

Strings on Celestial Sphere



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String Theory from a Worldsheet Perspective
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based on:

St.St., T.R. Taylor:
Strings on Celestial Sphere
arXiv:1806.05688
Nucl. Phys. B935 (2018) 388-411

Symmetries of Celestial Amplitudes
arXiv:1812.01080
to appear in Phys. Lett. B

+ work to appear

Recap: studying scattering amplitudes:
**deep connections between
gravity and gauge interactions**

e.g.: KLT, BCJ, EYM (double-copy-construction)
(*in momentum or twistor space*)

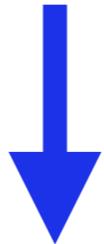
traditional momentum space description:

$$p_k^\mu, \quad k = 1, \dots, N$$
$$p_k^2 = -m_k^2$$

- amplitudes specified by asymptotic wave functions, which transform simply under space-time translations
- with manifest translation symmetry
- traditional amplitudes describe transitions between momentum eigenstates

D=4 Minkowski probably not the right space
to see **all** symmetries
of scattering amplitudes

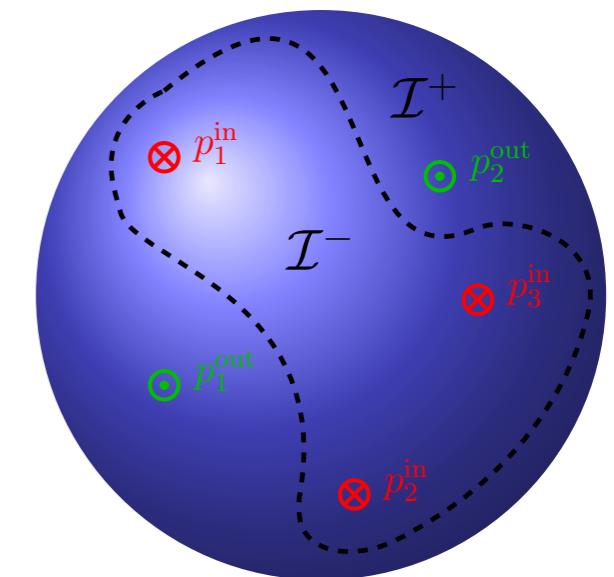
Lorentz group in $\mathbf{R}^{1,D+1}$ is identical
to Euclidean D-dimensional conformal group $\text{SO}(1,D+1)$



Scattering amplitudes in $\mathbf{R}^{1,D+1}$
interpretation
as Euclidean D-dimensional conformal correlators



D=2: celestial sphere



Can 2D CFT on celestial sphere offer some new insight into
gauge-gravity connections ?

N particles on celestial sphere

$$p_k \rightarrow (E_k, z_k, \bar{z}_k)$$

represent points z_k on CS^2

with:
$$z_k = \frac{p_k^1 + i p_k^2}{p_k^0 + p_k^3} , \quad E_k = p_k^0 , \quad (\vec{p}_k)^2 = (p_k^0)^2$$

$$p_k^\mu = E_k \left(1 , \frac{z_k + \bar{z}_k}{1 + |z_k|^2} , \frac{-i(z_k - \bar{z}_k)}{1 + |z_k|^2} , \frac{1 - |z_k|^2}{1 + |z_k|^2} \right)$$

$$:= \omega_k q_k^\mu \qquad \qquad \qquad \omega_k = \frac{2 E_k}{(1 + |z_k|^2)}$$

E.g.:

$$\langle ij \rangle = 2 (\omega_i \omega_j)^{1/2} (z_i - z_j)$$

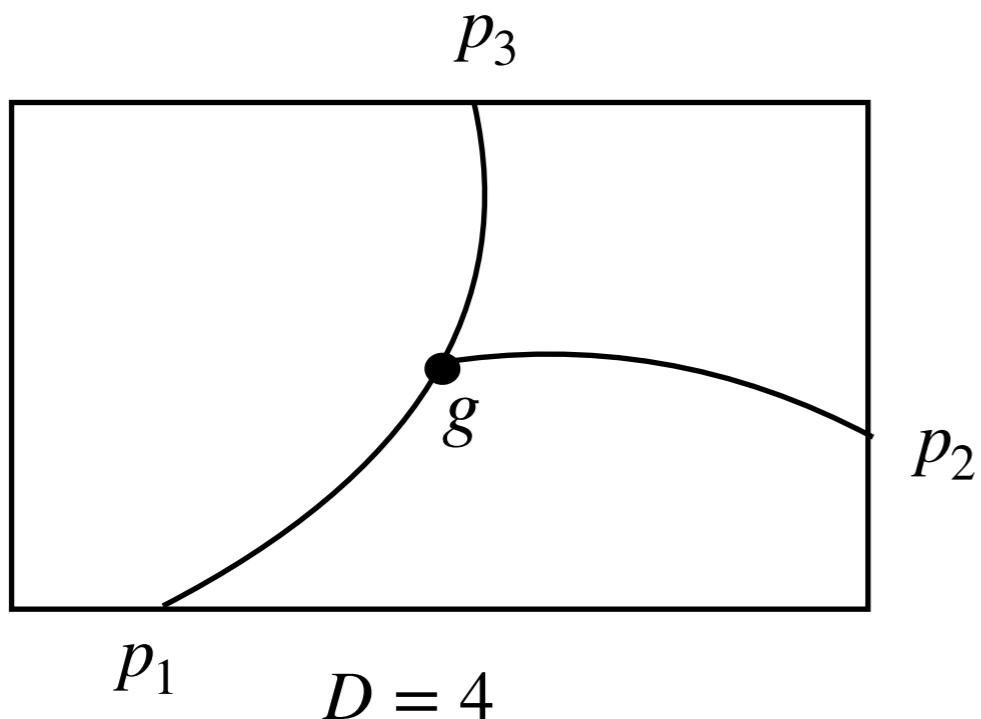
$$[ij] = 2 (\omega_i \omega_j)^{1/2} (\bar{z}_i - \bar{z}_j)$$

Lorentz symmetry:

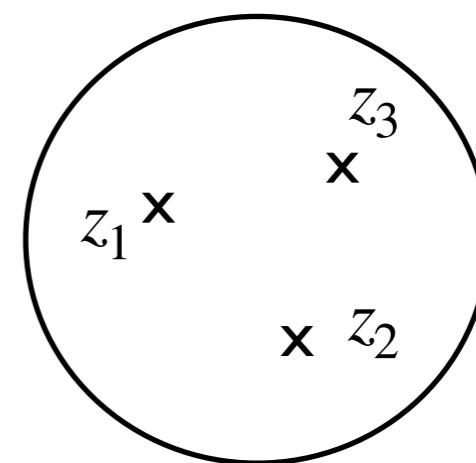
$$z_i \rightarrow \frac{az_i + b}{cz_i + d}$$

global conformal symmetry
on CS^2

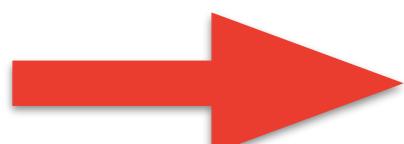
Amplitudes = conformal correlators of primary fields on CS^2



$$z_k = \frac{p_k^1 + ip_k^2}{p_k^0 + p_k^3} =$$



$$\sim \frac{g}{|z_1 - z_2|^{h_1+h_2-h_3} |z_2 - z_3|^{h_2+h_3-h_1} |z_1 - z_3|^{h_1+h_3-h_2}}$$



D=4 space-time QFT correlators

D=2 Euklidian CFT correlators

D=2 CFT correlators involve conformal wave packets

in momentum basis: plane waves with momentum $p = \omega q(z)$

in conformal basis: conformal primary wave functions $\Delta = h + \bar{h} \in \mathbb{C}$

construct complete set of on-shell wave functions in D=4:

solves D=4 wave (Maxwell) equations and transforms as $SL(2, \mathbb{Z})$ conformal primaries

Pasterski, Shao arXiv:1705.01027

bases	plane waves	conformal primary wavefunctions
vector fields	$A_{\mu\ell}(x; p) = \epsilon_{\mu\ell}(p) \exp\{\mp ip_\mu x^\mu\}$	$V_{\mu J}^{\Delta\pm}(x; z, \bar{z}) = (\partial_J q_\mu) (-q_\mu x^\mu \mp i\epsilon)^{-\Delta}$
3 continuous parameters	p^μ $p^2 = 0, \quad p > 0$	$\Delta = 1 + i\lambda \quad (\lambda \in \mathbb{R})$ $z \in CS^2$
2 discrete parameters	4d helicity $\ell = \pm 1$ incoming vs. outgoing	2d spin $J = \pm 1$ incoming vs. outgoing

in the massless case the change of basis is furnished

by **Mellin transform** of plane wave (or plus a shadow transform):

$$V_{\mu J}^{\Delta\pm}(x^\mu; z, \bar{z}) = \frac{\partial_J q_\mu}{\sqrt{2}} \int_0^\infty d\omega \omega^{\Delta-1} e^{\pm i\omega q \cdot x - \epsilon\omega}$$

$$= (\mp i)^\Delta \frac{\Gamma(\Delta)}{\sqrt{2}} \frac{\partial_J q_\mu}{(-q_\mu x^\mu \mp i\epsilon)^\Delta}$$

specified by x and
conformal dimension
 $\Delta = 1 + i\lambda, \quad \lambda \in \mathbb{R}$

no dependence on
D=4 momentum p^μ

$$\partial_J q^\mu = \begin{cases} \partial_{\bar{z}} q^\mu = \sqrt{2} \epsilon_+^\mu(q) = (\bar{z}, 1, -i, -\bar{z}) \\ \partial_z q^\mu = \sqrt{2} \epsilon_-^\mu(q) = (z, 1, +i, -z) \end{cases}$$

Pasterski, Shao, Strominger, 2017

N-point amplitude on celestial sphere

$$\mathcal{A}(\{p_i, \xi_j\}) = i(2\pi)^4 \delta^{(4)}\left(p_1 + p_2 - \sum_{k=3}^N p_k\right) \mathcal{M}(\{p_i, \xi_j\})$$

In the massless case, with or without spin,
transition from momentum space to conformal primary wavefunctions
with conformal dimension Δ_j
is implemented by Mellin transform:

$$\tilde{\phi}(\Delta) = \int_0^\infty d\omega \omega^{\Delta-1} \phi(\omega)$$

Mellin transform, with: $\Delta_j = 1 + i\lambda_j$

$$\begin{aligned} \tilde{\mathcal{A}}_{\{\lambda_n\}}(z_n, \bar{z}_n) &= \left(\prod_{n=1}^N \int_0^\infty \omega_n^{i\lambda_n} d\omega_n \right) \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=3}^N \omega_k q_k) \\ &\quad \times \mathcal{M}(\omega_n, z_n, \bar{z}_n) \end{aligned}$$

Three-point Amplitudes

(i) Mostly-plus three-gluon amplitude

$$\mathcal{M}(-, -, +) = \frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle} = \frac{\omega_1 \omega_2}{\omega_3} \frac{z_{12}^3}{z_{13} z_{23}}$$

$$\tilde{\mathcal{A}}(-, -, +) = 4 z_{21}^{1-i(\lambda_1+\lambda_2)} z_{23}^{i\lambda_1-1} z_{31}^{i\lambda_2-1} \delta(\bar{z}_{13}) \delta(\bar{z}_{23}) \underbrace{\int_0^\infty \omega_3^{i(\lambda_1+\lambda_2+\lambda_3)-1} d\omega_3}_{=2\pi \delta(\lambda_1+\lambda_2+\lambda_3)}$$

logarithmically divergent in the infra-red and in ultra-violet
any cutoff would violate SL(2,C) symmetry

conformal transformation properties, read off:

$$\left. \begin{array}{lcl} h_1 & = & \frac{i}{2} \lambda_1, \\ h_2 & = & \frac{i}{2} \lambda_2, \\ h_3 & = & 1 + \frac{i}{2} \lambda_3, \end{array} \quad \begin{array}{lcl} \bar{h}_1 = 1 + \frac{i}{2} \lambda_1, \\ \bar{h}_2 = 1 + \frac{i}{2} \lambda_2, \\ \bar{h}_3 = \frac{i}{2} \lambda_3, \end{array} \right\} \quad \begin{array}{l} \Delta_n = 1 + i\lambda_n \\ J_1 = J_2 = -1, \quad J_3 = +1 \\ \text{Pasterski, Shao, Strominger, 2017} \end{array}$$

(ii) Mostly-plus three-graviton amplitude

$$\mathcal{M}(-\text{--}, -\text{--}, +\text{+}) = \frac{\langle 12 \rangle^6}{\langle 13 \rangle^2 \langle 23 \rangle^2} = \frac{\omega_1^2 \omega_2^2}{\omega_3^2} \frac{z_{12}^6}{z_{13}^2 z_{23}^2}$$

$$\tilde{\mathcal{A}}(-\text{--}, -\text{--}, +\text{+}) = 4 z_{21}^{2-i(\lambda_1+\lambda_2)} z_{23}^{i\lambda_1-1} z_{31}^{i\lambda_2-1} \delta(\bar{z}_{13}) \delta(\bar{z}_{23}) \underbrace{\int_0^\infty \omega_3^{i(\lambda_1+\lambda_2+\lambda_3)} d\omega_3}_{= \text{UV divergent}}$$

The degree of this divergence will grow with the number of external gravitons, reflecting the violation of unitarity bounds at each order of perturbative Einstein's gravity

conformal transformation properties, read off:

$$\left. \begin{array}{lcl} h_1 & = & -\frac{1}{2} + \frac{i}{2}\lambda_1, \\ h_2 & = & -\frac{1}{2} + \frac{i}{2}\lambda_2, \\ h_3 & = & \frac{3}{2} + \frac{i}{2}\lambda_3, \end{array} \quad \begin{array}{lcl} \bar{h}_1 = \frac{3}{2} + \frac{i}{2}\lambda_1, \\ \bar{h}_2 = \frac{3}{2} + \frac{i}{2}\lambda_2, \\ \bar{h}_3 = -\frac{1}{2} + \frac{i}{2}\lambda_3 \end{array} \right\} \quad \begin{array}{l} \Delta_n = 1 + i\lambda_n \\ J_1 = J_2 = -2, \\ J_3 = +2 \end{array}$$

(iii) Mostly-plus EYM (one graviton, two gluon) amplitude

$$\mathcal{M}(-\text{--}, -\text{--}, +) = \frac{\langle 12 \rangle^4}{\langle 23 \rangle^2} = \frac{\omega_1^2 \omega_2}{\omega_3} \frac{z_{12}^4}{z_{23}^2}$$

$$\tilde{\mathcal{A}}(-\text{--}, -\text{--}, +) = 4 z_{21}^{1-i(\lambda_1+\lambda_2)} z_{23}^{i\lambda_1-1} z_{31}^{i\lambda_2} \delta(\bar{z}_{13}) \delta(\bar{z}_{23}) \int_0^\infty \omega_3^{i(\lambda_1+\lambda_2+\lambda_3)} d\omega_3$$

= UV divergent

three-point amplitudes in string-theory: same !

Four-point Gauge Amplitudes

$$r = \frac{z_{12} z_{34}}{z_{23} z_{41}}$$

conformal invariant
cross-ratio on CS^2

actually:

$$\frac{s_{23}}{s_{12}} = \frac{1}{r} = -\frac{u}{s} = \sin^2\left(\frac{\theta}{2}\right)$$

$$s = s_{12} = (p_1 + p_2)^2$$

$$u = -s_{23} = (p_2 - p_3)^2$$

θ = scattering angle in center of mass frame

$$\begin{aligned} \tilde{\mathcal{A}}(-, -, +, +) &= 8\pi \delta(r - \bar{r}) \delta\left(\sum_{n=1}^4 \lambda_n\right) \\ &\times \left(\prod_{i < j}^4 z_{ij}^{\frac{h}{3} - h_i - h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j} \right) r^{\frac{5}{3}} (r - 1)^{\frac{2}{3}} \theta(r - 1) \end{aligned}$$

type I superstring theory:

$$\begin{aligned}
 \tilde{\mathcal{A}}_I(-, -, +, +) &= 4 (\alpha')^\beta \delta(r - \bar{r}) \theta(r - 1) \left(\prod_{i < j}^4 z_{ij}^{\frac{h}{3} - h_i - h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j} \right) \\
 &\times r^{\frac{5-\beta}{3}} (r - 1)^{\frac{2-\beta}{3}} I(r, \beta)
 \end{aligned}$$

$$\beta := -\frac{i}{2} \sum_{n=1}^4 \lambda_n$$

$$I(r, \beta) = -\Gamma(1 - \beta) \frac{r}{2} \int_0^1 \frac{dx}{x} [r \ln x - \ln(1 - x)]^{\beta-1}$$

$$\begin{aligned}
 I(r, \beta) &= 2\pi \delta \left(\sum_{n=1}^4 \lambda_n \right) \\
 &+ \frac{i\pi}{2} (-r)^{\beta-1} \sinh \left(\frac{1}{2} \sum_{n=1}^4 \lambda_n \right)^{-1} \sum_{k=0}^{\infty} (-r)^{-k} \zeta \left(-\frac{i}{2} \sum_{n=1}^4 \lambda_n - k, \{1\}^k \right)
 \end{aligned}$$

Remarks:

- no α' - expansion (trivial dependence on α') !
- instead expansion in small scattering angle
$$r^{-1} = \sin^2\left(\frac{\theta}{2}\right)$$
- all heavy string modes participate on same footing
- field-theory is recovered in the limit of forward scattering $\theta = 0$

Question:

any
relation ?



$\left. \begin{matrix} \text{celestial CFT_2} \\ \text{string world-sheet CFT_2} \end{matrix} \right\}$

String world-sheet as celestial sphere

$$\mathcal{M}_I(-, -, +, +) = \mathcal{M}(-, -, +, +) F_I(s, u)$$

with string formfactor:

$$F_I(s, u) = -\alpha' s_{12} B(-\alpha' s_{12}, 1 + \alpha' s_{23}) = -s B(-s, 1 - u) = \frac{\Gamma(1 - s)\Gamma(1 - u)}{\Gamma(1 - s - u)}$$

consider high-energy limit:

$$B(-s, 1 - u) = \int_0^1 dx x^{-1-s} (1 - x)^{as}$$

saddle-point approximation:

$$x_0 = \frac{1}{1 - a} \in CS^2$$

$$a = r^{-1} < 0$$

world-sheet vertex position = point on celestial sphere
= solutions to scattering equations

celestial sphere =
world-sheet

CFT on celestial sphere related
to free world-sheet CFT

Four-point Closed String Amplitudes

heterotic gauge amplitude:

$$\begin{aligned}\tilde{\mathcal{A}}_H(-, -, +, +) &= 4(\alpha')^\beta \delta(r - \bar{r}) \theta(r - 1) \left(\prod_{i < j}^4 z_{ij}^{\frac{h}{3} - h_i - h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j} \right) \\ &\times r^{\frac{5-\beta}{3}} (r - 1)^{\frac{2-\beta}{3}} H(r, \beta)\end{aligned}$$

$$H(r, \beta) = -\Gamma(1 - \beta) \frac{r}{2\pi} \int_C \frac{d^2 z}{|z|^2 (1 - z)} \left[r \ln |z|^2 - \ln |1 - z|^2 \right]^{\beta-1}$$

$$\begin{aligned}H(r, \beta) &= 2\pi \delta \left(\sum_{n=1}^4 \lambda_n \right) \\ &+ \frac{i\pi}{2} (-r)^{\beta-1} \sinh \left(\frac{1}{2} \sum_{n=1}^4 \lambda_n \right)^{-1} \sum_{k=0}^{\infty} (-r)^{-k} S^C \left(-\frac{i}{2} \sum_{n=1}^4 \lambda_n - k - 1, k + 1 \right)\end{aligned}$$

Four-point Gravity Amplitudes

heterotic graviton amplitude:

$$\begin{aligned}\tilde{\mathcal{A}}_H(--, --, ++, ++) &= 4 (\alpha')^{\beta-1} \delta(r - \bar{r}) \theta(r - 1) \left(\prod_{i < j}^4 z_{ij}^{\frac{h}{3} - h_i - h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j} \right) \\ &\times r^{\frac{11-\beta}{3}} (r - 1)^{\frac{-1-\beta}{3}} G(r, \beta)\end{aligned}$$

$$G(r, \beta) = H(r, \beta - 1)$$

$$H(r, \beta) = -\Gamma(1 - \beta) \frac{r}{2\pi} \int_{\mathbb{C}} \frac{d^2 z}{|z|^2 (1 - z)} \left[r \ln |z|^2 - \ln |1 - z|^2 \right]^{\beta-1}$$

$$\begin{aligned}H(r, \beta) &= 2\pi \delta \left(\sum_{n=1}^4 \lambda_n \right) \\ &+ \frac{i\pi}{2} (-r)^{\beta-1} \sinh \left(\frac{1}{2} \sum_{n=1}^4 \lambda_n \right)^{-1} \sum_{k=0}^{\infty} (-r)^{-k} S^{\mathbf{c}} \left(-\frac{i}{2} \sum_{n=1}^4 \lambda_n - k - 1, k + 1 \right)\end{aligned}$$

Properties:

- Finite result for any r !
ultra-soft high energy behaviour of string formfactors
ensures the convergence of energy integrals
- UV completion provided by string theory
finite result for integration from IR to UV
- first calculation of graviton amplitudes in the conformal basis

Alert:

- Divergent for $r \rightarrow \infty$ (field-theory limit)
every order in the perturbative expansion of gravity
violates the unitarity bounds by growing powers of energy.
This uncontrollable growth at large energies poses an obstacle for transforming
gravitational amplitudes to celestial sphere

Single-valued Nielsen polylogarithms

Nielsen's polylogarithm functions (real):

$$S_{n,p}(t) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dx}{x} \ln^{n-1} x \ln^p(1-xt) , \quad t \in \mathbf{C}$$

in particular:

$$S_{n,p}(1) = \zeta(n+1, \{1\}^{p-1})$$

$$\zeta(n+1, \{1\}^{p-1}) = \underbrace{\zeta(n+1, 1, \dots, 1)}_{p-1} = \sum_{n_1 > n_2 > \dots > n_p} \frac{1}{n_1^{n+1} n_2 \cdots n_p}$$

Single-valued descendants:

$$S^c(n, p) = \pi^{-1} \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_C \frac{d^2 z}{|z|^2} (1-z)^{-1} \ln^{n-1} |z|^2 \ln^p |1-z|^2$$

$$S^c(n, p) = \text{sv } S_{n,p}(1) = \text{sv } \zeta(n+1, \{1\}^{p-1})$$

Concluding remarks

- explicit and compact expressions for string amplitudes on celestial sphere:
intriguing examples for the study of flat space holography
- string amplitudes on celestial sphere:
no α' - expansion (trivial dependence on α')
- all heavy string modes participate on same footing
- high-energy limit: string world-sheet = celestial sphere
- first calculation of graviton amplitudes in the conformal basis
(with gravity UV completed):
 - *ultra-soft high energy behaviour of string formfactors ensures the convergence of energy integrals*
 - *important for the soft graviton theorem in this basis*

Can 2D CFT on celestial sphere offer some new insight into gauge-gravity connections ?

on celestial sphere celestial gravitational amplitudes appear from space-time translations of pure gauge amplitudes,

$$\tilde{A}_{EYM}(1,2,\dots,N,G^{\pm\pm}) = \frac{\kappa}{g} \sum_{l=1}^{N-1} (\epsilon_G^\pm \cdot \mathcal{X}_l) \tilde{A}_{YM}(1,2,\dots,l,G^\pm,l+1,\dots,N)$$

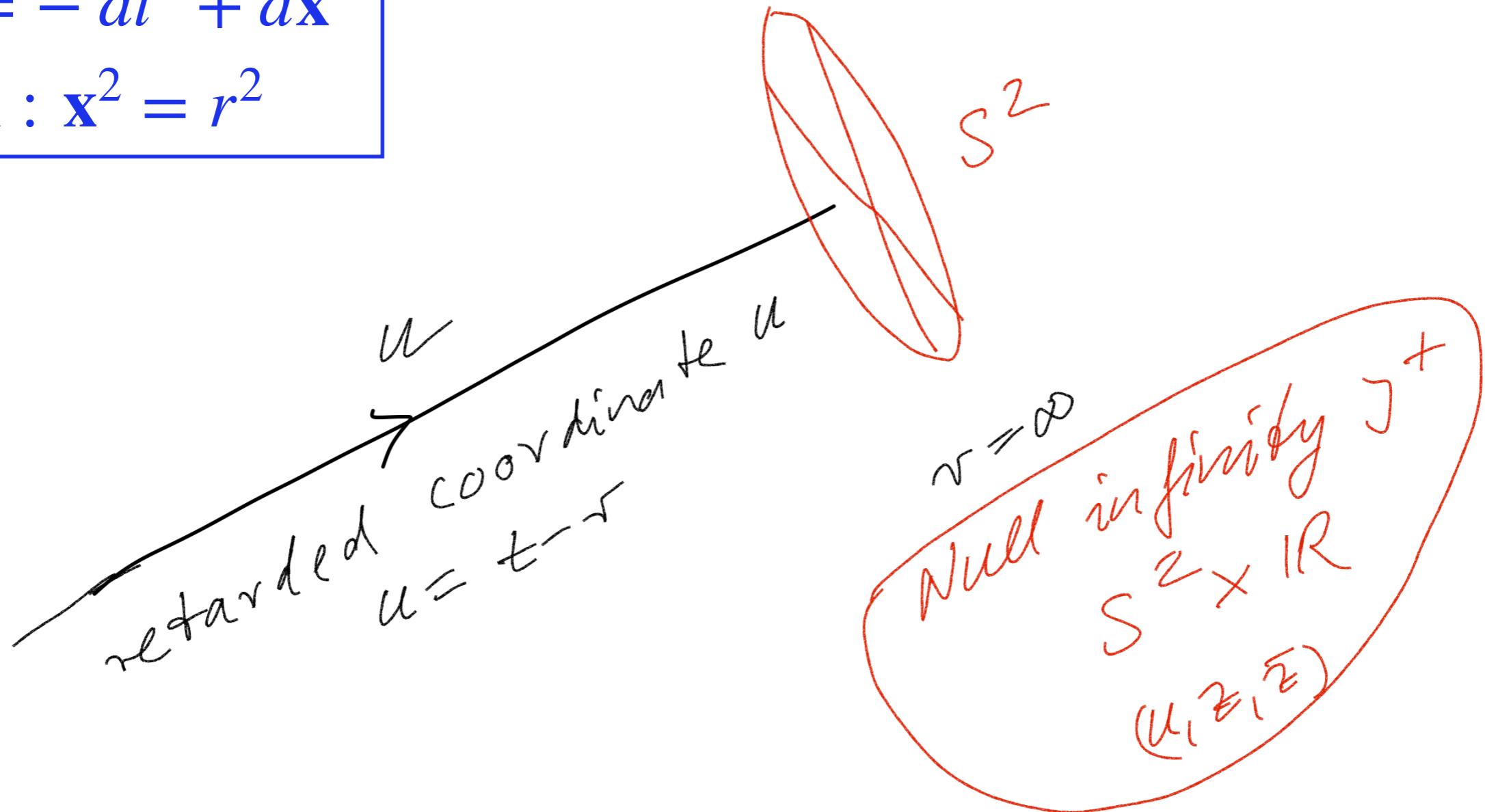
indicating that celestial CFT will be helpful in studying connections between gauge theories and gravity !



understanding the nature of 2D CFT on celestial sphere would enable a **holographic description of flat spacetime**

$$ds^2 = -dt^2 + d\mathbf{x}^2$$

$$\text{with : } \mathbf{x}^2 = r^2$$



$$ds^2 = \underbrace{-du^2 - 2 du dr}_{dr^2 - dt^2} + \underbrace{2 r^2 \gamma_{z\bar{z}} dz d\bar{z}}, \quad \gamma_{z\bar{z}} = 2(1 + |z|^2)^{-2}$$

$$dr^2 - dt^2 \quad S^2$$

$$u = t - r, \quad x^1 + ix^2 = -\frac{2rz}{1 + |z|^2}, \quad x^3 = -\frac{r(1 - |z|^2)}{1 + |z|^2}$$