Modular Forms of type IIB superstring theory, and $U(1)$-violating amplitudes

Congkao Wen
Queen Mary University of London

To appear
with Michael Green

String Theory from a Worldsheet Perspective
The Galileo Galilei Institute for Theoretical Physics
Introduction

- We are interested in low-energy effective action of type IIB superstring theory.

\[ \mathcal{L}_{\text{EFT}} \sim R + \alpha'^3 F_0^{(0)}(\tau) R^4 + \alpha'^5 F_0^{(2)}(\tau) d^4 R^4 
+ \alpha'^6 F_0^{(3)}(\tau) d^6 R^4 + \cdots \]

- Good understanding on the modular functions of the coefficient of \( R^4, d^4 R^4 \) and \( d^6 R^4 \). see Michael’s talk.

- We want to extend our understanding for general BPS-terms.
Outline

- Brief review on the known results.

- Derive these results from a different point of view: Using constraints from the consistency of superamplitudes.

- This allows us to extend the results to more general BPS interactions

\[ \mathcal{L}_{n_i}^{(p)} \sim F_{w_i}^{(p)}(\tau) d_{(i)}^{2p} P_{n}^{(w)}(\{\Phi\}), \]

where \( i \) denotes a possible degeneracy.
Review on non-holomorphic modular forms

Non-holomorphic modular forms are functions of $\tau$:

$$F_{w,w}'(\tau) \to (c\tau + d)^w (c\bar{\tau} + d)^{w'} F_{w,w}'(\tau)$$

under $SL(2,\mathbb{Z})$ transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}.$$ 

Covariant derivatives

$$\mathcal{D}_w F_{w,w}'(\tau) := \left( i\tau_2 \partial_{\tau} + \frac{w}{2} \right) F_{w,w}'(\tau) := F_{w+1,w'-1}^{(p)}(\tau),$$

$$\bar{\mathcal{D}}_{w'} F_{w,w}'(\tau) := \left( -i\tau_2 \partial_{\bar{\tau}} + \frac{w'}{2} \right) F_{w,w}'(\tau) := F_{w-1,w'+1}^{(p)}(\tau).$$

We will only consider the cases with $w' = -w$
Examples: Eisenstein series

- Well-known examples: non-holomorphic Eisenstein series

\[
E_w(s, \tau) = \sum_{(m,n) \neq (0,0)} \left( \frac{m + n\bar{\tau}}{m + n\tau} \right)^w \frac{\tau_2^s}{|m + n\tau|^{2s}}
\]

It has weight \((w, -w)\).

- Satisfies Laplace equation,

\[
\Delta_{-w}^{(w)} E_w(s, \tau) := 4\mathcal{D}_{w-1} \bar{\mathcal{D}}_{-w} E_w(s, \tau) = (s - w)(s + w - 1) E_w(s, \tau)
\]

or

\[
\Delta_{+w}^{(w)} E_w(s, \tau) := 4\bar{\mathcal{D}}_{-w-1} \mathcal{D}_w E_w(s, \tau) = (s + w)(s - w - 1) E_w(s, \tau)
\]
$d^{2p} R^4$ terms

- $R^4$ and $d^4 R^4$ \cite{Green, Gutperle + Vanhove, Green, Sethi, Basu, Pioline, Berkovits, Vafa}......

\[ E_0\left(\frac{3}{2}, \tau\right) R^4, \quad E_0\left(\frac{5}{2}, \tau\right) d^4 R^4 \]

- Perturbative expansions in large $\tau_2$ agree with explicit computations

\[ E_0\left(\frac{3}{2}, \tau\right) = 2\zeta(3)\tau_2^{\frac{3}{2}} + 4\zeta(2)\tau_2^{-\frac{1}{2}} + \text{instantons} \]

- The coefficient of $d^6 R^4$ satisfies an inhomogeneous Laplace equation \cite{Green, Vanhove, Yin, Wang}

\[ \left(\Delta^{(0)} - 12\right) F_0^{(3)}(\tau) = -E_0\left(\frac{3}{2}, \tau\right)^2 \]

as a consequence of first-order differential equations we will derive.
To compute scattering amplitudes, we expand the effective action around a fixed background $\tau^0$.

\[
F(\tau^0 + \delta \tau) = F(\tau^0) - 2i \tau_2 \partial_{\tau} F(\tau^0) \hat{\tau} + 2i \tau_2 \partial_{\bar{\tau}} F(\tau^0) \bar{\hat{\tau}}
\]

\[
- 2\tau_2^2 \partial_{\tau}^2 F(\tau^0) \hat{\tau}^2 - 2\bar{\tau}_2^2 \partial_{\bar{\tau}}^2 F(\tau^0) \bar{\hat{\tau}}^2 + \cdots
\]

here $\hat{\tau} := i\delta \tau/(2\tau_2^0)$.

Example: if $F_0^{(0)}(\tau^0 + \delta \tau)$ is the coefficient of $R^4$, the expansion generates five and higher-point interactions:
Each vertex from expansion is in the form of simple derivatives instead of $SL(2, \mathbb{Z})$ covariant derivatives:

- The fluctuation $\hat{\tau}$ does not transform properly under $U(1)$.

- Two-derivative classical action, when expanded around $\tau^0$ contains infinity set of $U(1)$-violating vertices. e.g.

\[
\frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{4\tau_2} = \partial_\mu \hat{\tau} \partial^\mu \bar{\tau} (1 + 2(\hat{\tau} + \bar{\tau}) + 3(\hat{\tau}^2 + 2\hat{\tau}\bar{\tau} + \bar{\tau}^2) + \cdots)
\]

they are all vanishing on-shell, no $U(1)$-violating amplitudes in type IIB supergravity.
SL(2, \mathbb{Z}) covariant derivatives

- The 2-derivative $U(1)$-violating vertices do contribute to higher-derivative amplitudes, by attaching them to higher-derivative vertices: $R^4\hat{\tau}$ etc.

These additional contributions precisely make the simple derivatives to be covariant derivatives.
Field redefinitions

- **A systematics:** A field redefinition that removes all these on-shell vanishing vertices:

  \[ B = - \frac{\hat{\tau}}{1 - \hat{\tau}} = \frac{\tau - \tau^0}{\tau - \bar{\tau}^0} \]

  the normal coordinate of the sigma model \( G/H \). Used in [Schwarz, 83'] for \( SU(1, 1) \) formulation of the classical theory.

- **The field \( B \) kills two birds with one stone:**
  - The \( B \) field transforms linearly
    \[ B \rightarrow \left( \frac{c\bar{\tau}^0 + d}{c\tau^0 + d} \right) B. \]
  - Removes all \( U(1) \)-violating (on-shell vanishing) vertices in the classical action.
Field redefinition and covariant derivatives

- Expanded in terms of $B$ fields:

$$ F(\tau^0 + \delta \tau) = F(\tau^0) - 2i\tau_2 \partial_\tau F(\tau^0) \hat{\tau} - 2\tau_2^2 \partial_\tau^2 F(\tau^0) \hat{\tau}^2 + O(\hat{\tau}^3) $$

$$ = F(\tau^0) - 2i\tau_2 \partial_\tau F(\tau^0) B + $$

$$ + 2 \left[ -\tau_2^2 \partial_\tau^2 F(\tau^0) + i\tau_2 \partial_\tau F(\tau^0) \right] B^2 + O(B^3) $$

Now each term is \textbf{covariant derivatives}

$$ i\tau_2 \partial_\tau F(\tau^0) = D_0 F(\tau^0) $$

$$ -\tau_2^2 \partial_\tau^2 F(\tau^0) + i\tau_2 \partial_\tau F(\tau^0) = D_1 D_0 F(\tau^0) $$
Field redefinition: other fields

- Similar field redefinition for other fields in theory.
- Example: a fermionic term

\[ \Lambda^a \gamma^\mu (\partial_\mu + iq_\Lambda Q_\mu) \bar{\Lambda}_a, \quad \text{with} \quad Q_\mu = \frac{\partial_\mu \tau_1}{2\tau_2}. \]

**The redefined field**

\[ \Lambda'_a = \Lambda_a \left( \frac{1 - B}{1 - \bar{B}} \right)^{q_\Lambda/2}. \]
Superamplitudes

- **Summary**: All interactions are manifestly $SL(2, \mathbb{Z})$ invariant, with appropriate choices of the fluctuation fields.

- **Ready to study scattering amplitudes**:
  - 10D spinor helicity and type IIB SUSY: [Boels and O’Connell]
    - Massless momentum
      \[ p^{BA} := (\gamma^\mu)^{BA} p_\mu = \lambda^{Ba} \lambda_a^A. \]
      
      $A = 1, \ldots, 16$ is the spinor of $SO(9,1)$ and $a = 1, \ldots, 8$ the $SO(8)$ little group index.
    
    - Supercharges
      \[ Q_n^A = \sum_{i=1}^n \lambda^A_{i,a} \eta^a_i, \quad \bar{Q}_n^A = \sum_{i=1}^n \lambda^A_{i,a} \frac{\partial}{\partial \eta^a_i}. \]
Superamplitudes

- The on-shell massless states:

\[ \Phi(\eta) = B + \eta^a \Lambda'_a + \frac{1}{2!} \eta^a \eta^b \phi_{ab} + \cdots + \frac{1}{8!} (\eta)^8 \bar{B}. \]

\[ q_B = -2, \quad q_{\Lambda'_a} = -\frac{3}{2}, \quad q_h = 0, \quad q_{\bar{B}} = 2, \quad \text{and} \quad q_\eta = -\frac{1}{2}. \]

- The super amplitudes

\[ A_n = \delta^{10} \left( \sum_{r=1}^{n} p_r \right) \delta^{16} (Q_n) \hat{A}_n(\eta, \lambda), \quad \text{with} \quad \bar{Q}_n^A \hat{A}_n(\eta, \lambda) = 0, \]

\[ U(1)\text{-conserved amplitudes} \hat{A}_n \sim \eta^{4(n-4)}. \]
Superamplitudes: maximal $U(1)$-violating

- Maximal $U(1)$-violating amplitudes [Boels]

\[ A_{n,i}^{(p)} = F_{n-4,i}(\tau)\delta^{16}(Q_n) \hat{A}_{n,i}^{(p)}(s_{ij}), \]

where $i$ denotes a possible degeneracy.

- The maximal $U(1)$-violating amplitudes have no poles.

- Therefore $\hat{A}_{n,i}^{(p)}(s_{ij})$ is a degree-$p$ symmetric polynomial of $s_{ij}$. They are super vertices.

- In 4D, they are KLT of $\text{MHV} \otimes \overline{\text{MHV}}$. 
Superamplitudes: maximal $U(1)$-violating

- Higher-point amplitudes are related to the lower-point ones by soft limits.

  - The coefficients are related by covariant derivatives,

    $$F^{(p)}_{n-4,i}(\tau) \sim D_{n-5} F^{(p)}_{n-5,i}(\tau)$$

  - The kinematics are related by soft limits (soft $B$ field)

    $$\hat{A}^{(p)}_{n,i}(s_{ij})\big|_{p_n \to 0} \to \hat{A}^{(p)}_{n-1,i}(s_{ij})$$

- Covariant derivative is a result of combination of soft dilaton
  $$(\tau_2 \partial_{\tau_2} A_n)$$ [Di Vecchia] and soft axion limit
  $$(w \sum_i R_i A_n).$$
Superamplitudes: maximal $U(1)$-violating

- $\hat{A}_{n,i}^{(0)}(s_{ij}) = 1$ is for dimension-8 interactions, related to $R^4$, (no degeneracy)

$$F_{n-4}^{(0)}(\tau) \sim D_{n-5} \cdots D_0 E\left(\frac{3}{2}, \tau\right) \sim E_{n-4}\left(\frac{3}{2}, \tau\right)$$

- $\hat{A}_{n,i}^{(2)}(s_{ij}) = \sum_{i<j} s_{ij}^2$ is for dimension-12 interactions, related to $d^4 R^4$, (no degeneracy)

$$F_{n-4}^{(2)}(\tau) \sim D_{n-5} \cdots D_0 E\left(\frac{5}{2}, \tau\right) \sim E_{n-4}\left(\frac{5}{2}, \tau\right)$$
Superamplitudes: maximal $U(1)$-violating

- $\hat{A}_{n,i}^{(3)}(s_{ij}) \sim s_{ij}^3$ for dimension-14 interactions are genuinely new: two independent kinematics (or interaction terms) for $n \geq 6$:

$$O_{6,1}^{(3)} = 10 \sum_{i<j} s_{ij}^3 + 3 \sum_{i<j<k} s_{ijk},$$

$$O_{6,2}^{(3)} = 2 \sum_{i<j} s_{ij}^3 - \sum_{i<j<k} s_{ijk} \sim \sum_{P} s_{12} s_{34} s_{56},$$

- $O_{6,1}^{(3)}$ appears at tree-level [Schlotterer], and goes to $O_{5}^{(3)}$ in the soft.

- $O_{6,2}^{(3)}$ is constructed to vanish in soft limit, it starts at one loop. Soft implies no “naked” $\tau$, i.e. only appears as $\partial_\mu \tau$. 
Superamplitudes: constraints and differential eqs.

- The coefficient of $O_{6,1}^{(3)}$ is then $\mathcal{E}_{2,1}^{(3)} \sim D_1 \mathcal{E}_1^{(3)}$. ($\mathcal{E}_1^{(3)}$ coefficient of $d^6 R^4 B$).

- $\mathcal{E}_1^{(3)} \sim D_0 \mathcal{E}_0^{(3)}$ ($\mathcal{E}_0^{(3)}$ coefficient of $d^6 R^4$).

- The coefficients are constrained from the consistency of superamplitudes:
  
  - Consider six-point amplitude, e.g. $A_6(h, h, h, h, B, \bar{B})$.
  
  - The corresponding superamplitude (with $\leq p^{14}$) cannot have a contact term, so it’s uniquely determined by factorizations.

  - Implies a linear relation among the coefficient of contact diagram and those of factorization diagrams of the component amplitude.$[\text{Yin, Wang}^2 \ [\text{Chen, Huang, C.W.}]]}$
Superamplitudes: constraints and differential eqs.

- Contributions to the $A_6(h, h, h, h, B, \bar{B})$ at order $p^{14}$

\[
\bar{D}\mathcal{E}_1^{(3)} - d^6 R^4 B \bar{B} + c_1 \mathcal{E}_0^{(3)} - d^6 R^4 \bar{B} + B E_0(\frac{3}{2}) E_0(\frac{3}{2}) = 0.
\]

- The absence of supersymmetric contact terms requires

\[
\bar{D}\mathcal{E}_1^{(3)} + c_1 \mathcal{E}_0^{(3)} + c_2 E_0(\frac{3}{2}) E_0(\frac{3}{2}) = 0.
\]
Superamplitudes: constraints and differential eqs.

- The constants $c_1, c_2$ are in principle computable from superamplitude, or use known perturbation results:

  $c_1 = -3, \quad c_2 = \frac{1}{4}.$

- The first-order equation leads to the well-known inhomogeneous Laplace equation [Green, Vanhove]

  $$\left(\Delta^{(0)} - 12\right) E_0^{(3)}(\tau) = -E\left(\frac{3}{2}, \tau\right)^2,$$

  and for $E_1^{(3)}(\tau)$

  $$\left(\Delta^{(1)} - 12\right) E_1^{(3)}(\tau) = -\frac{1}{2} E_1\left(\frac{3}{2}\right) E_0\left(\frac{3}{2}\right).$$
Superamplitudes: constraints and differential eqs.

- To study $\mathcal{E}_{2,1}^{(3)}$, $\mathcal{E}_{2,2}^{(3)}$ of $\mathcal{O}_{6,1}^{(3)}$, $\mathcal{O}_{6,2}^{(3)}$, consider the seven-point $A_7(h, h, h, h, B, B, \bar{B})$ at order $p^{14}$
Superamplitudes: constraints and differential eqs.

- Now, the super-amplitude constraint is

\[ \bar{D}\xi_{2,1}^{(3)} + a_1 \xi_1^{(3)} + a_2 E_0(\frac{3}{2})E_1(\frac{3}{2}) = 0, \]

\[ \bar{D}\xi_{2,2}^{(3)} + b_1 \xi_1^{(3)} + b_2 E_0(\frac{3}{2})E_1(\frac{3}{2}) = 0. \]

Two independent equations due to two independent kinematics.

- We actually know \( \xi_{2,1}^{(3)} \sim D_1 \xi_1^{(3)} \), so \( a_1, a_2 \) are known

\[ \bar{D}\xi_{2,1}^{(3)} - \frac{1}{2} \xi_1^{(3)} + \frac{1}{40} E_0(\frac{3}{2})E_1(\frac{3}{2}) = 0. \]

- The equation for \( \xi_{2,2}^{(3)} \) is more interesting.
Superamplitudes: constraints and differential eqs.

- No tree-level term in $\mathcal{E}_{2,2}^{(3)}$ fixes one constant:

$$
\bar{D}\mathcal{E}_{2,2}^{(3)} + c_1' \left( \mathcal{E}_1^{(3)} - \frac{1}{12} E_0(\frac{3}{2})E_1(\frac{3}{2}) \right) = 0,
$$

and an inhomogeneous Laplace equation

$$
\left( \Delta^{(2)} - 10 \right) \mathcal{E}_{2,2}^{(3)} = -c_1 \left( E_0(\frac{3}{2})E_2(\frac{3}{2}) - E_1(\frac{3}{2})E_1(\frac{3}{2}) \right).
$$

- $c_1$ can be determined by the 7-point superamplitude, or 6-point string amplitude at one loop.

- Explicit solution: perturbative terms:

$$
\mathcal{E}_{2,2}(\tau) \sim \zeta(2)\zeta(3)\tau_2 - \frac{4}{15} \zeta(2)^2 \tau_2^{-1} + \frac{1}{15} \zeta(6) \tau_2^{-3} + (e^{-2\pi\tau_2}).
$$
Higher-point BPS terms

- There are two sets of dimension-14 terms: $\mathcal{O}^{(3)}_{n,1}$ and $\mathcal{O}^{(3)}_{n,2}$

$$
\mathcal{O}^{(3)}_{n,1} = \frac{1}{32} \left( (28 - 3n) \sum_{i<j} s_{ij}^3 + 3 \sum_{i<j<k} s_{ijk}^3 \right),
$$

$$
\mathcal{O}^{(3)}_{n,2} = (n - 4) \sum_{i<j} s_{ij}^3 - \sum_{i<j<k} s_{ijk}^3.
$$

- They are constructed such that

$$
\mathcal{O}^{(3)}_{n,1} \mid_{p_n \rightarrow 0} \rightarrow \mathcal{O}^{(3)}_{n-1,1}
$$

$$
\mathcal{O}^{(3)}_{n,2} \mid_{p_n \rightarrow 0} \rightarrow \mathcal{O}^{(3)}_{n-1,2}
$$
Higher-point BPS terms

- $O^{(3)}_{n,1}$ is related to $d^6 R^4$ via soft limits. The coefficients are related by covariant derivatives: they are all determined.

- $O^{(3)}_{n,2}$ is related to $O^{(3)}_{6,2}$ via soft limits. We know all the coefficients, up to, one constant.

- The constant can be fixed either by a one-loop six-point computation in type IIB string theory or the unique seven-point superamplitude ($A_7(h, h, h, h, B, B, \bar{B})$).
Conclusion and remarks

- In general, interactions can be separated into different sets. These of the same set are related by soft limits and covariant derivative $D_w$.

- Consistency of superamplitudes imposes first-order $\bar{D}_w$ eqs. on the modular forms of BPS terms.

- Interesting predictions for IIB superstring amplitudes, e.g.
  - $O_{6,2}$ appears at one loop but vanishes at tree level.
  - $O_{6,1} \sim d^6 R^4 B^2$ has tree and 1, 3 loops, but not 2 loops.
Thank you!