Asymptotics of higher genus string integrands

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Modular graph functions beyond genus one I

- In perturbative closed string theory, scattering amplitudes of N external states at h loop involve an integral over the moduli space $\mathcal{M}_{h,N}$ of genus h curves Σ with N punctures $z_1, \dots z_N$. At fixed complex structure, this reduces to an integral over N copies of Σ.
- At genus one, the low energy expansion of the resulting integrand produces an infinite family of real-analytic modular forms labelled by certain graphs, known as modular graph functions. They exhibit remarkable asymptotics near the cusp (*finite Laurent polynomial plus exponential corrections*) and transcendentality properties.
- My goal will be to describe some hints of a similar structure at higher genus, primarily for $N = 4, g = 2$.

String integrand I

- Recall that the genus h contribution to N-point scattering amplitude is of the form

$$\mathcal{A}_h(\{k_i,\epsilon_i\}) = \int_{\mathcal{M}_{h,N}} \mathcal{I}_h(\{k_i,\epsilon_i\})$$

  where $\mathcal{I}_h(\{k_i,\epsilon_i\}) \sim \langle \prod_{i=1}^N V_{k_i,\epsilon_i}(z_i) \prod_{j=1}^{3h-3} |(b,\mu)|^2 \rangle$ is a correlator in the worldsheet conformal field theory.
- The moduli space $\mathcal{M}_{h,N}$ is fibered over $\mathcal{M}_h \equiv \mathcal{M}_{h,0}$, with fiber $\Sigma_1 \times \cdots \times \Sigma_N$. The vertex operators V_{k_i,ϵ_i} are $(1,1)$-forms over Σ_i, while the ghost part produces a volume form on the base $\mathcal{M}_h$.
- I will focus on $1 \le h \le 3$, where $\mathcal{M}_h$ is isomorphic to a fundamental domain $\mathcal{F}_h$ for the action of $Sp(h,\mathbb{Z})$ on the Siegel upper half-plane $\mathcal{H}_h$ (away from some divisors).

String integrand II

- In the RNS formalism, $\mathcal{I}_h$ originates from an integrand $\mathfrak{I}_h$ over the moduli space of super Riemann curves $\mathfrak{M}_{h,N}$, after summing over spin structures and integrating over fermionic moduli. There is no canonical projection $\mathfrak{M}_{h,N} \to \mathcal{M}_{h,n}$, leading to total derivative ambiguities *[Donagi Witten'13]*. State of the art: $h = 2, N = 4$ *[D'Hoker Phong '05]*
- In Berkovits' pure spinor formalism, b is a composite field involving powers of $(\lambda\bar{\lambda})^{-1}$, leading to potential divergences from the region $\lambda\bar{\lambda} \to 0$. State of the art: idem, with partial results for $h = 2, N = 5$ *[Gomez Mafra Schlotterer '15]* and $h = 3, N = 4$ *[Gomez Mafra '13]*

Modular graph functions beyond genus one I

- Although we do not know the integrand $\mathcal{I}_h$ in general, for type II strings on T^d we expect a result of the form

$$\mathcal{I}_h = \delta(\sum k_i)\, Z_{KN}(\{k_i\})\, Z_{T^d}\, P_h(\{k_i, \epsilon_i, z_i\})\, \mu_h$$

where

  - $Z_{KN}(\{k_i, z_i\}) = \exp(\sum_{i<j} s_{ij} G(z_i, z_j))$ is the Koba-Nielsen factor, with $s_{ij} = -\frac{\alpha'}{2} k_i \cdot k_j$ the Mandelstam variables and $G(z_i, z_j)$ the scalar Green function.
  - Z_{T^d} is the genus-h Siegel theta series for a lattice of signature (d, d)
  - $P_h(\{k_i, \epsilon_i, z_i\})$ is a (1,1)-form in each position z_i, and a Lorentz invariant homogeneous polynomial in k_i, ϵ_i
  - μ_h is the Siegel volume form over $\mathcal{M}_h \simeq \mathcal{F}_h$

Canonical forms on compact curves I

- Let A_I, B_I ($I = 1 \dots h$) be a canonical basis of $H_1(\Sigma)$, such that $A_I \cap A_J = B_I \cap B_J = 0$, $A_I \cap B_J = \delta_{IJ}$. Choose ω_I a basis of holomorphic differentials on Σ such that $\int_{A_I} \omega_J = \delta_{IJ}$. The period matrix $\tau_{IJ} = \int_{B_I} \omega_J$ satisfies $\tau > 0$, hence $\tau \in \mathcal{H}_h$.
- Under $Sp(h, \mathbb{Z})$ change of basis, $\tau \mapsto (a\tau + b)(c\tau + d)^{-1}$. The Siegel volume form $\mu_h = \frac{\prod_{I \leq J} |\mathrm{d}\tau_{IJ}|^2}{(\det \mathrm{Im}\tau)^{h+1}}$ is invariant.
- An example of (1,1) form on Σ_i is the canonical Kähler form $\kappa(z_i) = \frac{i}{2h} \mathrm{Im}\tau^{IJ} \omega_I(z_i) \bar{\omega}_J(z_i)$.
- Examples of $(1,1)$ forms on $\Sigma_i \times \Sigma_j$ are $\kappa(z_i)\kappa(z_j)$ and $|\nu(z_i, z_j)|^2$ with $\nu(z_i, z_j) = \mathrm{Im}\tau^{IJ} \omega_I(z_i) \overline{\omega}_J(z_j)$. Another example on h copies of Σ is $|\Delta(z_1, \dots z_h)|^2$ where $\Delta(z_1, \dots z_h) = \epsilon^{I_1 \dots I_h} \omega_{I_1}(z_1) \dots \omega_{I_h}(z_h)$ is a (1,0)-form in each variable.

Canonical forms on compact curves II

- The Arakelov Green function $G(z,w)$ is a symmetric function on $\Sigma\times\Sigma$ defined by

$$\begin{aligned}\partial_z\bar\partial_z G(z,w) =& -\pi\delta^{(2)}(z,w)+\pi\,\kappa(z)\ ,\\ \int_\Sigma G(z,w)\,\kappa(w) =& 0\ ,\end{aligned}$$

  The r.h.s. integrates to zero thanks to $\int_\Sigma \kappa=1$, allowing G to be globally well-defined.

- In genus one, setting $v=\int_z^w \omega=\alpha+\beta\tau$,

$$G(z,w)=-\log\left|\frac{\vartheta_1(v,\tau)}{\eta}\right|^2+\frac{2\pi}{\tau_2}(\mathrm{Im} v)^2=\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\tau_2\,e^{2\pi\mathrm{i}(m\beta-n\alpha)}}{\pi|m+n\tau|^2}$$

Genus-one modular graph functions I

- After Taylor expanding in powers of s_{ij}, and integrating over the positions z_i of the vertex operators, the integrand is a linear combination of genus one modular graph functions, of the type

$$\mathcal{B}_\Gamma(\tau) = \int_{\Sigma^N/\Sigma} \prod_{(i,j)\in\Gamma} G(z_i, z_j) \prod_i \kappa(z_i) \qquad \kappa(z) = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\mathrm{i}\,\mathrm{Im}\tau}$$

  where (i,j) runs over the edges of the graph Γ (possibly dressed with derivatives wrt z_i) *[d'Hoker Green Gurdogan Vanhove '15]*

- By construction, $\mathcal{B}_\Gamma$ are real analytic modular functions. e.g.

$$D_k(\tau) = \int_\Sigma G(v)^k \kappa(v)$$

  gives E_2 for $k=2$, $E_3+\zeta(3)$ for $k=3$.

Genus-one modular graph functions II

- Modular graph functions satisfy an intricate set of algebraic and differential equations. Their expansion near the cusp $\tau \to \mathrm{i}\infty$ has the remarkable form

$$\mathcal{B}_\Gamma(\tau) = \sum_{n=w}^{1-w} a_n \, (\pi\tau_2)^n + \mathcal{O}(e^{-2\pi\tau_2})$$

where w is the number of edges (i.e. degree in G). The coefficients a_n are single valued multizeta values of transcendentality degree $w - n$.

d'Hoker Green Gurdogan Vanhove Zerbini Kaidi Basu Kleinschmidt Verschinin Gerken Schlotterer Duke . . .

Genus-one modular graph functions III

- After integrating over $\tau \in \mathcal{F}_1$, the coefficient of the effective interaction $\sigma_2^p \sigma_3^q \mathcal{R}^4$ at one-loop (where $\sigma_n = s^n + t^n + u^n$) is given by a regularized theta lifting,

$$\mathcal{E}^{(1,d)}_{(p,q)} = \sum_{\Gamma} \alpha_\Gamma \int_{\mathcal{F}_1(L)} \mathcal{B}_\Gamma(\tau)\, Z_{T^d}(G_{ij}, B_{ij}; \tau)\, \mu_1$$

  where G_{ij}, B_{ij} parametrize the Narain moduli space $\frac{O(d,d)}{O(d)\times O(d)}$ and $\mathcal{F}_1(L) = \mathcal{F}_1 \cap \{\tau_2 < L\}$ is the truncated fundamental domain.
- The powerlike terms in $\mathcal{B}_\Gamma(\tau)$ are responsible for infrared divergences as $L \to \infty$. The full amplitude including non-analytic terms is finite when $D > 4$. More on this later.

Higher genus modular graph functions I

- For $h = 2$, $N = 4$, the precise integrand is

d'Hoker Phong '05

$$\mathcal{I}_2 = \delta(\sum k_i)\, \mathcal{R}^4\, |\mathcal{Y}|^2\, Z_{KN}\, Z_{T^d}\, \det \mathrm{Im}\tau\, \mu_2$$

where

$$\mathcal{Y} = (t-u)\Delta(z_1, z_2)\,\Delta(z_3, z_4) + \mathrm{perm}$$

$$Z_{KN} = e^{s\,[G(z_1,z_2)+G(z_3,z_4)]+t\,[G(z_1,z_4)+G(z_2,z_3)]+u[G(z_1,z_3)+G(z_2,z_4)]}$$

- At leading order, $Z_{KN} \simeq 1$ and $\int_{\Sigma^4} |\Delta(1,2)\,\Delta(3,4)|^2 \propto 1/\det \mathrm{Im}\tau$. The coefficient of the $\nabla^4\mathcal{R}^4$ interaction at genus 2 is then

$$\mathcal{E}^{(d,2)}_{(1,0)} = \frac{\pi}{2} \int_{\mathcal{F}_2(L)} Z_{T^d}\, \mu_2 \propto E^{O(d,d)}_{\frac{d-3}{2}\Lambda_2}$$

Higher genus modular graph functions II

- At next-to-leading order, one power of $G(z_1, z_2)$ comes down from Z_{KN}. The integral over z_3, z_4 is easy. The coefficient of the $\nabla^6\mathcal{R}^4$ interaction at genus-two is *[d'Hoker Green'13]*

$$\mathcal{E}_{(0,1)}^{(d,2)} = \pi \int_{\mathcal{F}_2(L)} \varphi_{KZ}\ Z_{T^d}\ \mu_2$$

where φ_{KZ} is the Kawazumi-Zhang invariant, defined by

$$\varphi_{KZ} = -\frac{1}{4} \int_{\Sigma\times\Sigma} |\nu(z_1, z_2)|^2\, G(z_1, z_2)$$

where $\nu(z_1, z_2) = \mathrm{Im}\tau^{IJ}\omega_I(z_1)\overline{\omega}_J(z_2)$.

- This is the simplest genus-two modular graph function, associated to the graph 

(but the measure need to be specified)

Higher genus modular graph functions III

- Using standard theory of complex structure deformations, one may show that φ_{KZ} is an eigenmode of the Laplacian on $\mathcal{M}_2$,

$$\left[\Delta_{Sp(4)} - 5\right] \varphi_{KZ} = -2\pi \det \mathrm{Im}\tau \, \delta^{(2)}(v)$$

  which makes it east to compute $\int_{\mathcal{F}_2} \varphi_{KZ} \, \mu_2$ *[d'Hoker Green BP Russo'14]*
- Integrating against Z_{T^d}, it follows that *[BP Russo'15]*

$$\left[\Delta_{O(d,d)} - (d+2)(5-d)\right] \mathcal{E}^{(d,2)}_{(0,1)} = -\left[\mathcal{E}^{(d,1)}_{(0,0)}\right]^2 \propto \left[E^{O(d,d)}_{\frac{d-2}{2}\Lambda_1}\right]^2$$

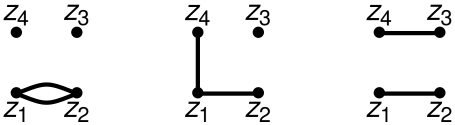
  hence $\mathcal{E}^{(d,2)}_{(0,1)}$ is not an Eisenstein series.
- Combined with information about asymptotics (see below), this shows that φ_{KZ} can be obtained as a Borcherds' type theta lift of the weak Jacobi form θ_1^2/η^6 from $SL(2)$ to $SO(3,2) = Sp(4)$. This gives access to the full Fourier expansion *[BP'15]*.

- At NNLO, one needs to bring down two Green functions from Z_{KN}. The coefficient of the $\nabla^8\mathcal{R}^4$ interaction at genus two is then

$$\mathcal{E}^{(d,2)}_{(2,0)} = \frac{\pi}{64}\int_{\mathcal{F}_2(L)} \mathcal{B}_{(2,0)}\, Z_{T^d}\, \mu_2$$

where $\mathcal{B}_{(2,0)}$ is a linear combination of three genus-two modular graph functions,

$$\mathcal{B}_{(2,0)} = \int_{\Sigma^4} \frac{|\Delta(1,3)\,\Delta(2,4)|^2}{(\det \mathrm{Im}\tau)^2} \left\{ \begin{array}{c} G(1,2)^2 - 2G(1,2)\,G(1,4) \\ +G(1,2)G(3,4) \end{array} \right\}$$



Higher genus modular graph functions V

- At NNNLO order, the $\nabla^{10}\mathcal{R}^4$ interaction involves modular graph functions with 4 vertices and 3 edges, etc.
- For higher point scattering amplitudes, one expects a similar hierarchy of integrands with N vertices and $k \geq 0$ edges, but the integration measure on Σ^N is not known for $N > 5$. Presumably it should be built from holomorphic one-forms ω_I, derivatives $\partial G(z, w)$ and their complex conjugate.
- For $h = 3$, $N = 4$, the leading $\nabla^6\mathcal{R}^4$ integrand computed in pure spinor formalism is given by *[Gomez Mafra 2013]*

$$\mathcal{E}^{(3,d)}_{(0,1)} = \frac{5}{16}\int_{\mathcal{F}_3(L)} Z_{T^d}\,\mu_3 \propto E^{O(d,d)}_{\frac{d-4}{2}\Lambda_3}$$

but the $\nabla^8\mathcal{R}^4$ integrand remains to be computed *[Gomez Mafra 20??]*.

Arakelov-Green function at higher genus I

- Recall that the Arakelov Green function $G(z,w)$ is defined by

$$\partial_z\bar\partial_z G(z,w) = -\pi\delta^{(2)}(z,w) + \pi\,\kappa(z)\,, \qquad \int_\Sigma G(z,w)\,\kappa(w) = 0$$

- After cutting Σ along $2h$ cycles to make it simply connected, $G(z,w)$ can be expressed in terms of the string Green function

$$\mathcal{G}(z,w) = -\log|E(z,w)|^2 + 2\pi\mathrm{i}\,\mathrm{Im}v_I\,\mathrm{Im}\tau^{IJ}\,\mathrm{Im}v_J$$

where $E(z,w)$ is the prime form and $v_I = \int_z^w \omega_I$. $\mathcal{G}(z,w)$ satisfies

$$\partial_z\bar\partial_z \mathcal{G}(z,w) = -\pi\delta^{(2)}(z,w) + \pi h\,\kappa(z)$$

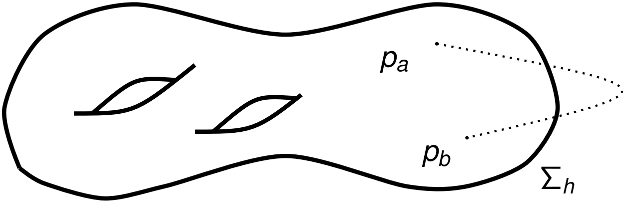
- $G(z,w)$ is obtained from $\mathcal{G}(z,w)$ via

$$G(z,w) = \mathcal{G}(z,w) - \gamma(z) - \gamma(w) + \gamma_0$$

with $\gamma(z) = \int_\Sigma \mathcal{G}(z,w)\,\kappa(w)$ and $\gamma_0 = \int_\Sigma \gamma(z)\kappa(z)$.

Arakelov-Green function at higher genus II

- Using the theta series representation of the prime form $E(z, w)$, one can obtain accurate asymptotics of $G(z, w)$ near boundaries of moduli space.
- In the non-separating degeneration, a genus $h + 1$ curve Σ_{h+1} degenerates into a genus h curve Σ_h with two marked points p_a, p_b, linked by a thin handle of proper length $t = \det \mathrm{Im}\tau_{h+1} / \det \mathrm{Im}\tau$.

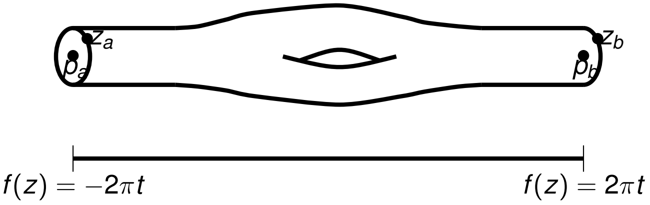


Arakelov-Green function at higher genus III

- Theorem *[Wentworth'91, d'H G BP '17]*: In the limit $t \to \infty$,

$$G_{h+1}(z,w) = \frac{\pi\, t}{3(h+1)^2} + \Big[G_h(z,w) + \frac{G_h(p_a,p_b)}{(h+1)^2}$$
$$- \frac{1}{2(h+1)} \left(G_h(z,p_a) + G_h(z,p_b) + G_h(w,p_a) + G_h(w,p_b) \right) \Big]$$
$$+ \frac{1}{8\pi t} \left[\frac{f(z)^2 + f(w)^2}{(h+1)^2} - 2 f(z) f(w) - \frac{2h}{(h+1)^2} \int_{\Sigma_h} f(x)^2\, \kappa(x) \right] + \mathcal{O}(e^{-2\pi t})$$

- Here $f(z) := G_h(z,p_b) - G_h(z,p_a)$ is a single-valued real function on Σ_h, interpolating from $f(p_a) = -2\pi t$ to $f(p_b) = +2\pi t$.



Asymptotics of higher genus integrands I

- This gives access to the asymptotics of genus-two string invariants, e.g. for $\tau = \begin{pmatrix} \rho & v \\ v & \sigma_1 + \mathrm{i}(t + \frac{v_2^2}{\rho_2}) \end{pmatrix}$, $t \to \infty$,

$$\varphi_{KZ}(\tau) = \frac{\pi t}{6} + \frac{1}{2} g_1(v) + \frac{5}{4\pi t} \left(g_2(v) - E_2 \right) + \mathcal{O}(e^{-2\pi t})$$

  where $g_1(v, \rho)$ is the genus-one Arakelov Green function, and $g_2(v, \rho) = \int_{\Sigma_1} g_1(v - w) g_1(w)\, \kappa(w)$.

- These asymptotics can be derived independently from the theta lift representation of φ_{KZ}. Different powers of t come from different orbits *[BP'15]*. The eigenvalue equation $[\Delta_{Sp(4)} - 5]\, \varphi_{KZ} = 0$ holds order by order in t.

Asymptotics of higher genus integrands II

- More generally,the w-th term in the Taylor expansion of the $N = 4, h = 2$ amplitude (a homogeneous polynomial of degree $w + 2$ in s_{ij})

$$\mathcal{B}_w(s_{ij};\tau) := \int_{\Sigma^4} \frac{|\mathcal{Y}|^2}{(\det \mathrm{Im}\tau)^2} \left(\sum_{1\le i<j\le 4} s_{ij}\, G(z_i,z_j) \right)^w$$

has an asymptotic expansion of the form *[d'H G BP '17]*

$$\mathcal{B}_w(s_{ij};\tau) = \sum_{k=-w}^{w} b_w^{(k)}(s_{ij};v,\rho)\,(\pi t)^k + \mathcal{O}(e^{-2\pi t})$$

as $t \to \infty$. Here $b_w^{(k)}(s_{ij};v,\rho)$ are generalized genus-one modular graph functions.

Asymptotics of higher genus integrands III

- For $w = 2$, corresponding to $\nabla^8\mathcal{R}^4$ coupling, one finds

$$\begin{aligned}\mathcal{B}_{(2,0)}(\Omega) =& \frac{8\pi^2 t^2}{45} + \frac{2\pi t}{3} g_1 + \left(3E_2 + g_1^2 - \frac{2}{3}F_2\right) \\ &+ \frac{1}{\pi t}\left(\frac{3}{2}D_3 - \frac{1}{2}D_3^{(1)} - g_3 - g_1 F_2 + \frac{1}{8\pi}\Delta_v\left(F_2^2 + 2F_4\right)\right) \\ &+ \frac{1}{16\pi^2 t^2}\left[(\Delta_\tau + 8)\left(F_2^2 + 4F_4\right) + \mathcal{K}^c\right] + \mathcal{O}(e^{-2\pi t})\end{aligned}$$

where

$$\begin{aligned} g_{k+1}(z) &= \int_{\Sigma_1} \kappa_1(w)\, g_1(z-w)\, g_k(w) \\ F_k(z) &= \frac{1}{k!}\int_{\Sigma_1} \kappa_1(w)\, [g_1(w-z) - g_1(w)]^k \\ D_k^{(1)}(z) &= \int_{\Sigma_1} \kappa_1(w)\, g_1(z-w)\, g_1^{k-1}(w)\end{aligned}$$

Tropical limit I

- By iterating non-separating degenerations, one may extract the asymptotics in the maximal degeneration (aka tropical limit) where the string wordsheet is reduced to a graph.
- For $h = 2$, in the limit $V \equiv [\det \mathrm{Im}\tau]^{-1/2} \to 0$ one obtains 

$$
\begin{aligned}
\mathcal{B}^{(2)}_{(0,1)} &= \frac{10\pi}{3V} A_{10} + \frac{5\zeta(3)}{\pi^2} V^2 + \mathcal{O}(e^{-V^{-1/2}}) \\
\mathcal{B}^{(2)}_{(2,0)} &= \frac{\pi^2}{V^2} \left[-\frac{26}{189} A_{00} + \frac{20}{189} A_{02} + \frac{20}{99} A_{11} + \frac{64}{45} A_{20} \right] \\
&\quad + \frac{V\,\zeta(3)}{\pi} \left[\frac{9}{5} A_{01} + \frac{8}{3} A_{10} \right] + \frac{3\zeta(5)}{2\pi^3} V^3 A_{01} + \beta \frac{\zeta(3)^2}{\pi^4} V^4 + \mathcal{O}(e^{-V^{-1/2}}
\end{aligned}
$$

where A_{ij} are modular functions of $\hat{\tau} = \frac{v_2 + \mathrm{i}\sqrt{\det \mathrm{Im}\tau}}{\tau_2}$.

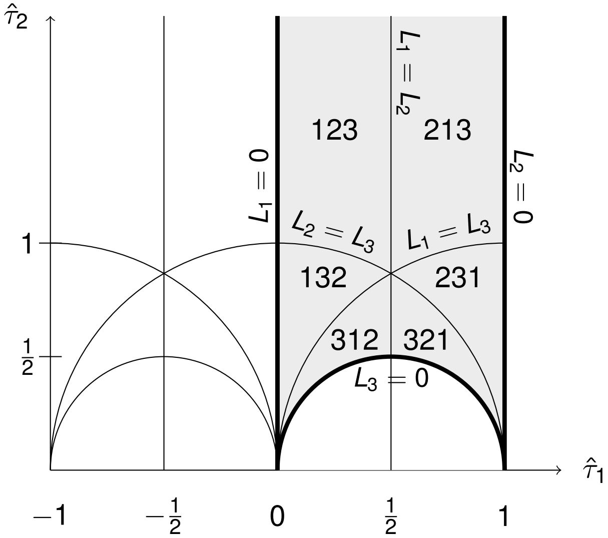
Tropical limit II

- More precisely, in the fundamental domain $\mathcal{F} = \{0 < \mathrm{Re}\hat\tau < 1, |\tau - \frac12| > \frac12\}$ for the action of $GL(2,\mathbb{Z})$ on $\hat\tau$,

$$A_{ij} := D^{(n)}_{-2n}\left[\left(\hat\tau^2(\hat\tau-1)^2\right)^i\left(\hat\tau^2-\hat\tau+1\right)^j\right]\ , \quad n = 3i+j$$

  where $D^{(n)}_{-2n}$ is the n-th iterated Maass raising operator.
- The leading terms coincides with the two-loop supergravity integrand in Schwinger time representation. How about subleading terms ?

Tropical limit III



Infrared and ultraviolet divergences I

- Unlike supergravity, perturbative string theory has no ultraviolet divergences, since all boundaries of $\mathcal{M}_{h,N}$ correspond to nodal singularities, which are equivalent to long tubes.
- Infrared divergences originate from massless particle exchange and are captured by supergravity, supplemented by an infinite set of higher derivative interactions induced by massive string modes.
- One way to construct the low energy effective action is to separate the moduli space $\mathcal{M}_{h,n}$ into a disjoint union $\mathcal{M}_{h,N}(L) \cup \mathcal{N}_{h,N}(L)$, where $\mathcal{M}_{h,N}(L)$ retains curves with handles of length $t < L$
- *More precisely, $\mathcal{M}_{h,N}(L)$ is such that the length of the shortest geodesic measured w.r.t. the metric of constant curvature is bigger that $\epsilon = \pi/t$.*

Infrared and ultraviolet divergences II

- The integral over $\mathcal{M}_{h,N}(L)$ is manifestly convergent and analytic as $s_{ij} \to 0$. Its Taylor expansion provides L-dependent local interactions in the Wilsonian effective action for the massless modes.
- The integral over $\mathcal{N}_{h,N}(L)$ generates the supergravity amplitude with UV cut-off $\Lambda = 1/\sqrt{\alpha' L}$. Amplitudes are manifestly independent of the sliding scale L.
- It follows that the L/Λ-dependence of the integral over $\mathcal{M}_{h,N}(L)$ must match the UV divergence of the supergravity amplitude. In particular, a term $a \log L$ in the string integral must match a pole a/ϵ in supergravity in dimension $D - 2\epsilon$.

Green Russo Vanhove'10; BP'18

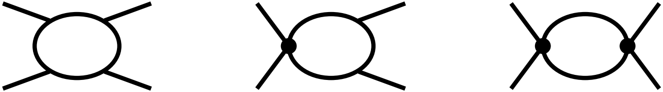
Infrared and ultraviolet divergences III

- For genus-one amplitudes, a power-like term $a_n \tau_2^n \in \mathcal{B}(\tau)$ leads to

$$\mathcal{I}_1 = \int_{\mathcal{F}_1(L)} \mathcal{B}(\tau)\, Z_{T^d}\, \mu_1 \sim a_n \frac{L^{d/2+n-1}}{d/2+n-1}$$

or $\mathcal{I}_1 \sim a_n \log L$ in dimension $D = 8 + 2n$.

- The leading integrand $\mathcal{B}_{(0,0)} \sim 1$ reproduces the one-loop $\mathcal{R}^4$ divergence in $D = 8$ SUGRA
- Subleading integrands $\mathcal{B}_{(1,0)} \ni \zeta(3)/\tau_2$, $\mathcal{B}_{(0,1)} \ni \zeta(5)/\tau_2^2$, $\mathcal{B}_{(2,0)} \ni \zeta(3)^2/\tau_2^2$ reproduce form factor divergences in $D = 6$, $D = 4$.



Infrared and ultraviolet divergences IV

- For genus-two amplitudes, the integral $\mathcal{I}_2 = \int_{\mathcal{F}_2(L)} \mathcal{B}(\tau)\, Z_{T^d} \mu_2$ has several sources of infrared divergences:

1) the minimal non separating degeneration $t \to \infty$: a term $t^n \mathcal{C}_n(\rho, v)$ in $\mathcal{B}(\tau)$ leads to the one-loop subdivergence

$$\mathcal{I}_2 \sim \frac{L^{d/2+n-2}}{d/2+n-2} \int_{\mathcal{F}_1} \mu_1 \left[\int_{\Sigma_1} \kappa(v)\, \mathcal{C}_n(\rho, v) \right] Z_{T^d}$$

or a log divergence in $D = 6 + 2n$ if $\int_{\Sigma_1} \kappa(v)\, \mathcal{C}_n(\rho, v) \neq 0$;

- In $D = 6$ and $D = 4$, this combines with the genus one divergence of the $\nabla^4 \mathcal{R}^4$ and $\nabla^6 \mathcal{R}^4$ form factors as expected

2) the minimal separating degeneration $v \to 0$ does not lead to any divergence, since the integrand grows like a power of $\log |v|$

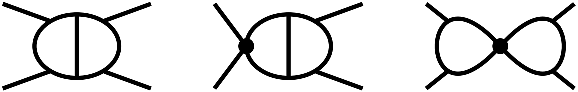
Infrared and ultraviolet divergences V

3) the maximal separating degeneration $V \to 0$ where $V \equiv [\det \mathrm{Im}\tau]^{-1/2}$: a term $V^{-n}\hat{\mathcal{C}}_n(\hat{\tau}) \in \mathcal{B}(\tau)$ leads to

$$\mathcal{I}_2 \sim \frac{L^{d+n-3}}{d+n-3} \int_{\mathcal{H}_1/GL(2,\mathbb{Z})} \hat{\mu}_1\, \hat{\mathcal{C}}_n(\hat{\tau})\, \hat{\tau}_2^{3-n-d}$$

hence a log divergence in $D = 7 + n$.

- The leading integrand $\mathcal{B}_{(1,0)} \sim 1$ reproduces the two-loop $\nabla^4 \mathcal{R}^4$ divergence in $D = 7$ SUGRA *[Bern et al '98]*, while the subleading terms $\mathcal{B}_{(0,1)} \sim \zeta(3) V^2$, $\mathcal{B}_{(2,0)} \sim \zeta(3) V + \zeta(5) V^3 + \zeta(3)^2 V^4$ reproduce form factor divergences in $D = 5$, $D = 6, 4, 3$.



Outlook I

- Genus-two string integrands provide a source of real-analytic Siegel modular forms with remarkably simple asymptotics. Are they secretly holomorphic, or theta lifts of holomorphic objects ? Do they satisfy interesting algebraic or differential identities ?
- Can one use information about these integrands to guess non-perturbative expressions for unprotected effective interactions, e.g. $\nabla^8\mathcal{R}^4$ in type II strings ? Information about the genus three integrand would be very useful.
- More generally, it is desirable to improve our understanding of string perturbation theory at the practical level, either in the RNS or PS framework, and break the $h = 2, N = 4$ barrier.

Outlook II

- Thanks for your attention !