu} k_{j\nu} \partial_z \mathcal{G}_0(z, z_i) \partial_{\bar{z}} \mathcal{G}_0(z, z_j)$$

- Except for an extra term and the measure, the two amplitudes have the same form in terms of the Green functions.
- The two Green functions $\mathcal{G}_h$ and $\mathcal{G}_0$ are of course different.

- Using their general properties

$$\partial_{\bar{z}}\partial_z \mathcal{G}_h(z,w) = \pi\delta^{(2)}(z-w) + T(z)$$
$$\int_{\Sigma_h} d^2z \partial_{\bar{z}}\partial_z \mathcal{G}_h(z) = 0 \Longrightarrow \int_{\Sigma_h} d^2z T(z,w) = -2\pi$$
$$\mathrm{e}^{\frac{\alpha'}{2}k_i q \mathcal{G}_h(z_i,z_i)} \sim |E(z_i,z_i)|^{\alpha' k_i q} = 0$$

we can show that, except for an extra term, the soft theorem is exactly as at the tree level.

- The soft theorem for the graviton is exactly as at the tree level, but the extra term modifies the soft behaviour of the dilaton.
- A delicate point is that, in the bosonic string, the integral over the moduli is IR divergent.
- Our result is valid if we regularise (in the same way) both $M_{n+1}^{(h)}$ and $M_n^{(h)}$ with an IR cutoff and we keep D not too low ($D > 4$).

- We get ($m_i^2 = -\frac{4}{\alpha'}$):

$$M_{n;\phi}^{(h)}(k_i;q) = \frac{\kappa_D}{\sqrt{D-2}} \Bigg[-\sum_{i=1}^{n} \frac{m_i^2}{k_i q} \mathrm{e}^{q\partial_{k_i}} + 2 - \sum_{i=1}^{n} \hat{D}_i + h(D-2) + q_\mu \sum_{i=1}^{n} \hat{K}_i^\mu \Bigg] M_n^{(h)} + \mathcal{O}(q^2)\,,$$

where

$$\hat{D}_i = k_i \cdot \frac{\partial}{\partial k_i}\,, \qquad \hat{K}_i^\mu = \frac{1}{2} k_i^\mu \frac{\partial^2}{\partial k_{i\nu} \partial k_i^\nu} - k_i^\rho \frac{\partial^2}{\partial k_i^\rho \partial k_{i\mu}}\,,$$

are the generators of the space-time dilatations and special conformal transformations.
- The contribution in red comes from the extra term for the dilaton.
- The extra term gives also a contribution of order q that is vanishing if we use the Arakelov Green function for $\mathcal{G}_h$.

- The h-loop N-tachyon amplitude is equal to

$$M_n^{(h)} = \left(\frac{8\pi}{\alpha'}\right)^{1-h} \left(\frac{\kappa_D}{2\pi}\right)^{2(h-1)} \frac{1}{(2\pi\alpha')^{\frac{hD}{2}}} \left(\frac{\kappa_D}{2\pi}\right)^n$$

$$\times \int dM \int \prod_{i=1}^{n} d^2 z_i \prod_{i<j} \mathrm{e}^{\frac{\alpha'}{2} k_i k_j \mathcal{G}_h(z_i, z_j)}$$

- It has the following form:

$$M_n^{(h)} = \sqrt{\alpha'}^{(2-D)h-2} \kappa_D^{2(h-1)+n} G\left(\sqrt{\alpha'} k_i, R/\sqrt{\alpha'}\right) ,$$

where

$$\kappa_D = (2\pi)^{\frac{D-3}{2}} \sqrt{2^{-9}} g_s \sqrt{\alpha'}^{\frac{D-2}{2}} \left(\frac{\sqrt{\alpha'}}{R}\right)^{\frac{26-D}{2}}$$

- $M_n^{(h)}$ satisfies the following condition:

$$\left(-\sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}} + \sum_{i=1}^{n} k_{i\mu}\frac{\partial}{\partial k_{i\mu}} - h(D-2) - 2 + \frac{D-2}{2}(n+2(h-1)) - R\frac{\partial}{\partial R}\right) M_n^{(h)} = 0$$

that is equal to

$$\left(-\sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}} + \sum_{i=1}^{n} k_{i\mu}\frac{\partial}{\partial k_{i\mu}} - h(D-2) - 2 + \frac{D-2}{2} g_s\frac{\partial}{\partial g_s} - R\frac{\partial}{\partial R}\right) M_n^{(h)} = 0$$

- Since the subleading (regular) term of $\mathcal{O}(q^0)$ of the soft dilaton amplitude is equal to

$$M_{n+1}^{(h)}|_{q^0} = \frac{\kappa_D}{\sqrt{D-2}} \left(- \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} + 2 + h(D-2) \right) M_n^{(h)}$$

then, for an arbitrary loop, we have

$$M_{n+1}^{(h)}|_{q^0} = \frac{\kappa_D}{\sqrt{D-2}} \left[-\sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} + \frac{D-2}{2} g_s \frac{\partial}{\partial g_s} - R \frac{\partial}{\partial R} \right] M_n^{(h)} \ ,$$

precisely as at the tree level!

- The other terms of order q^{-1}, q^0, q^1 are as at tree level.

Infrared Divergences

- The bosonic string has no UV divergences, but has IR divergences due to the tachyon and dilaton tadpoles.
- Let us see how they appear at one-loop for the n tachyon amplitude:

$$T_N^{(1)} = C_1 N_0^N \int_{\mathcal{F}} d^2\tau \; \mu(\tau,\bar\tau) \prod_{i=1}^{N-1} \left[\int d^2\nu_i \right]$$

$$\times \prod_{i<j} \left| \frac{\sin \pi\nu_{ij}}{\pi} \prod_{n=1}^{\infty} \frac{(1-k^n \mathrm{e}^{2\pi i \nu_{ij}})(1-k^n \mathrm{e}^{-2\pi i \nu_{ij}})}{(1-k^n)^2} \mathrm{e}^{-\pi \frac{(Im(\nu_i-\nu_j))^2}{Im\tau}} \right|^{\alpha' k_i k_j}$$

where $\nu_N = 0$, $k = \mathrm{e}^{2\pi i\tau}$ and

$$\mu(\tau,\bar\tau) = (2\pi)^2 \mathrm{e}^{4\pi Im\tau} \prod_{n=1}^{\infty} \left[\frac{1}{|1-\mathrm{e}^{2\pi i\tau n}|^{48}} \right] \frac{(F(\tau,\bar\tau))^{26-D}}{(Im\tau)^{D/2}}$$

- Let us consider the region of the moduli space where all ν_i are very close to each other and to $\nu_N = 0$.
- It can be reached by introducing the variables η_i, ϵ, ϕ as follows:

$$\mathrm{e}^{i\phi}\epsilon\eta_i = \nu_i \ \ , \ i = 1 \dots N-2 \ \ ; \ \ \epsilon\mathrm{e}^{i\phi} = \nu_{N-1} \ \ ; \ \ \eta_{N-1} = 1$$

  where $\epsilon \sim 0$.
- For small ϵ we can neglect the product over n in the Green function and in terms of the new variables we get

$$T_N^{(1)} = C_1 N_0^N \int_{\mathcal{F}} d^2\tau \mu(\tau, \bar{\tau}) \prod_{i=1}^{N-2} \int d^2\eta_i \prod_{i<j} \left|\eta_{ij}\right|^{\alpha' k_i k_j}$$

$$\times \int_0^{2\pi} d\phi \int_0^1 \frac{d\epsilon}{\epsilon^{3-\frac{\alpha'}{2}q^2}}$$

$$\times \left[1 - \alpha' \sum_{i<j} k_i k_j \frac{\pi\epsilon^2}{\tau_2} \left(\sin\phi(Re(\eta_i - \eta_j) + cos\phi Im(\eta_i - \eta_j)\right)^2 + ..\right]$$

  where the divergence for $\epsilon \sim 0$ has been regularized by the substitution $-3 \to -3 + \frac{\alpha'}{2}q^2$.

- Performing the integrals over ϕ and ϵ we arrive to

$$T_N^{(1)} = C_1 N_0^N \int_{\mathcal{F}} d^2\tau \mu(\tau, \bar{\tau}) \prod_{i=1}^{N-2} \int d^2\eta_i \prod_{i<j} \left|\eta_{ij}\right|^{\alpha' k_i k_j}$$

$$\times \left[\frac{2\pi}{2 - \frac{\alpha'}{2} q^2} - \frac{2\pi}{q^2} \frac{\pi}{\tau_2} \sum_{i<j} k_i k_j \eta_i \bar{\eta}_j + \dots \right]$$

- It can be written as follows:

$$T_N^{(1)} = \frac{C_1}{C_0 N_0} \int_{\mathcal{F}} d^2\tau \mu(\tau, \bar{\tau}) \left[\frac{2\pi T_{(N+1)tach}^{(Tree)}}{2 - \frac{\alpha'}{2} q^2} + \frac{(2\pi)^2 T_{Ntach+1dil}^{(Tree)}}{\alpha' q^2 Im\tau} + \dots \right] =$$

- Also the integral over τ must be regularized in the infrared (for $Im\tau \to \infty$) to get a finite expression.

- The previous IR divergences do not depend on the number of external legs.
- Therefore they can be regularized in the same way both in $M_n^{(h)}$ and in $M_{n+1}^{(h)}$
- Therefore the soft operator connecting the two is left unchanged with respect to the tree level (apart from the extra term for the dilaton).
- Those IR divergences do not appear in superstring theories.
- Therefore, we expect that the soft operator is unchanged with respect to the tree level.
- However, if $D = 4$, we get extra infrared divergences, in the limit of $Im\tau \to \infty$ that depend on the number of external legs. [Green, Schwarz and Brink, 1982]
- Those infrared divergences change drastically the soft operator.

- Finally, string amplitudes involving massive states are plagued by divergences that require mass renormalization [Weinberg (1985) ; Pius, Rutra and Sen, 1311.1257, 1401.7014]
- They can be regularized by not allowing the Koba-Nielsen variables to get close to each other in certain configurations.
- Since they depend on the number of external massive particles, we don't expect that they modify the soft operator.

Arakelov metric and Green function

- The h-loop Green function is equal to

$$\mathcal{G}_h(z_i, z_j) = \log \frac{|E(z_i, z_j)|^2}{|V_i'(0) V_j'(0)|} + \mathrm{Re}\left(\int_{z_j}^{z_i} \omega_I\right) (2\pi \mathrm{Im}\tau)^{-1}_{IJ} \mathrm{Re}\left(\int_{z_i}^{z_j} \omega_J\right)$$

- $V_i(z)$, satisfying the condition $V_i(0) = z_i$, parametrize the coordinates around the punctures.
- When the external states are on-shell physical states, the dependence on the $V_i(z)$ drops out, because of momentum conservation.
- In [D'Hoker and Phong (1988)] the regularized Green function in the conformal gauge is defined as

$$-\mathcal{G}_h(z_i, z_j) = \log \frac{|E(z_i, z_j)|^2}{(\rho_i \rho_j)^{-1/2}} + \mathrm{Re}\left(\int_{z_j}^{z_i} \omega_I\right) (2\pi \mathrm{Im}\tau)^{-1}_{IJ} \mathrm{Re}\left(\int_{z_i}^{z_j} \omega_J\right)$$

where the metric of the Riemann surface in the conformal gauge is chosen to be $ds^2 = \rho(z) dz d\bar{z}$.

- It is invariant under transport along the homology cycles of the Riemann surface.
- Under a conformal transformation $(\rho_i)^{-1/2}$ and $V_i'(0)$ transform in the same way and can be identified:

$$\frac{1}{\rho_i(z_i)} = |V_i'(0)|^2 \;\; ; \;\; \rho = 2g_{z\bar{z}}$$

- The Arakelov metric is defined by the following relation:

$$g^A_{z\bar{z}} R(g^A) = -\partial_z \partial_{\bar{z}} \log g^A_{z\bar{z}} = 8\pi(1-h)K_{z\bar{z}}$$
$$K_{z\bar{z}} = \frac{1}{4\pi h}\omega_I(z)(2\pi \mathrm{Im}\tau)^{-1}_{IJ}\bar{\omega}_J(\bar{z})$$

- The Arakelov Green function satisfies the following property:

$$\partial_z \partial_{\bar{z}} \mathcal{G}^A_h(z,w) = \pi \delta^{(2)}(z-w) - 2\pi K_{z\bar{z}}$$

that implies

$$\int d^2 z \partial_z \partial_{\bar{z}} \mathcal{G}^A_h(z,w) = 0$$

- There is still an arbitrariness in the choice of $g_{z\bar{z}}$ that allows one to choose $g_{z\bar{z}}$ so that $\mathcal{G}_h$ satisfies the relation:

$$\int d^2z \sqrt{g^a} R(g^A) \mathcal{G}_h(z, w) = 0$$

- This condition implies that the contribution of order q of the extra term actually vanishes.

- From the N-string vertex one gets the off-shell N tachyon amplitude:

$$A_N = \int \prod_{i=1}^{n} d^2 z_i \prod_{i=1}^{n} |V_i'(0)|^{2(\frac{\alpha' p_i^2}{4}-1)} \prod_{i<j} |z_i - z_j|^{\alpha' p_i p_j}$$

- Performing the functional integral à la Polyakov and regularizing the divergent terms with a conformal invariant cut-off $\sqrt{g(z_i)}|\Delta z_i| = K$ one gets:

$$\int \prod_{i=1}^{n} d^2 z_i \prod_{i=1}^{n} [\rho(z_i)]^{1-\frac{\alpha' p_i^2}{4}} \prod_{i<j} |z_i - z_j|^{\alpha' p_i p_j}$$

- Comparing the two expressions one gets again:

$$|V_i'(0)|^2 = \frac{1}{\rho(z_i)}$$

String dilaton versus field theory dilaton

- We have seen that the low energy behavior of the string dilaton involves the generators of the conformal group $\hat{\mathcal{D}}$ and $\hat{\mathcal{K}}_\mu$.
- String theory is not conformal invariant because it contains a dimensional constant α'.
- Why then the soft behavior of the string dilaton contains the generators of scale and special conformal transformations?
- In the literature the word dilaton is used for both the string dilaton and the Nambu-Goldstone boson of spontaneously broken conformal invariance.
- To distinguish them, we call the second one field theory dilaton.
- Recently the identification of the two dilatons has been proposed in a paper by [R. Boels and W. Wormsbecher, 1507.08162]
- They proposed that the two dilatons have the same soft behavior.

- This is, in general, only correct at the tree level because, for the field theory dilaton, the loop corrections break explicitly conformal invariance.
- There are theories, as $\mathcal{N} = 4$ super Yang-Mills, that remain conformal invariant also at the quantum level.
- In these theories we expect the dilaton soft theorems to be valid also at loop level.
- For the string dilaton, instead, the soft behavior is mantained also at the loop level provided that the theory is free from UV and IR divergences (or with an IR cutoff).
- In the following we derive the soft behavior of the field theory dilaton that follows from the conformal WT identities.

- As an example consider a conformal invariant version of the Higgs potential:

$$L = -\frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2}\partial_\mu \Xi \partial^\mu \Xi - \frac{\lambda^2}{2}\left(H^2 - \Xi^2\right)^2$$

- Flat direction corresponding to $\langle H \rangle = \langle \Xi \rangle = a$.
- If $a \neq 0$ then conformal invariance is spontaneously broken.
- In terms of the fields:

$$\frac{\Xi + H}{\sqrt{2}} = r + \sqrt{2}a \;\; ; \quad \frac{\Xi - H}{\sqrt{2}} \equiv s$$

- one gets the following Lagrangian ($m = 2\sqrt{2}a\lambda$):

$$L = -\frac{1}{2}\partial_\mu r \partial^\mu r - \frac{1}{2}\partial_\mu s \partial^\mu s - \frac{1}{2}m^2 s^2 - 4\sqrt{2}a\lambda^2 r s^2 - 2\lambda^2 r^2 s^2$$

  s is massive and r is a massless dilaton.
- What is the soft behavior of a dilaton?

- By explicit calculation one gets:

$$T_{n+1} \sim \frac{1}{\sqrt{2}a}\left[-\sum_{i=1}^{n}\frac{m_i^2\left(1+q^\mu\frac{\partial}{\partial k_i^\mu}\right)}{k_i q}+4-n-\sum_i k_i\frac{\partial}{\partial k_i}\right]T_n+\mathcal{O}(q)$$

- This result is more general: it follows from the WT identity of the dilatational current and from the eq. $-\sqrt{2}\,a\,\partial^2 r = T_\mu^\mu$.
- For the string dilaton we get:

$$M_{n+1} \sim \kappa_d\left[-\sum_{i=1}^{n}\frac{m_i^2\left(1+q^\mu\frac{\partial}{\partial k_i^\mu}\right)}{k_i q}+2-\sum_i k_i\frac{\partial}{\partial k_i}\right]M_n+\mathcal{O}(q)$$

- The two soft behaviors are similar but not equal. Why?
- In the case of a NG boson all dimensional factors are rescaled by a scale transformation and one gets $D - n\frac{D-2}{2} \to 4-n$ (for $D=4$) that is the dimension of the amplitude.
- In string theory one rescales only the factor $\frac{1}{\alpha'}$ keeping κ_D fixed.

Soft theorem from Ward Identities

- We consider a field theory whose action is invariant under some global transformation.
- We call j^μ the corresponding Nöther current and we consider the following matrix element:

$$T^* \langle 0 | j^\mu(x) \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

- Taking the derivative with respect to x and then performing a Fourier transform, we get:

$$\int d^D x \, e^{-iq\cdot x} \Big[-\partial_\mu \, T^* \langle 0 | j^\mu(x) \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

$$+ T^* \langle 0 | \partial_\mu j^\mu(x) \phi(x_1) \dots \phi(x_n) | 0 \rangle \Big]$$

$$= - \sum_{i=1}^{n} e^{-iq\cdot x_i} \, T^* \langle 0 | \phi(x_1) \dots \delta\phi(x_i) \dots \phi(x_n) | 0 \rangle \, ,$$

where $\delta\phi$ is the infinitesimal transformation of the field ϕ under the generators of the symmetry.

- More precisely, the last term is equal to

$$-\sum_{i=1}^{n} \int d^D x e^{-iq\cdot x} \, T^* \langle 0|\phi(x_1) \ldots [j^0, \phi(x_i)]\delta(x^0 - x_i^0) \ldots \phi(x_n)|0\rangle \, .$$

- Since the equal-time commutator is proportional to $\delta^{D-1}(\vec{x} - \vec{x}_i)$, the previous expression becomes

$$-\sum_{i=1}^{n} e^{-iq\cdot x_i} \, T^* \langle 0|\phi(x_1) \ldots [Q, \phi(x_i)]\delta(x^0 - x_i^0) \ldots \phi(x_n)|0\rangle \, .$$

where

$$\delta\phi(x) = [Q, \phi(x)]$$

- For a scale transformation:

$$j^\mu_{\mathcal{D}} = x_\nu T^{\mu\nu} \;\; ; \;\; \partial_\mu j^\mu_{\mathcal{D}} = T_\mu{}^\mu \;\; ; \;\; \mathcal{D} = \int d^{D-1}x \; j^0_{\mathcal{D}},$$
$$\delta\phi(x) = [\mathcal{D}, \phi(x)] = i\,(d + x^\mu \partial_\mu)\,\phi(x)\,,$$

  where $T^{\mu\nu}$ is the energy-momentum tensor of the theory.
- We can neglect the first term in the Ward identity by keeping terms up to $\mathcal{O}(q^0)$ (provided that there is no pole) and we can use the following equation

$$T_\mu{}^\mu(x) = -v\,\partial^2\,\xi(x) \;\; ; \;\; \partial_\mu j^\mu_{\mathcal{D}}(x) = v\,(-\partial^2)\,\xi(x)$$

  where v is related to the vev of the dilaton field, denoted by $\langle\xi\rangle$.
- In particular scalar theories we find:

$$v = \frac{D-2}{2}\langle\xi\rangle$$

- We introduce the LSZ operator:

$$\Big[\mathrm{LSZ}\Big] \equiv i^n \left(\prod_{j=1}^{n} \lim_{k_j^2 \to -m_j^2} \int d^D x_j \, \mathrm{e}^{-ik_j \cdot x_j} (-\partial_j^2 + m_j^2) \right) ,$$

where the limits $k_j^2 \to -m_j^2$ put the external states on-shell, which has to be performed only at the end.

- We apply it in the second term of the Ward identity:

$$\Big[\mathrm{LSZ}\Big] \int d^D x \, \mathrm{e}^{-iq\cdot x} \, T^* \langle 0 | \partial_\mu j^\mu_{\mathcal{D}}(x) \phi(x_1) \ldots \phi(x_n) | 0 \rangle$$

$$= (-i) \, v \, (2\pi)^D \delta^{(D)}(\sum_{j=1}^{n} k_j + q) \mathcal{T}_{n+1}(q; k_1, \ldots, k_n) \, ,$$

where we have Fourier transformed and extracted the poles of the correlation function to identify the amplitude $\mathcal{T}_{n+1}$ (a dilaton with momentum q and the other particles with momentum k_i).

- Performing the same operation with the last term of the Ward identity:

$$\left[\text{LSZ}\right]\left(-\sum_{i=1}^{n} e^{-iq\cdot x_i}\, T^*\, \langle 0|\, \phi(x_1)\cdots\delta\phi(x_i)\cdots\phi(x_n)\,|0\rangle\right)$$

$$= -\sum_{i=1}^{n}\Bigg[\lim_{k_i^2\to -m_i^2}(k_i^2+m_i^2)\, i\left(d - D - (k_i+q)^\mu \frac{\partial}{\partial k_i^\mu}\right)$$

$$\times \frac{(2\pi)^D \delta^{(D)}\left(\sum_{j=1}^{n} k_j + q\right)}{(k_i+q)^2+m_i^2}\mathcal{T}_n(k_1,\ldots,k_i+q,\ldots,k_n)\Bigg],$$

where all states $j \neq i$ have already been amputated and put on-shell.

- The next step is to commute the differential operator passing the $i^{\rm th}$ propagator and the δ-function, using the identity:

$$\sum_{i=1}^{n} k_i^{\mu} \frac{\partial}{\partial k_{i\,\nu}} \left[\delta^{(D)}(\sum_{j=1}^{n} k_j) \mathcal{T}_n(k_1, \ldots, k_n) \right] = \delta^{(D)}(\sum_{j=1}^{n} k_j)$$

$$\times \left(-\eta^{\mu\nu} + \sum_{i=1}^{n} k_i^{\mu} \frac{\partial}{\partial k_{i\,\nu}} \right) \mathcal{T}_n(k_1, \ldots, -\sum_{j=1}^{n-1} k_j) \, .$$

- It is necessary to enforce momentum conservation whenever a derivative is acting on the amplitude.
- We will denote this procedure for brevity by $\bar{k}_n = -\sum_{i=1}^{n-1} k_i$.

- Expanding $\mathcal{T}_n$ in the soft momentum q and following through with this procedure, we find:

$$
\begin{aligned}
& - i(2\pi)^D \delta^{(D)}(\sum_{j=1}^{n} k_j + q) \Bigg\{ D - n\, d - \sum_{i=1}^{n} k_i^\mu \frac{\partial}{\partial k_i^\mu} \\
& - \sum_{i=1}^{n} \lim_{k_i^2 \to -m_i^2} \frac{2 m_i^2 (k_i^2 + m_i^2)}{\left[(k_i + q)^2 + m_i^2\right]^2} \left(1 + q^\mu \frac{\partial}{\partial k_i^\mu}\right) \Bigg\} \\
& \times \mathcal{T}_n(k_1, \ldots, \bar{k}_n) \,,
\end{aligned}
$$

where we have used $d = (D-2)/2$, and neglected terms of $\mathcal{O}(q^1)$.

- We arrive at:

$$v\mathcal{T}_{n+1}(0;k_i) = \left[-\sum_{i=1}^{n} \frac{m_i^2}{k_i q}\left(1 + q^{\mu}\frac{\partial}{\partial k_i^{\mu}}\right) + D - nd - \sum_{i=1}^{n} k_i^{\mu}\frac{\partial}{\partial k_i^{\mu}}\right]$$
$$\times \mathcal{T}_n(k_i)$$

  up to terms of order q^0.
- One can repeat the same calculation with the current corresponding to the special conformal transformation:

$$j^{\mu}{}_{(\lambda)} = T^{\mu\nu}(2x_\nu x_\lambda - \eta_{\nu\lambda}x^2)$$
$$\partial_\mu j^{\mu}{}_{(\lambda)} = 2\, x_\lambda\, T^{\mu}{}_{\mu} = 2\, v\, x_\lambda(-\partial^2)\,\xi(x)$$

- The calculation gives the term of order q^1 in the soft behavior.

- The final result is

$$
\begin{aligned}
v\,\mathcal{T}_{n+1}(q;k_1,\ldots,k_n) = \Bigg\{ & -\sum_{i=1}^{n}\frac{m_i^2}{k_i\cdot q}\left(1+q^\mu\frac{\partial}{\partial k_i^\mu}+\frac{1}{2}q^\mu q^\nu\frac{\partial^2}{\partial k_i^\mu\partial k_i^\nu}\right) \\
& + D - nd - \sum_{i=1}^{n} k_i^\mu\frac{\partial}{\partial k_i^\mu} \\
& - q^\lambda\sum_{i=1}^{n}\left[\frac{1}{2}\left(2\,k_i^\mu\frac{\partial^2}{\partial k_i^\mu\partial k_i^\lambda}-k_{i\,\lambda}\frac{\partial^2}{\partial k_{i\nu}\partial k_i^\nu}\right)\right. \\
& \left. + d\,\frac{\partial}{\partial k_i^\lambda}\right]\Bigg\}\,\mathcal{T}_n(k_1,\ldots,\bar{k}_n)+\mathcal{O}(q^2)\,.
\end{aligned}
$$

- In the gravity dilaton $D - nd \to 2$ and there is no term in red appearing in the last line.

Double-soft behavior

- We have used the Ward identities with two currents to compute the non-singular (regular) terms of the double-soft behavior.
- The result is

$$f_\xi^2 T_{n+2}(q_1, q_2, k_1, \ldots, \bar{k}_n) = \Bigg[\left(D - d + \sum_{i=1}^n \hat{D}_i \right) \left(D + \sum_{i=1}^n \hat{D}_i \right) + (q_1^\lambda + q_2^\lambda) \sum_{i=1}^n \hat{K}_{k_i,\lambda} \left(D - d + \sum_{i=1}^n \hat{D}_i \right) \Bigg] T_n(k_1, \ldots, \bar{k}_n) + \mathcal{O}(q_1^2, q_2^2, q_1 q_2) \quad ; \quad d \equiv \frac{D-2}{2} \quad ; \quad d_i = d + \eta_i$$

where $f_\xi = v$ is the dilaton decay constant and

$$\hat{D}_i = -\left(d_i + k_i \cdot \partial_{k_i} \right) , \qquad \hat{K}_{k_i,\mu} = \frac{1}{2} k_{i\mu} \partial_{k_i}^2 - (k_i \cdot \partial_{k_i}) \partial_{k_i,\mu} - d_i \, \partial_{k_i,\mu}$$

- It is the same result that one obtains by performing two single-soft limits one after the other.
[R. Marotta, M. Mojaza and PDV, JHEP **1709** (2017) 001.]

Conclusions and Outlook

- By imposing the conditions required by gauge invariance

$$q^{\mu}\left(M_{\mu\nu} - f\eta_{\mu\nu}\right) = q^{\nu}\left(M_{\mu\nu} - f\eta_{\mu\nu}\right) = 0$$

  and giving the structure of the three-point amplitude, one can fix the three leading terms in the soft behavior of both the graviton and the dilaton and the leading term of the $B_{\mu\nu}$.
- In general, f can be an arbitrary function of the external momenta.
- This would prevent us to get a soft behavior for the dilaton.
- By performing explicit calculations in string theory, it turns out that f has a very simple form.
- Its presence has an important physical meaning when including open strings and discussing loops.
- This result, obtained by imposing the previous relations, agrees with explicit string calculations with (bosonic+heterotic) string corrections to the term of order q^1.

- Those string corrections appear only in the soft behavior of the graviton.
- The soft behavior of the dilaton has no string corrections and it is the same in all string theories: it is universal.
- The string corrections are a direct consequence of the fact that the three-graviton vertex has string corrections with respect to the one of the Einstein-Hilbert action.
- The soft behavior of the dilaton contains the generators of scale and special conformal transformations.
- We have then considered a spontaneous broken conformal theory.
- We have shown that the Ward identities of scale and special conformal transformations fix the three leading terms of the soft behavior of the Nambu-Goldstone boson, that we call field theory dilaton.
- The soft behs. of the two dilatons are similar but not exactly equal.
- This comparison has been done at the tree level.

- Conformal invariance is, in general, explicitly broken by loop corrections that make the field theory dilaton to acquire a mass proportional to the beta-function of the theory (used in Higgs as a dilaton).
- Soft behavior of the field theory dilaton in $\mathcal{N} = 4$ super Yang-Mills that stays conformal invariant also at the quantum level.
- On the other hand, the gravity dilaton stays massless in string perturbation theory and seems to have the same soft behavior at the loop level as in the tree diagrams.
- Why does the soft behavior of the string dilaton contain the generators of scale and special conformal transformations?
- Why is the function f so simple?