Modular Forms of type IIB superstring theory, and $U(1)$-violating amplitudes

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Introduction

- We are interested in low-energy effective action of type IIB superstring theory.

$$\mathcal{L}_{\mathrm{EFT}} \sim R + \alpha'^3 F_0^{(0)}(\tau) R^4 + \alpha'^5 F_0^{(2)}(\tau) d^4 R^4 \\ + \alpha'^6 F_0^{(3)}(\tau) d^6 R^4 + \cdots\cdots$$

- Good understanding on the modular functions of the coefficient of R^4, d^4R^4 and d^6R^4. see Michael's talk.

- We want to extend our understanding for general BPS-terms.

Outline

- Brief review on the known results.
- Derive these results from a different point of view: Using constraints from the consistency of superamplitudes.
- This allows us to extend the results to more general BPS interactions

$$\mathcal{L}^{(p)}_{n\,i} \sim F^{(p)}_{w\,i}(\tau)\, d^{2p}_{(i)}\, \mathcal{P}^{(w)}_{n}(\{\Phi\})\,,$$

where i denotes a possible degeneracy.

Review on non-holomorphic modular forms

- Non-holomorphic modular forms are functions of τ:

$$F^{(p)}_{w,w'}(\tau) \to (c\tau+d)^{w}\,(c\bar{\tau}+d)^{w'}\,F^{(p)}_{w,w'}(\tau)$$

under $SL(2,\mathbb{Z})$ transformation

$$\tau \to \frac{a\tau+b}{c\tau+d}\,.$$

- Covariant derivatives

$$\mathcal{D}_w F^{(p)}_{w,w'}(\tau) := \left(i\tau_2\partial_\tau + \frac{w}{2}\right) F^{(p)}_{w,w'}(\tau) := F^{(p)}_{w+1,w'-1}(\tau)\,,$$

$$\bar{\mathcal{D}}_{w'} F^{(p)}_{w,w'}(\tau) := \left(-i\tau_2\partial_{\bar{\tau}} + \frac{w'}{2}\right) F^{(p)}_{w,w'}(\tau) := F^{(p)}_{w-1,w'+1}(\tau)\,.$$

- We will only consider the cases with $w' = -w$

Examples: Eisenstein series

- Well-known examples: non-holomorphic Eisenstein series

$$E_w(s,\tau) = \sum_{(m,n)\neq(0,0)} \left(\frac{m+n\bar{\tau}}{m+n\tau}\right)^w \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

It has weight $(w,-w)$.

- Satisfies Laplace equation,

$$\begin{aligned}\Delta_-^{(w)} E_w(s,\tau) &:= 4\mathcal{D}_{w-1}\bar{\mathcal{D}}_{-w} E_w(s,\tau) \\ &= (s-w)(s+w-1)\, E_w(s,\tau)\end{aligned}$$

or

$$\begin{aligned}\Delta_+^{(w)} E_w(s,\tau) &:= 4\bar{\mathcal{D}}_{-w-1}\mathcal{D}_{w} E_w(s,\tau) \\ &= (s+w)(s-w-1)\, E_w(s,\tau)\end{aligned}$$

$d^{2p} R^4$ terms

- R^4 and $d^4 R^4$ [Green, Gutperle + Vanhove][Green, Sethi][Basu][Pioline][Berkovits, Vafa]......

$$E_0(\tfrac{3}{2}, \tau) R^4 \,, \qquad E_0(\tfrac{5}{2}, \tau) d^4 R^4$$

- Perturbative expansions in large τ_2 agree with explicit computations

$$E_0(\tfrac{3}{2}, \tau) = 2\zeta(3)\tau_2^{\frac{3}{2}} + 4\zeta(2)\tau_2^{-\frac{1}{2}} + \text{instantons}$$

- The coefficient of $d^6 R^4$ satisfies an inhomogeneous Laplace equation [Green, Vanhove][Yin, Wang]

$$\left(\Delta_-^{(0)} - 12\right) F_0^{(3)}(\tau) = -E_0(\tfrac{3}{2}, \tau)^2$$

as a consequence of first-order differential equations we will derive.

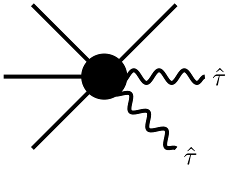
On-shell amplitudes and fluctuations

- To compute scattering amplitudes, we expand the effective action around a fixed background τ^0.

$$F(\tau^0 + \delta\tau) = F(\tau^0) - 2i\tau_2 \partial_\tau F(\tau^0)\hat{\tau} + 2i\tau_2 \partial_{\bar{\tau}} F(\tau^0)\bar{\hat{\tau}} - 2\tau_2^2 \partial_\tau^2 F(\tau^0)\hat{\tau}^2 - 2\bar{\tau}_2^2 \partial_{\bar{\tau}}^2 F(\tau^0)\bar{\hat{\tau}}^2 + \cdots$$

here $\hat{\tau} := i\delta\tau/(2\tau_2^0)$.

- Example: if $F_0^{(0)}(\tau^0 + \delta\tau)$ is the coefficient of R^4, the expansion generates five and higher-point interactions:



$SL(2,\mathbb{Z})$ covariant derivatives

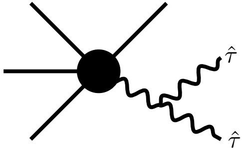
- Each vertex from expansion is in the form of simple derivatives instead of $SL(2,\mathbb{Z})$ covariant derivatives:
  - The fluctuation $\hat{\tau}$ does not transform properly under $U(1)$.
  - Two-derivative classical action, when expanded around τ^0 contains infinity set of $U(1)$-violating vertices. e.g.

$$\frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{4\tau_2} = \partial_\mu \hat{\tau} \partial^\mu \bar{\hat{\tau}} \left(1 + 2(\hat{\tau} + \bar{\hat{\tau}}) + 3(\hat{\tau}^2 + 2\hat{\tau}\bar{\hat{\tau}} + \bar{\hat{\tau}}^2) + \cdots\right)$$

    they are all vanishing on-shell, no $U(1)$-violating amplitudes in type IIB supergravity.

$SL(2,\mathbb{Z})$ covariant derivatives

- The 2-derivative $U(1)$-violating vertices do contribute to higher-derivative amplitudes, by attaching them to higher-derivative vertices: $R^4\hat{\tau}$ etc.



These additional contributions precisely make the simple derivatives to be covariant derivatives.

Field redefinitions

- A systematics: A field redefinition that removes all these on-shell vanishing vertices:

$$B = -\frac{\hat{\tau}}{1-\hat{\tau}} = \frac{\tau - \tau^0}{\tau - \bar{\tau}^0}$$

the normal coordinate of the sigma model G/H. Used in [Schwarz, 83'] for $SU(1,1)$ formulation of the classical theory.

- The field B kills two birds with one stone:
  - The B field transforms linearly

$$B \to \left(\frac{c\bar{\tau}^0 + d}{c\tau^0 + d}\right) B\,.$$

  - Removes all $U(1)$-violating (on-shell vanishing) vertices in the classical action.

Field redefinition and covariant derivatives

- Expanded in terms of B fields:

$$\begin{aligned} F(\tau^0 + \delta\tau) &= F(\tau^0) - 2i\tau_2 \partial_\tau F(\tau^0)\hat{\tau} - 2\tau_2^2 \partial_\tau^2 F(\tau^0)\hat{\tau}^2 + \mathcal{O}(\hat{\tau}^3) \\ &= F(\tau^0) - 2i\tau_2 \partial_\tau F(\tau^0) B + \\ &+ 2\left[-\tau_2^2 \partial_\tau^2 F(\tau^0) + i\tau_2 \partial_\tau F(\tau^0)\right] B^2 + \mathcal{O}(B^3) \end{aligned}$$

Now each term is covariant derivatives

$$\begin{aligned} i\tau_2 \partial_\tau F(\tau^0) &= \mathcal{D}_0 F(\tau^0) \\ -\tau_2^2 \partial_\tau^2 F(\tau^0) + i\tau_2 \partial_\tau F(\tau^0) &= \mathcal{D}_1 \mathcal{D}_0 F(\tau^0) \end{aligned}$$

Field redefinition: other fields

- Similar field redefinition for other fields in theory.
- Example: a fermionic term

$$\Lambda^a \gamma^\mu (\partial_\mu + i q_\Lambda Q_\mu) \bar{\Lambda}_a, \quad \text{with} \quad Q_\mu = \frac{\partial_\mu \tau_1}{2\tau_2} \,.$$

The redefined field

$$\Lambda'_a = \Lambda_a \left(\frac{1 - B}{1 - \bar{B}} \right)^{q_\Lambda/2} \,.$$

Superamplitudes

- Summary: All interactions are manifestly $SL(2,\mathbb{Z})$ invariant, with appropriate choices of the fluctuation fields.
- Ready to study scattering amplitudes:
  - 10D spinor helicity and type IIB SUSY: [Boels and O'Connell] massless momentum

$$p^{BA} := (\gamma^{\mu})^{BA}\, p_{\mu} = \lambda^{Ba}\lambda^{A}_{a}\,.$$

    $A = 1, \ldots, 16$ is the spinor of $SO(9,1)$ and $a = 1, \ldots, 8$ the $SO(8)$ little group index.
  - Supercharges

$$Q^{A}_{n} = \sum_{i=1}^{n} \lambda^{A}_{i,a}\,\eta^{a}_{i}\,, \qquad \bar{Q}^{A}_{n} = \sum_{i=1}^{n} \lambda^{A,a}_{i}\,\frac{\partial}{\partial \eta^{a}_{i}}\,.$$

Superamplitudes

- The on-shell massless states:

$$\Phi(\eta) = B + \eta^a \Lambda'_a + \frac{1}{2!}\eta^a \eta^b \phi_{ab} + \cdots + \frac{1}{8!}(\eta)^8 \bar{B}\,.$$

$q_B = -2, q_{\Lambda'_a} = -\frac{3}{2}, \cdots, q_h = 0, \cdots, q_{\bar{B}} = 2$, and $q_\eta = -\frac{1}{2}$.

- The super amplitudes

$$A_n = \delta^{10}\left(\sum_{r=1}^{n} p_r\right) \delta^{16}(Q_n)\, \hat{A}_n(\eta, \lambda)\,, \quad \text{with} \quad \bar{Q}^A_n \hat{A}_n(\eta, \lambda) = 0\,,$$

$U(1)$-conserved amplitudes $\hat{A}_n \sim \eta^{4(n-4)}$.

Superamplitudes: maximal $U(1)$-violating

- Maximal $U(1)$-violating amplitudes [Boels]

$$A_{n,i}^{(p)} = F_{n-4,i}^{(p)}(\tau)\delta^{16}(Q_n)\,\hat{A}_{n,i}^{(p)}(s_{ij})\,,$$

where i denotes a possible degeneracy.

- The maximal $U(1)$-violating amplitudes have no poles.
- Therefore $\hat{A}_{n,i}^{(p)}(s_{ij})$ is a degree-p symmetric polynomial of s_{ij}. They are super vertices.
- In 4D, they are KLT of $\mathrm{MHV} \otimes \overline{\mathrm{MHV}}$.

Superamplitudes: maximal $U(1)$-violating

- Higher-point amplitudes are related to the lower-point ones by soft limits.
  - The coefficients are related by covariant derivatives,

$$F^{(p)}_{n-4,i}(\tau) \sim \mathcal{D}_{n-5} F^{(p)}_{n-5,i}(\tau)$$

  - The kinematics are related by soft limits (soft B field)

$$\hat{A}^{(p)}_{n,i}(s_{ij})\Big|_{p_n \to 0} \to \hat{A}^{(p)}_{n-1,i}(s_{ij})$$

- Covariant derivative is a result of combination of soft dilaton ($\tau_2 \partial_{\tau_2} A_n$) [Di Vecchia][Di Vecchia, Marotta, Mojaza, Nohle] and soft axion limit ($w \sum_i R_i A_n$).

Superamplitudes: maximal $U(1)$-violating

- $\hat{A}^{(0)}_{n,i}(s_{ij}) = 1$ is for dimension-8 interactions, related to R^4, (no degeneracy)

$$F^{(0)}_{n-4}(\tau) \sim \mathcal{D}_{n-5} \cdots \mathcal{D}_0 E(\tfrac{3}{2}, \tau) \sim E_{n-4}(\tfrac{3}{2}, \tau)$$

- $\hat{A}^{(2)}_{n,i}(s_{ij}) = \sum_{i<j} s_{ij}^2$ is for dimension-12 interactions, related to d^4R^4, (no degeneracy)

$$F^{(2)}_{n-4}(\tau) \sim \mathcal{D}_{n-5} \cdots \mathcal{D}_0 E(\tfrac{5}{2}, \tau) \sim E_{n-4}(\tfrac{5}{2}, \tau)$$

Superamplitudes: maximal $U(1)$-violating

- $\hat{A}^{(3)}_{n,i}(s_{ij}) \sim s_{ij}^3$ for dimension-14 interactions are genuinely new: two independent kinematics (or interaction terms) for $n \geq 6$:

$$\mathcal{O}^{(3)}_{6,1} = 10 \sum_{i<j} s_{ij}^3 + 3 \sum_{i<j<k} s_{ijk}^3 \,,$$

$$\mathcal{O}^{(3)}_{6,2} = 2 \sum_{i<j} s_{ij}^3 - \sum_{i<j<k} s_{ijk}^3 \sim \sum_{P} s_{12} s_{34} s_{56} \,,$$

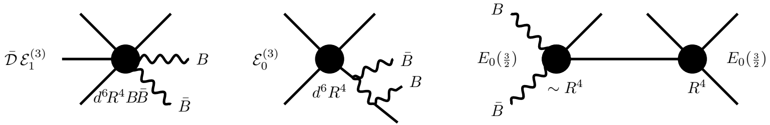
- $\mathcal{O}^{(3)}_{6,1}$ appears at tree-level [Schlotterer], and goes to $\mathcal{O}^{(3)}_{5}$ in the soft.
- $\mathcal{O}^{(3)}_{6,2}$ is constructed to vanish in soft limit, it starts at one loop. Soft implies no "naked" τ, i.e. only appears as $\partial_\mu \tau$.

Superamplitudes: constraints and differential eqs.

- The coefficient of $\mathcal{O}_{6,1}^{(3)}$ is then $\mathcal{E}_{2,1}^{(3)} \sim \mathcal{D}_1\mathcal{E}_1^{(3)}$. ($\mathcal{E}_1^{(3)}$ coefficient of d^6R^4B).
- $\mathcal{E}_1^{(3)} \sim \mathcal{D}_0\mathcal{E}_0^{(3)}$ ($\mathcal{E}_0^{(3)}$ coefficient of d^6R^4) .
- The coefficients are constrained from the consistency of superamplitudes:
  - Consider six-point amplitude, e.g. $A_6(h,h,h,h,B,\bar{B})$.
  - The corresponding superamplitude (with $\leq p^{14}$) cannot have a contact term, so it's uniquely determined by factorizations.
  - Implies a linear relation among the coefficient of contact diagram and those of factorization diagrams of the component amplitude.[Yin, Wang]^2 [Chen, Huang, C.W.]

Superamplitudes: constraints and differential eqs.

- Contributions to the $A_6(h, h, h, h, B, \bar{B})$ at order p^{14}



- The absence of supersymmetric contact terms requires

$$\bar{\mathcal{D}}\mathcal{E}_1^{(3)} + c_1\mathcal{E}_0^{(3)} + c_2 E_0(\tfrac{3}{2})E_0(\tfrac{3}{2}) = 0\,.$$

Superamplitudes: constraints and differential eqs.

- The constants c_1, c_2 are in principle computable from superamplitude, or use known perturbation results:

$$c_1 = -3, \qquad c_2 = \frac{1}{4}\,.$$

- The first-order equation leads to the well-known inhomogeneous Laplace equation [Green, Vanhove]

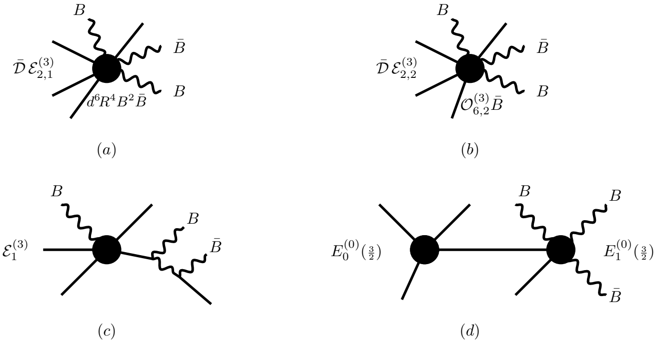
$$\left(\Delta_-^{(0)} - 12\right)\,\mathcal{E}_0^{(3)}(\tau) = -E(\tfrac{3}{2},\tau)^2\,,$$

and for $\mathcal{E}_1^{(3)}(\tau)$

$$\left(\Delta_-^{(1)} - 12\right)\mathcal{E}_1^{(3)}(\tau) = -\frac{1}{2}\,E_1(\tfrac{3}{2})\,E_0(\tfrac{3}{2})\,.$$

Superamplitudes: constraints and differential eqs.

- To study $\mathcal{E}^{(3)}_{2,1}$, $\mathcal{E}^{(3)}_{2,2}$ of $\mathcal{O}^{(3)}_{6,1}$, $\mathcal{O}^{(3)}_{6,2}$, consider the seven-point $A_7(h, h, h, h, B, B, \bar{B})$ at order p^{14}



Superamplitudes: constraints and differential eqs.

- Now, the super-amplitude constraint is

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1} + a_1\mathcal{E}^{(3)}_{1} + a_2 E_0(\tfrac{3}{2})E_1(\tfrac{3}{2}) = 0\,,$$
$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,2} + b_1\mathcal{E}^{(3)}_{1} + b_2 E_0(\tfrac{3}{2})E_1(\tfrac{3}{2}) = 0\,.$$

Two independent equations due to two independent kinematics.

- We actually know $\mathcal{E}^{(3)}_{2,1} \sim \mathcal{D}_1\mathcal{E}^{(3)}_{1}$, so a_1, a_2 are known

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1} - \frac{1}{2}\mathcal{E}^{(3)}_{1} + \frac{1}{40}E_0(\tfrac{3}{2})E_1(\tfrac{3}{2}) = 0\,.$$

- The equation for $\mathcal{E}^{(3)}_{2,2}$ is more interesting.

Superamplitudes: constraints and differential eqs.

- No tree-level term in $\mathcal{E}^{(3)}_{2,2}$ fixes one constant:

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,2} + c_1' \left(\mathcal{E}^{(3)}_1 - \frac{1}{12} E_0(\tfrac{3}{2}) E_1(\tfrac{3}{2}) \right) = 0\,,$$

and an inhomogeneous Laplace equation

$$\left(\Delta^{(2)}_- - 10 \right) \mathcal{E}^{(3)}_{2,2} = -c_1 \left(E_0(\tfrac{3}{2}) E_2(\tfrac{3}{2}) - E_1(\tfrac{3}{2}) E_1(\tfrac{3}{2}) \right)\,.$$

- c_1 can be determined by the 7-point superamplitude, or 6-point string amplitude at one loop.
- Explicit solution: perturbative terms:

$$\mathcal{E}_{2,2}(\tau) \sim \zeta(2)\zeta(3)\tau_2 - \frac{4}{15}\zeta(2)^2 \tau_2^{-1} + \frac{1}{15}\zeta(6)\tau_2^{-3} + (e^{-2\pi\tau_2})\,.$$

Higher-point BPS terms

- There are two sets of dimension-14 terms: $\mathcal{O}^{(3)}_{n,1}$ and $\mathcal{O}^{(3)}_{n,2}$

$$\mathcal{O}^{(3)}_{n,1} = \frac{1}{32}\left((28-3n)\sum_{i<j} s_{ij}^3 + 3\sum_{i<j<k} s_{ijk}^3\right),$$
$$\mathcal{O}^{(3)}_{n,2} = (n-4)\sum_{i<j} s_{ij}^3 - \sum_{i<j<k} s_{ijk}^3 .$$

- They are constructed such that

$$\mathcal{O}^{(3)}_{n,1}\Big|_{p_n\to 0} \to \mathcal{O}^{(3)}_{n-1,1}$$
$$\mathcal{O}^{(3)}_{n,2}\Big|_{p_n\to 0} \to \mathcal{O}^{(3)}_{n-1,2}$$

Higher-point BPS terms

- $\mathcal{O}^{(3)}_{n,1}$ is related to $d^6 R^4$ via soft limits. The coefficients are related by covariant derivatives: they are all determined.

- $\mathcal{O}^{(3)}_{n,2}$ is related to $\mathcal{O}^{(3)}_{6,2}$ via soft limits. We know all the coefficients, up to, one constant.

- The constant can be fixed either by a one-loop six-point computation in type IIB string theory or the unique seven-point superamplitude $(A_7(h, h, h, h, B, B, \bar{B}))$.

Conclusion and remarks

- In general, interactions can be separated into different sets. These of the same set are related by soft limits and covariant derivative $\mathcal{D}_w$.
- Consistency of superamplitudes imposes first-order $\bar{\mathcal{D}}_w$ eqs. on the modular forms of BPS terms.
- Interesting predictions for IIB superstring amplitudes, e.g.
  - $\mathcal{O}_{6,2}$ appears at one loop but vanishes at tree level.
  - $\mathcal{O}_{6,1} \sim d^6 R^4 B^2$ has tree and $1, 3$ loops, but not 2 loops.

Thank you!