Triality in Little String Theories

Stefan Hohenegger

GGI Workshop: String Theory from a Worldsheet Perspective

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based on work in collaboration with: Brice Bastian, Amer Iqbal and Soo-Jong Rey

hep-th 1610.07916 , hep-th 1710.02455
hep-th 1711.07921, hep-th 1807.00186, hep-th 1810.05109, hep-th 1811.03387

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UCB Lyon 1

Study of Quantum Theories in Higher Dimensions

Strong Motivation to study supersymmetric/-conformal quantum theories in dimensions > 4:

* at the heart of key structures in M-theory and string theory
  (flagship example: world-volume theory of multiple M5-branes)
* encode topological invariants and data of underlying string geometry
* connection to supersymmetric gauge theories in 4 dimensions (AGT relations)
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Geometrically:

Calabi Yau manifolds

(refined) topological string

Brane Configurations

(Non-)compact M5/M2-systems

D5-NS5-brane configurations

World-sheet description

M-strings

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use 'stringy' tools to compute quantities in quantum field theory

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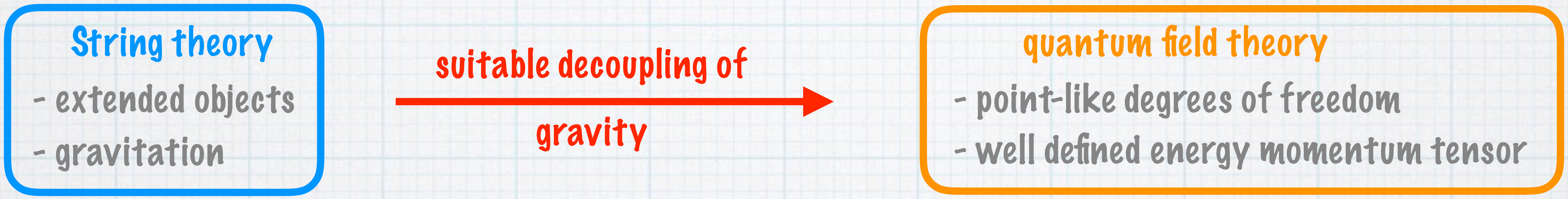

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Rich class of examples realised in 11-dimensional **M-theory** through systems of parallel M5-branes with M2-branes stretched between them

[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa 2013]
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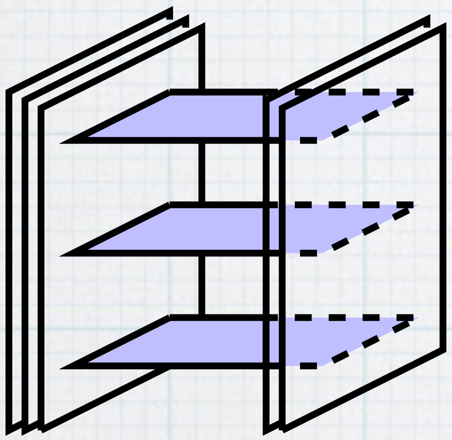

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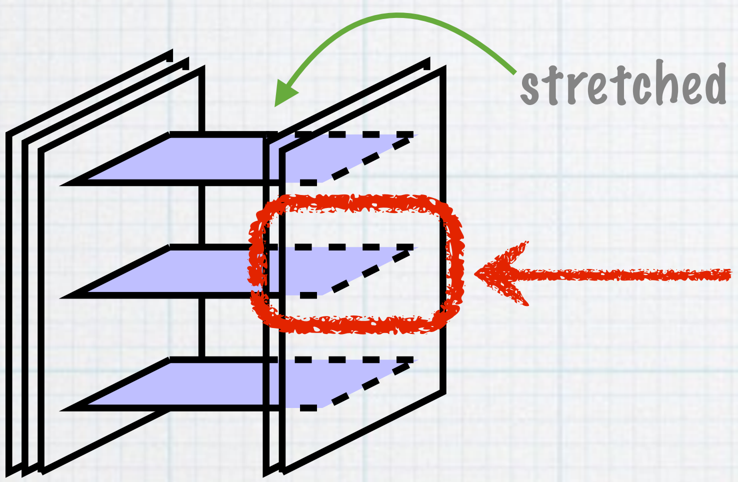

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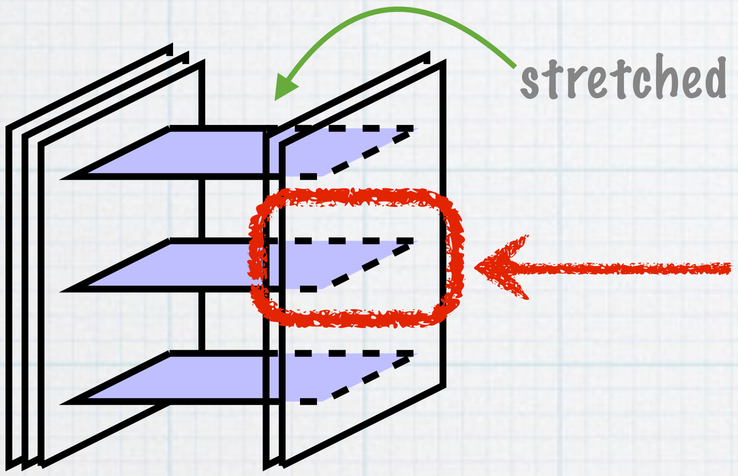

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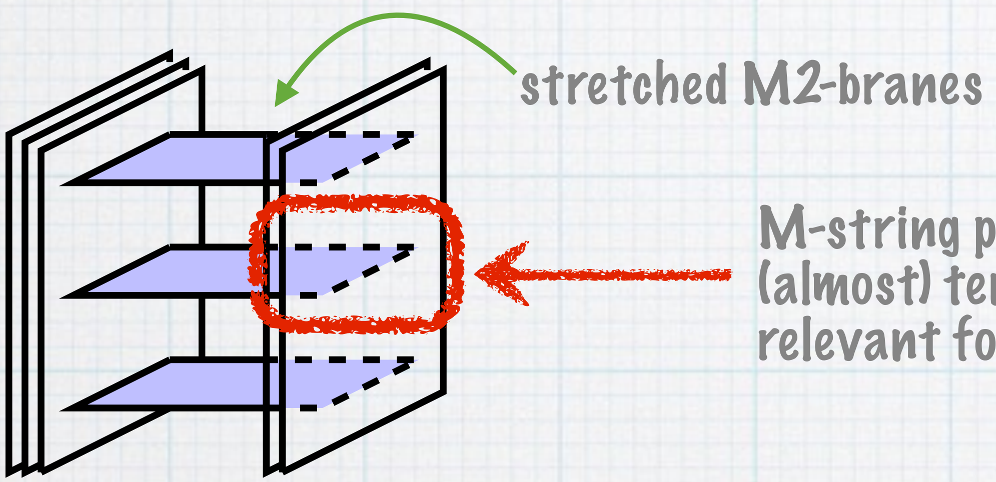

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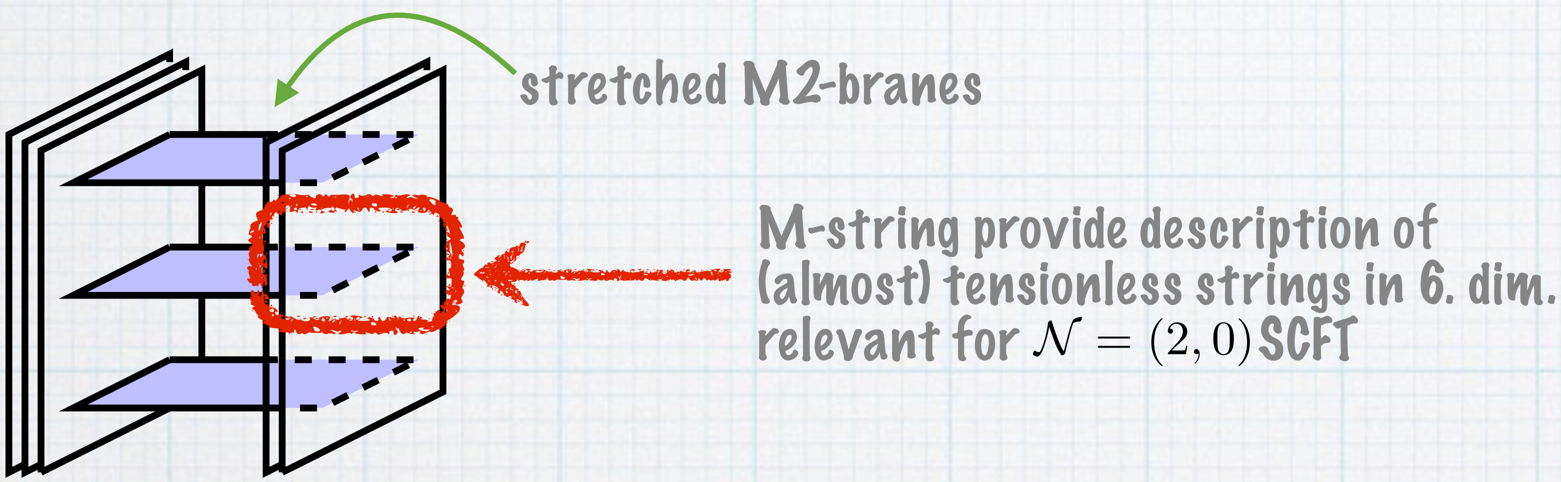

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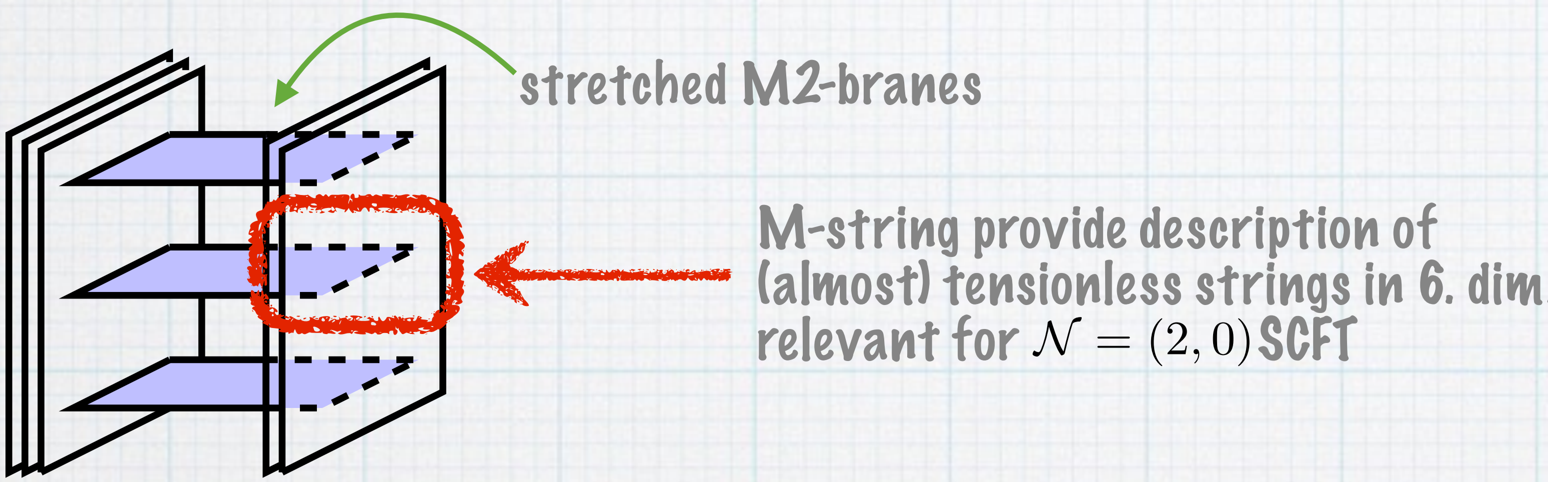

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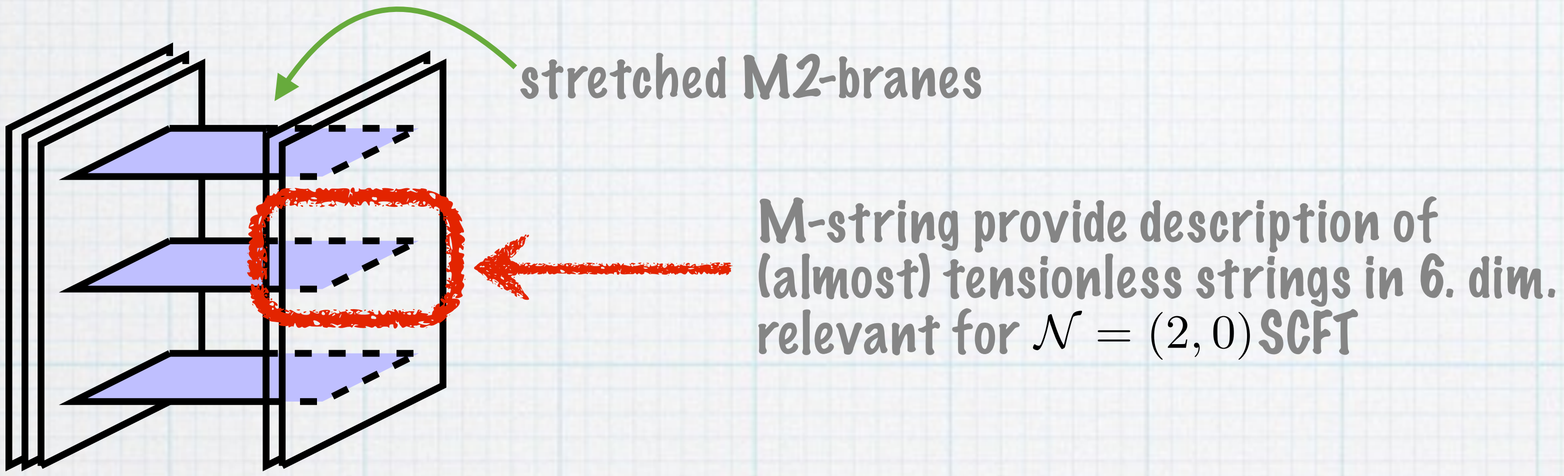

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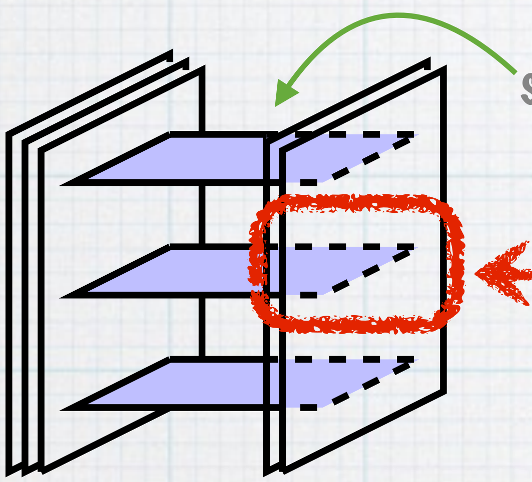

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in this talk: **triality**

6-dimensional systems:
- gravity is decoupled
- have an intrinsic string scale
- obtained from type II string theory through the decoupling limit

$$g_{\mathrm{st}} \to 0 \text{ while } \ell_{\mathrm{st}} = \text{fixed}$$

Little String Theories with 16 supercharges (A-series)

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BPS states from the point of view of M5-branes correspond to M2-branes ending on them

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Little String Theories with 8 supercharges: **particular class** obtained as

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in this talk:
further dualities

Brane Configurations

The most general configuration of branes in M-theory in 11 dimensions looks like

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M5-branes	●	●	●	●	●	●					
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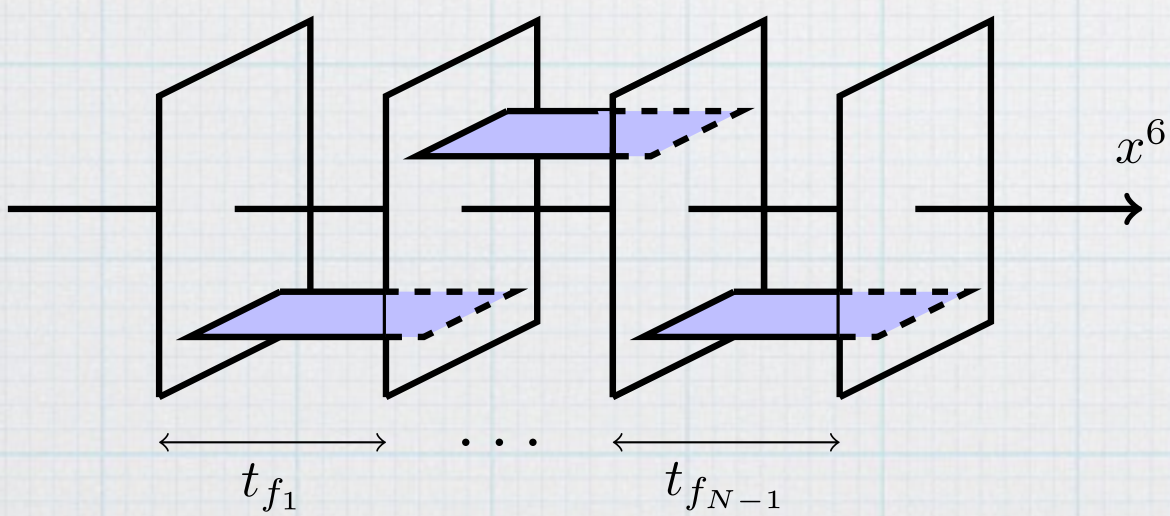
non-compact case: $\mathbb{R}$

M5-branes distributed along non-comp. (6)-direction

with M2-branes stretched between them



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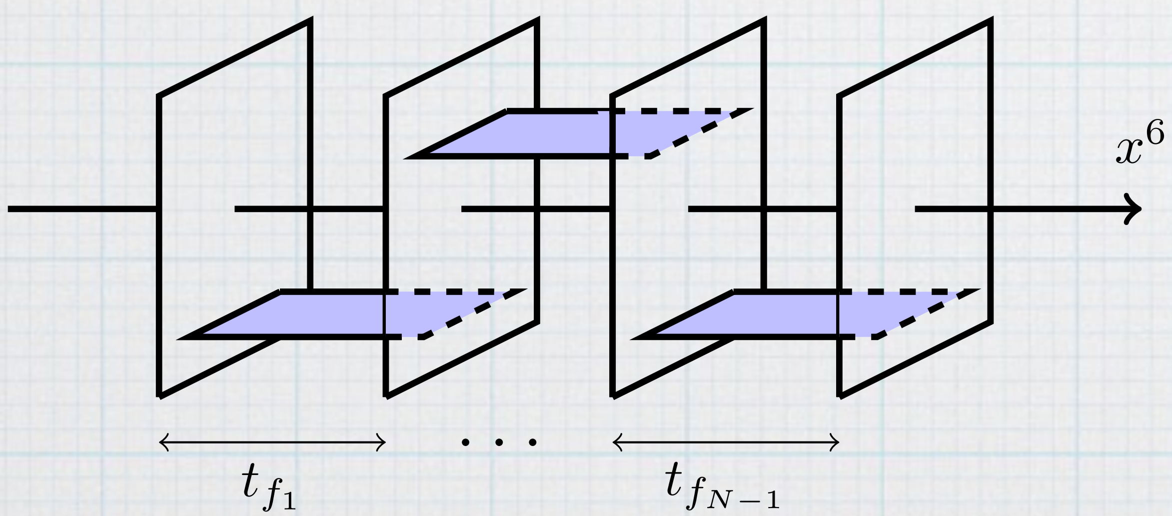
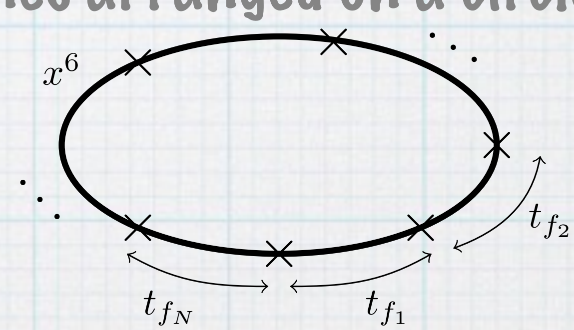


compact case: S^1

M5-branes arranged on a circle $R_6 = \dfrac{\rho}{2\pi i}$



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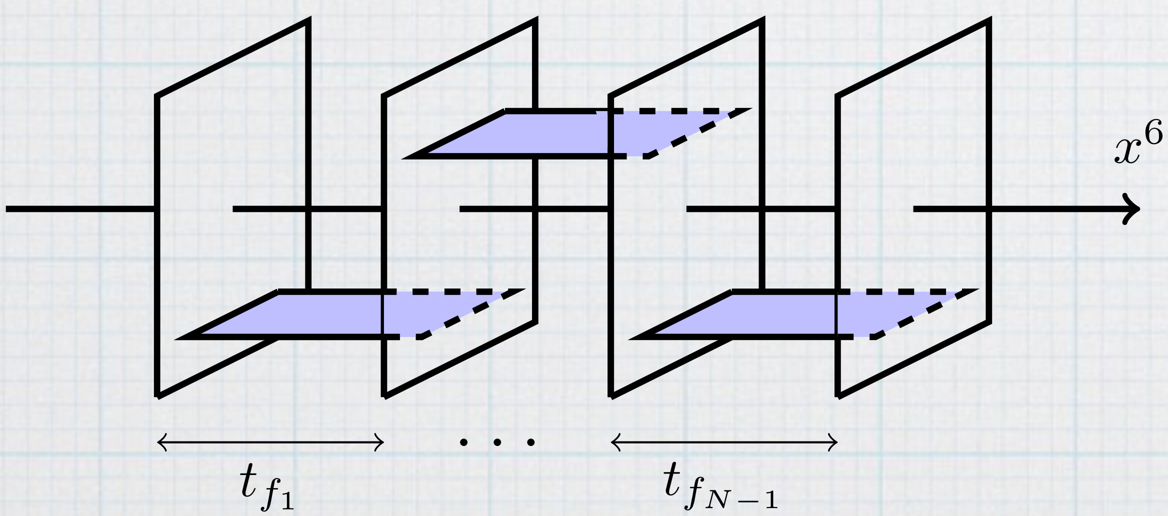
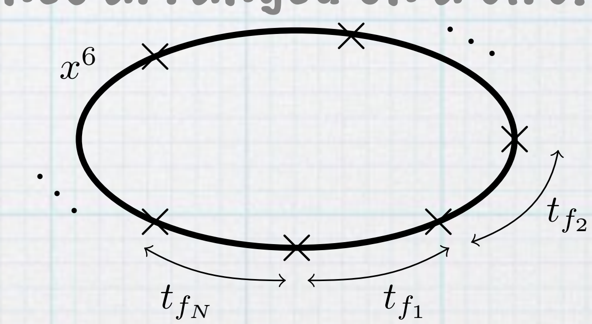
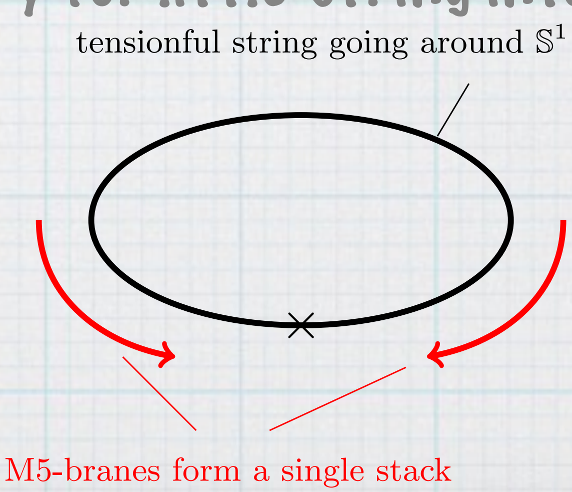


compact case: S^1

M5-branes arranged on a circle $R_6 = \frac{\rho}{2\pi i}$



necessary for little-string interpretation



Brane Configurations

The most general configuration of branes in M-theory in 11 dimensions looks like

	(0)	(1)	2	3	4	5	6	7	8	9	10
M5-branes	●	●	●	●	●	●					
M2-branes	●	●					●				

$\underbrace{(0)\ (1)}_{T^2 \sim S^1 \times S^1}$ $\underbrace{2\ 3\ 4\ 5}_{\mathbb{R}^4_{||}}$ $\underbrace{7\ 8\ 9\ 10}_{\mathrm{ALE}_{A_{M-1}} \sim \mathbb{R}^4/\mathbb{Z}_M}$

Compactification: Compactify (0,1) to $T^2 \sim S^1 \times S^1$ with radii R_0 and $R_1 =: \frac{\tau}{2\pi i}$

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ϵ_1			○	○				○	○	○	○
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Deformations: there are two types of deformations with respect to the compactified (0,1)-directions introducing complex coordinates $(z_1, z_2) = (x_2 + ix_3, x_4 + ix_5)$ and $(w_1, w_2) = (x_7 + ix_8, x_9 + ix_{10})$

(0)-direct.: $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} : (z_1, z_2) \to (e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2) \quad \text{and} \quad (w_1, w_2) \to (e^{-i\pi(\epsilon_1+\epsilon_2)} w_1, e^{-i\pi(\epsilon_1+\epsilon_2)} w_2)$

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gauge theory: Omega-background [Nekrasov 2012]

mass-deformation

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For vanishing mass deformation ($m = 0$) the M-brane configuration is dual to D5-NS5-branes in IIB

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NS5 branes	•	•	•	•	•	–	•			
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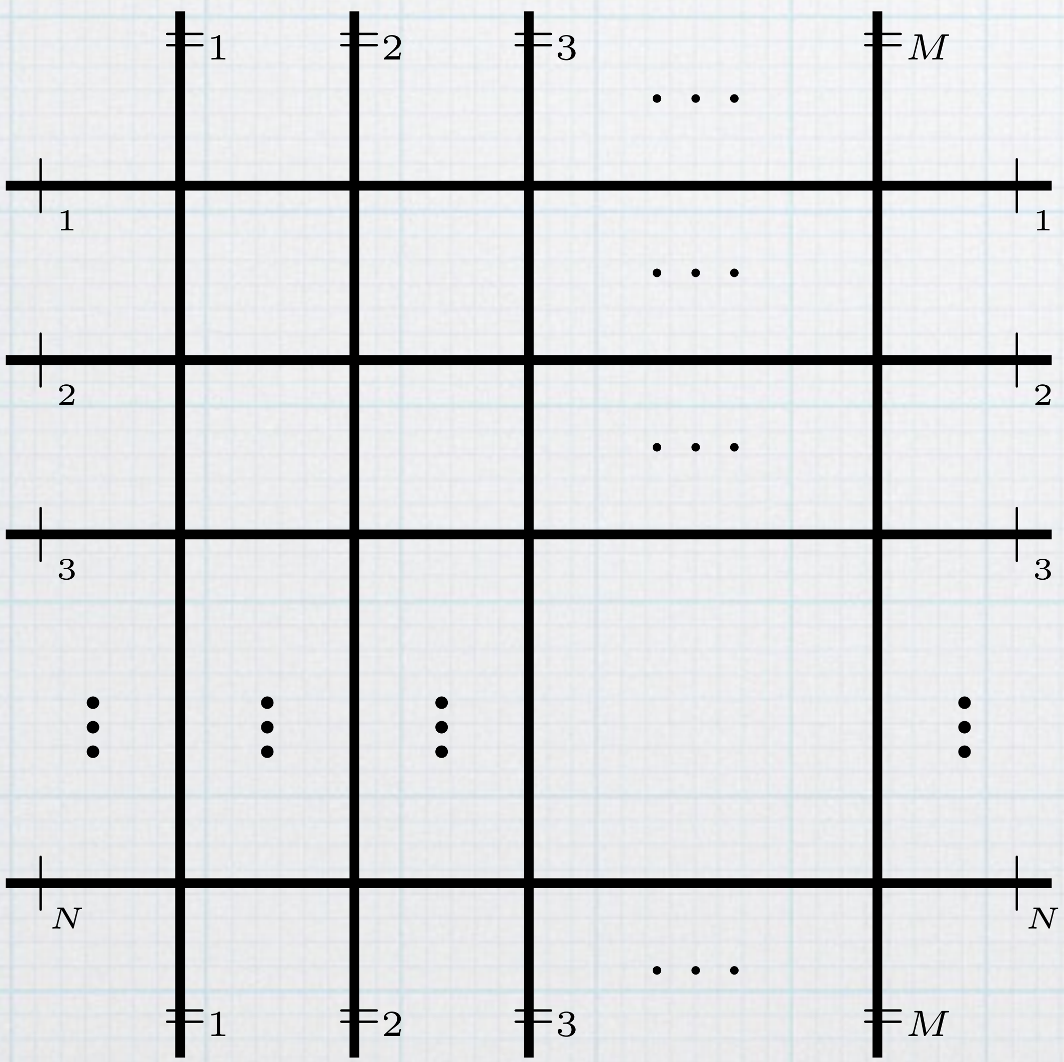

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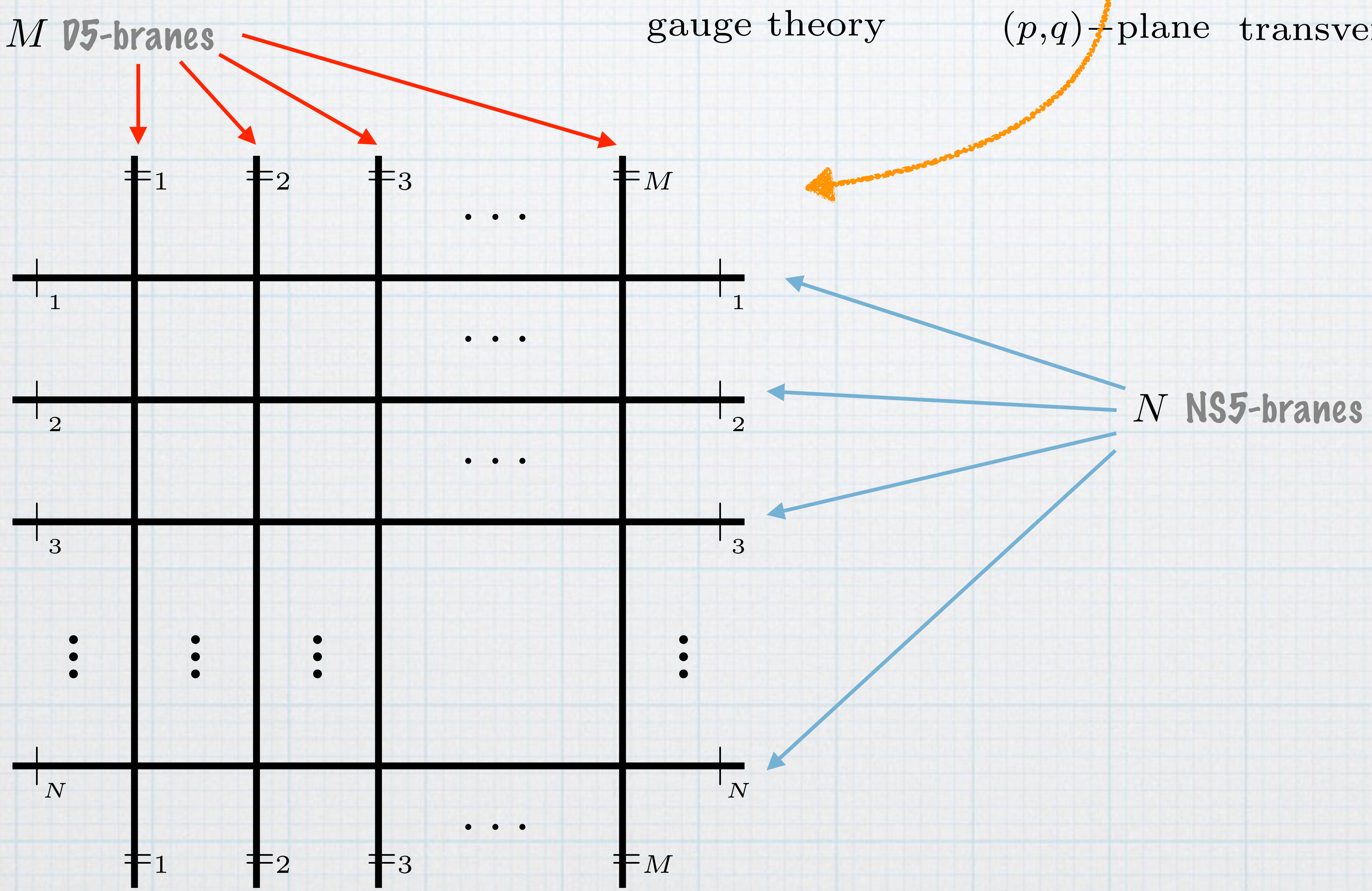

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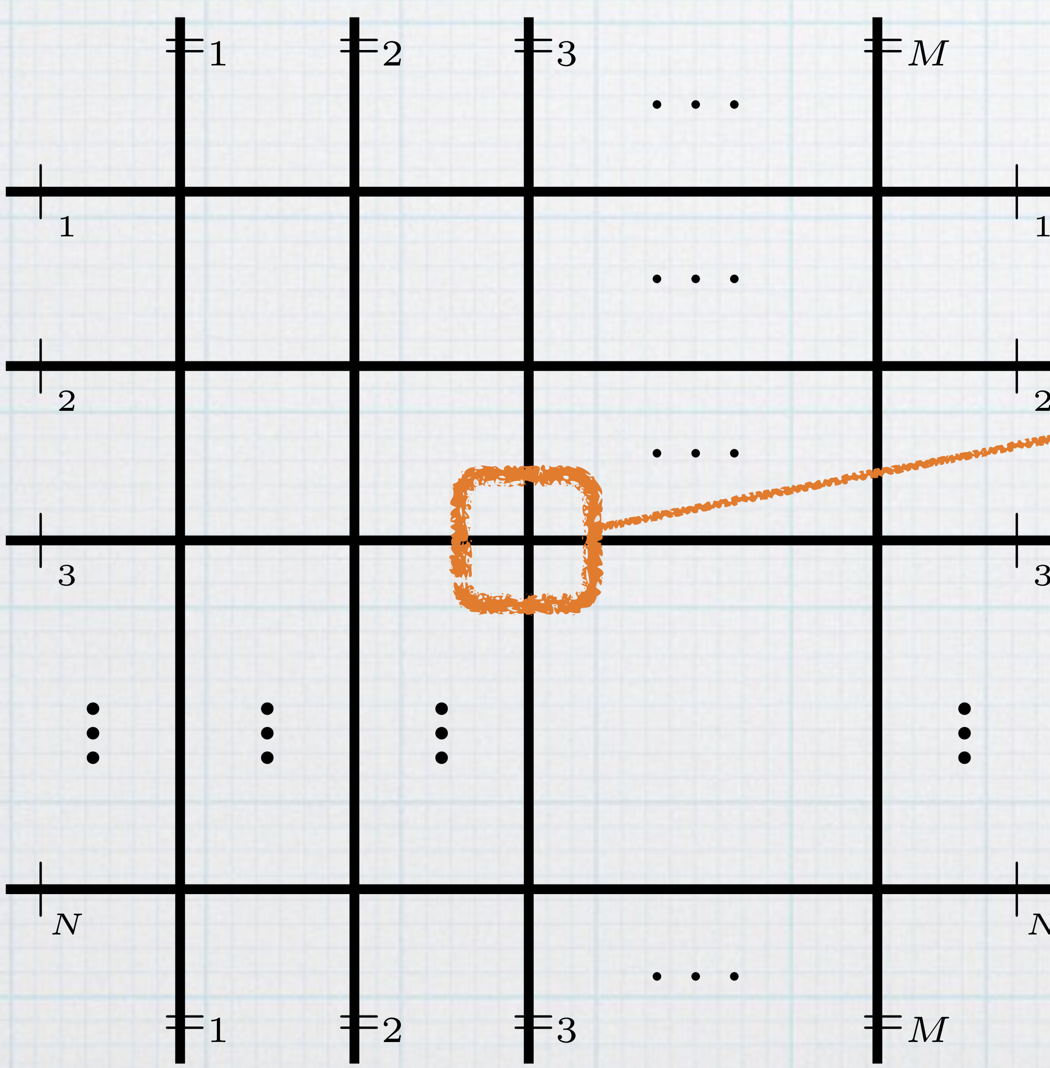

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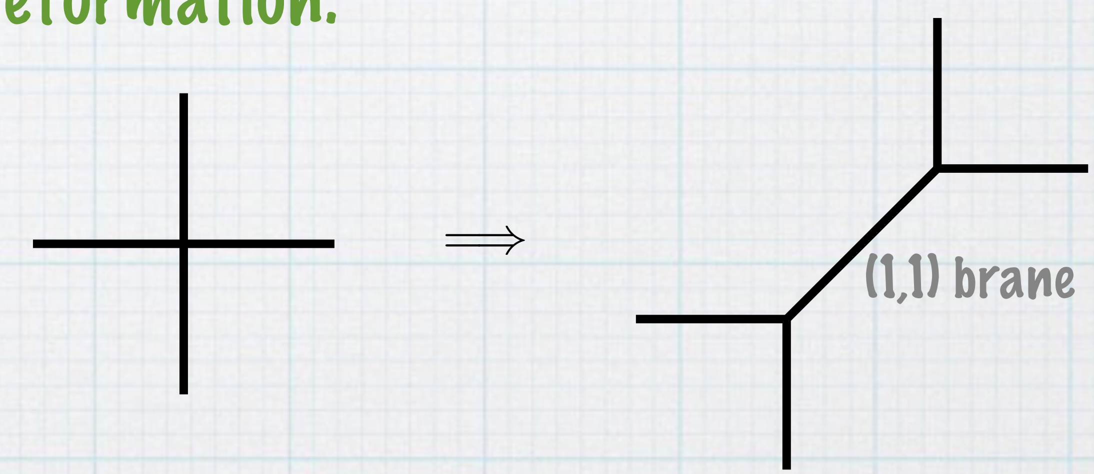

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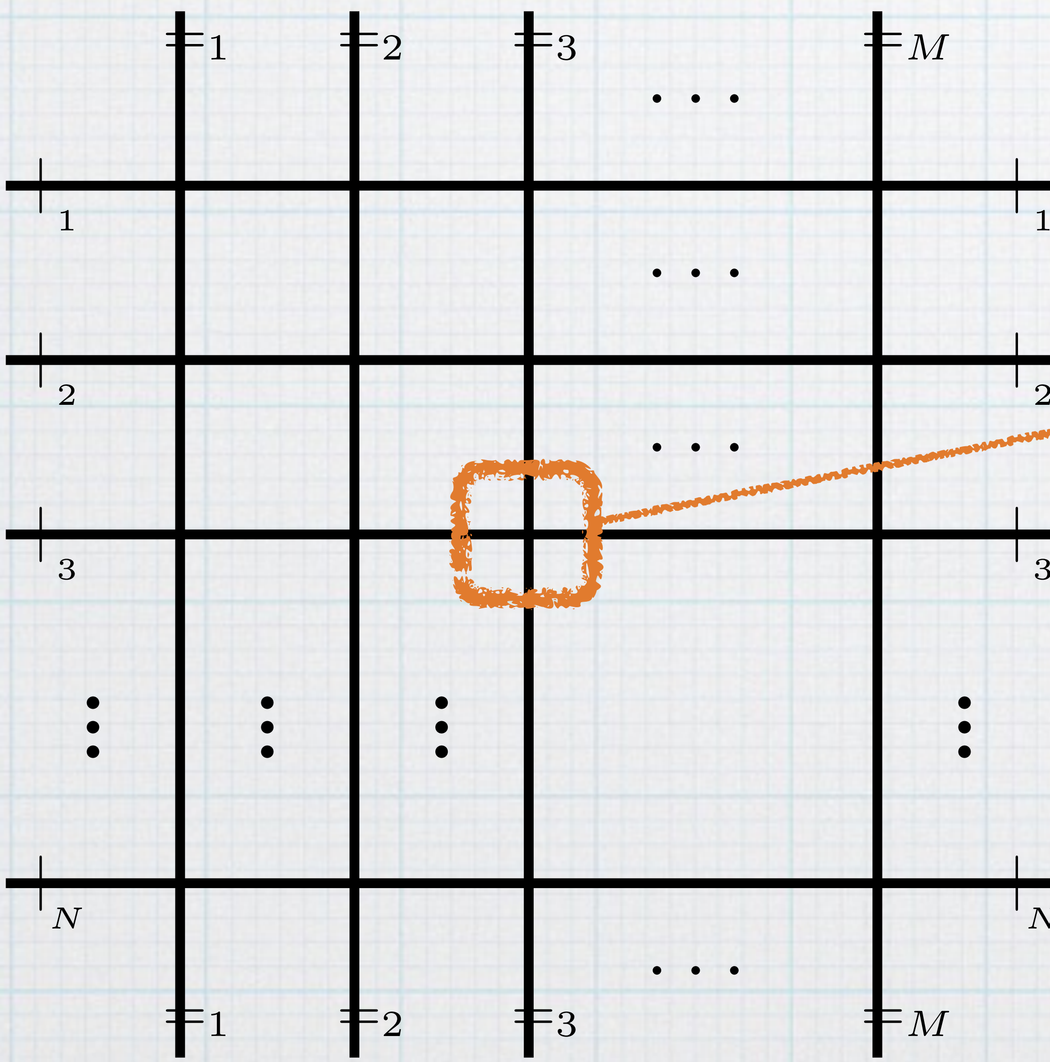

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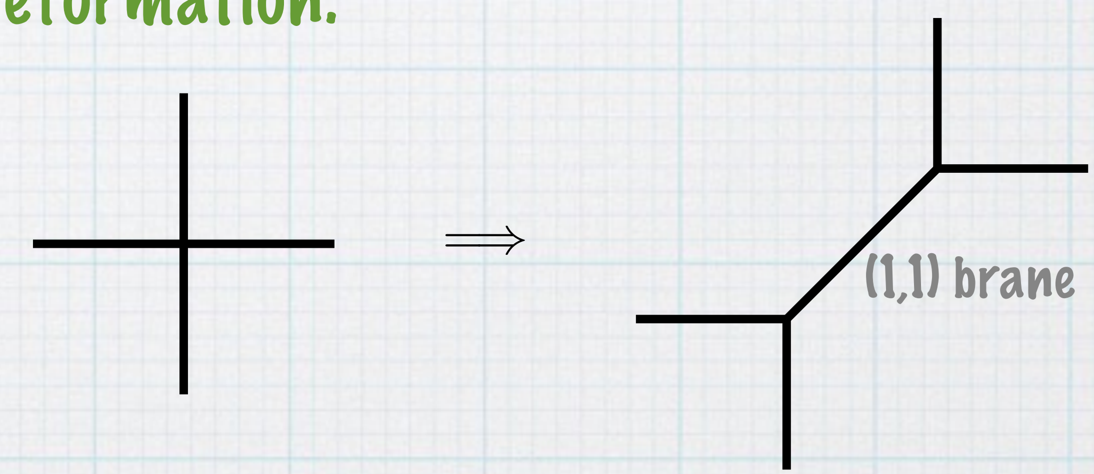

uplift the deformed type II configuration to M-th.
on an elliptically fibered Calabi-Yau threefold $X_{N,M}$

[Leung, Vafa 1997]

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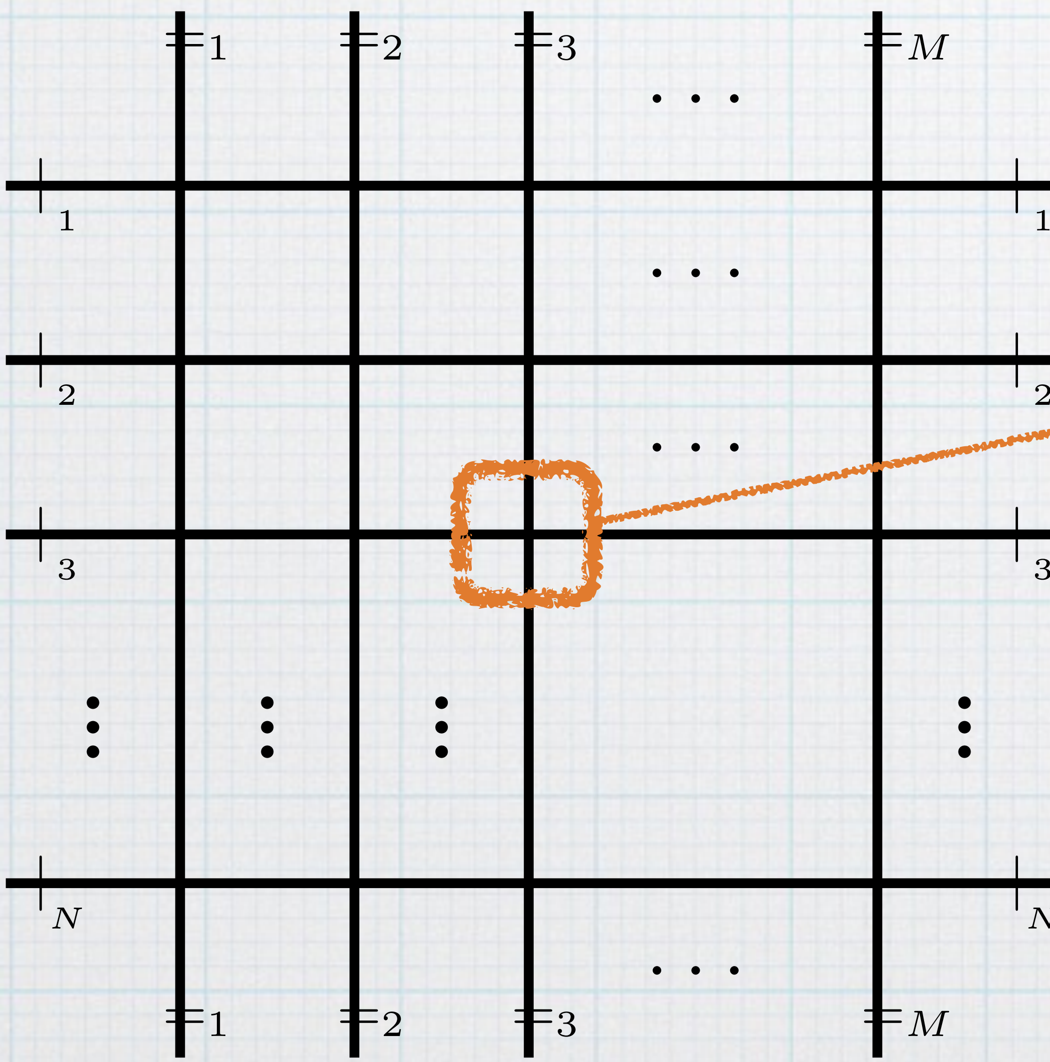

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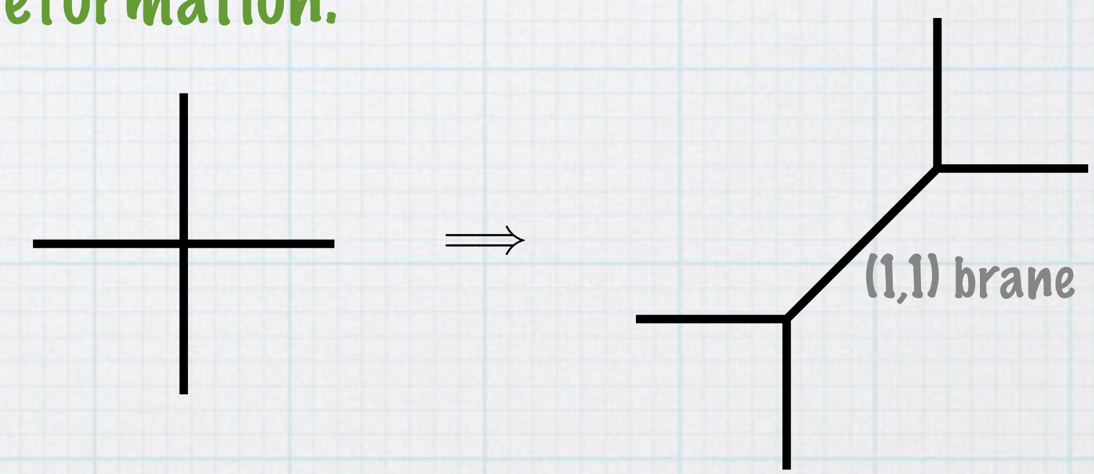

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topic diagram of $X_{N,M}$ same as deformed brane web

Dual Construction of LSTs: Toric Calabi-Yau 3folds

Specific, 2-parameter series of toric, double elliptically fibered Calabi-Yau threefolds $X_{N,M}$

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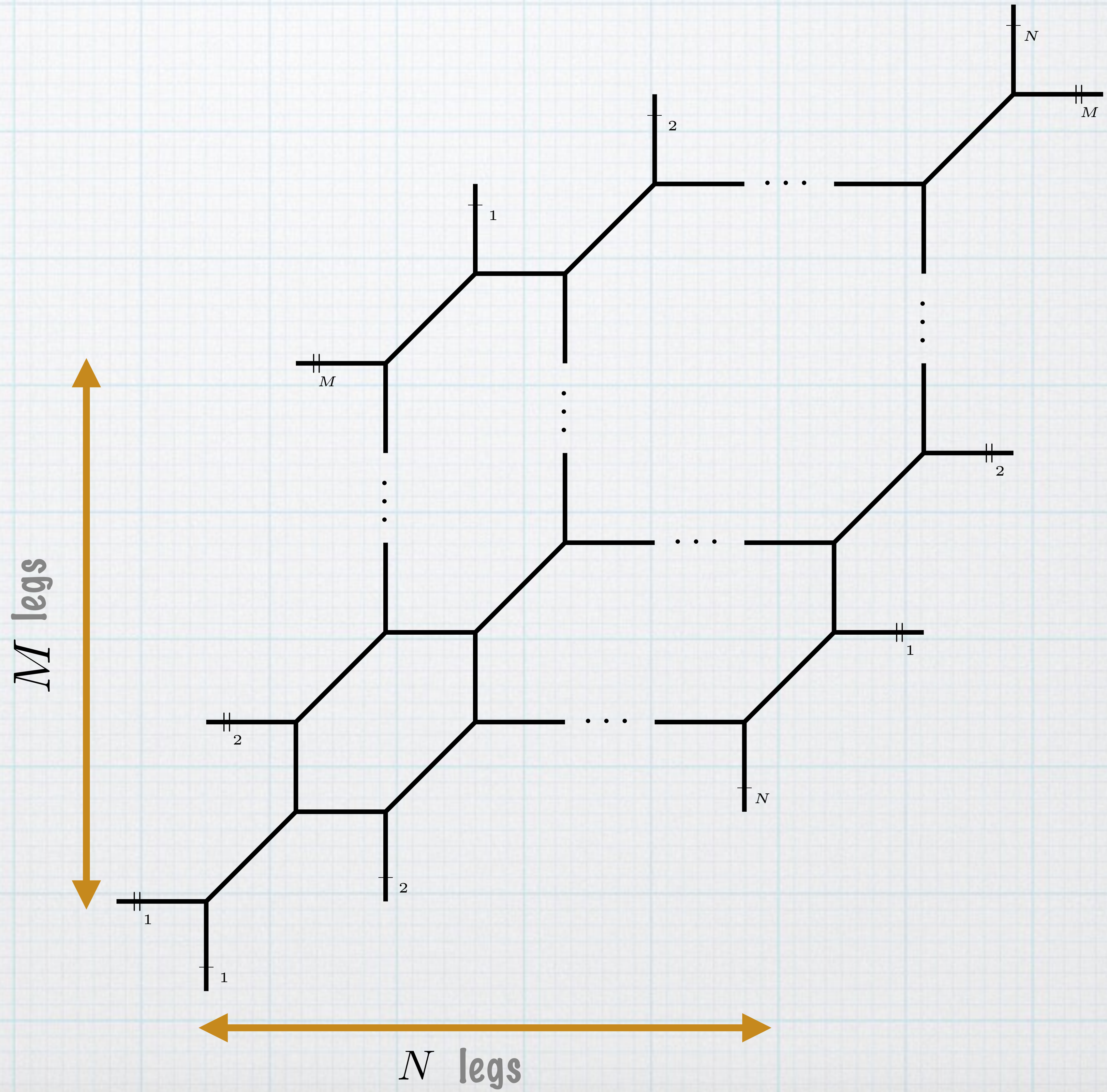

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Toric Web Diagram:

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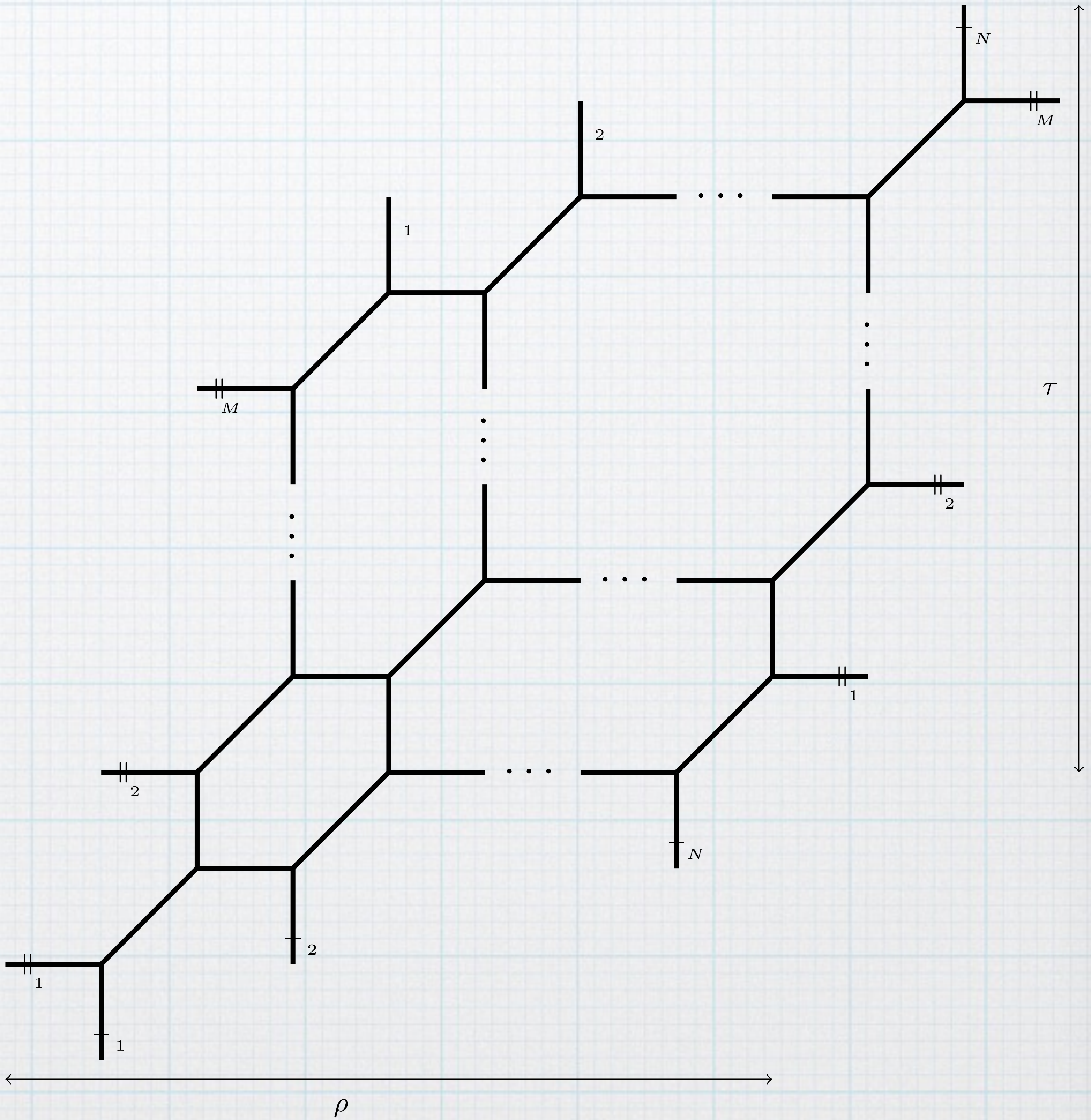

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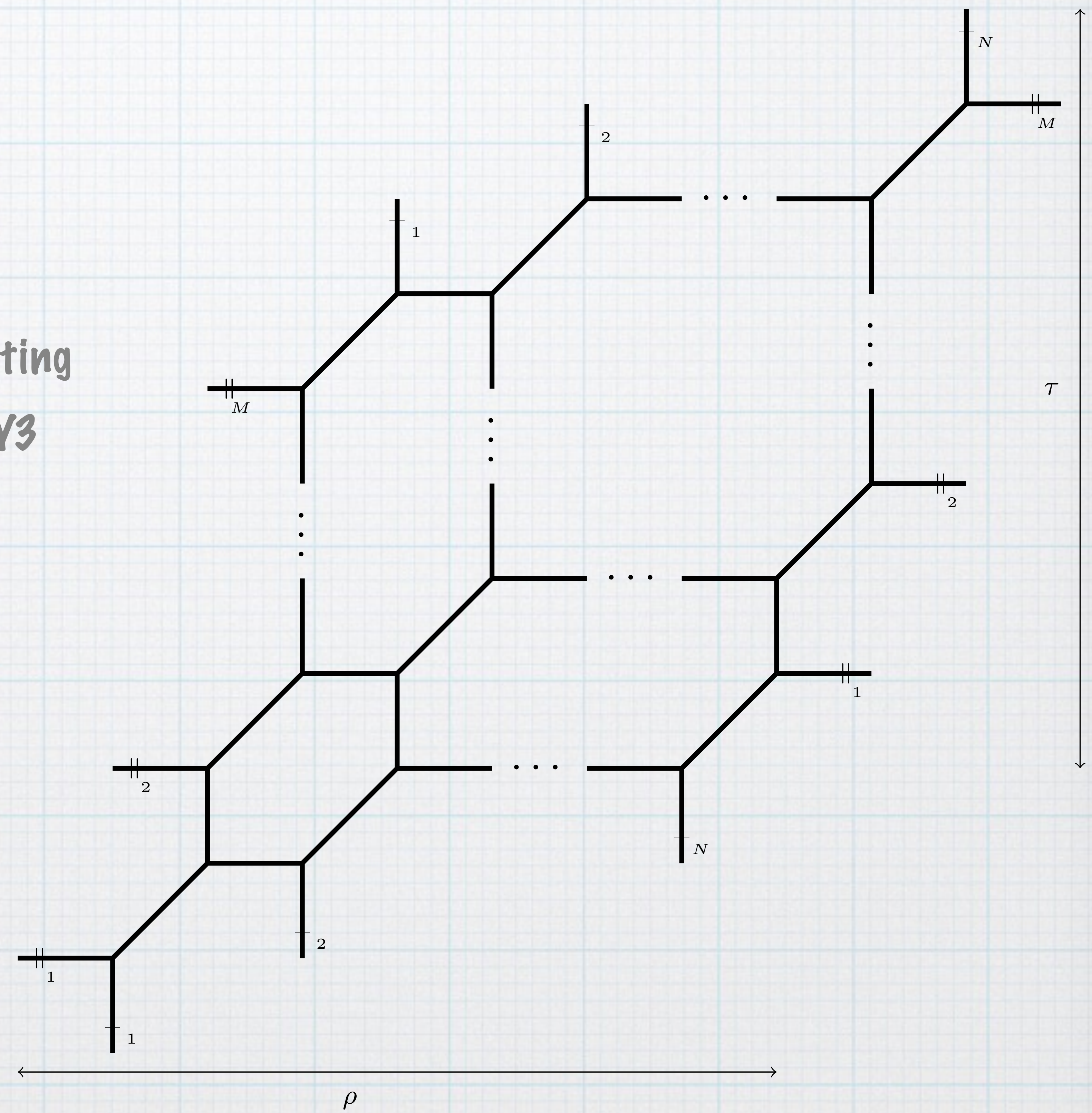
* $3NM$ different parameters representing the area of various curves C of the CY3

$$d = \int_C \omega$$



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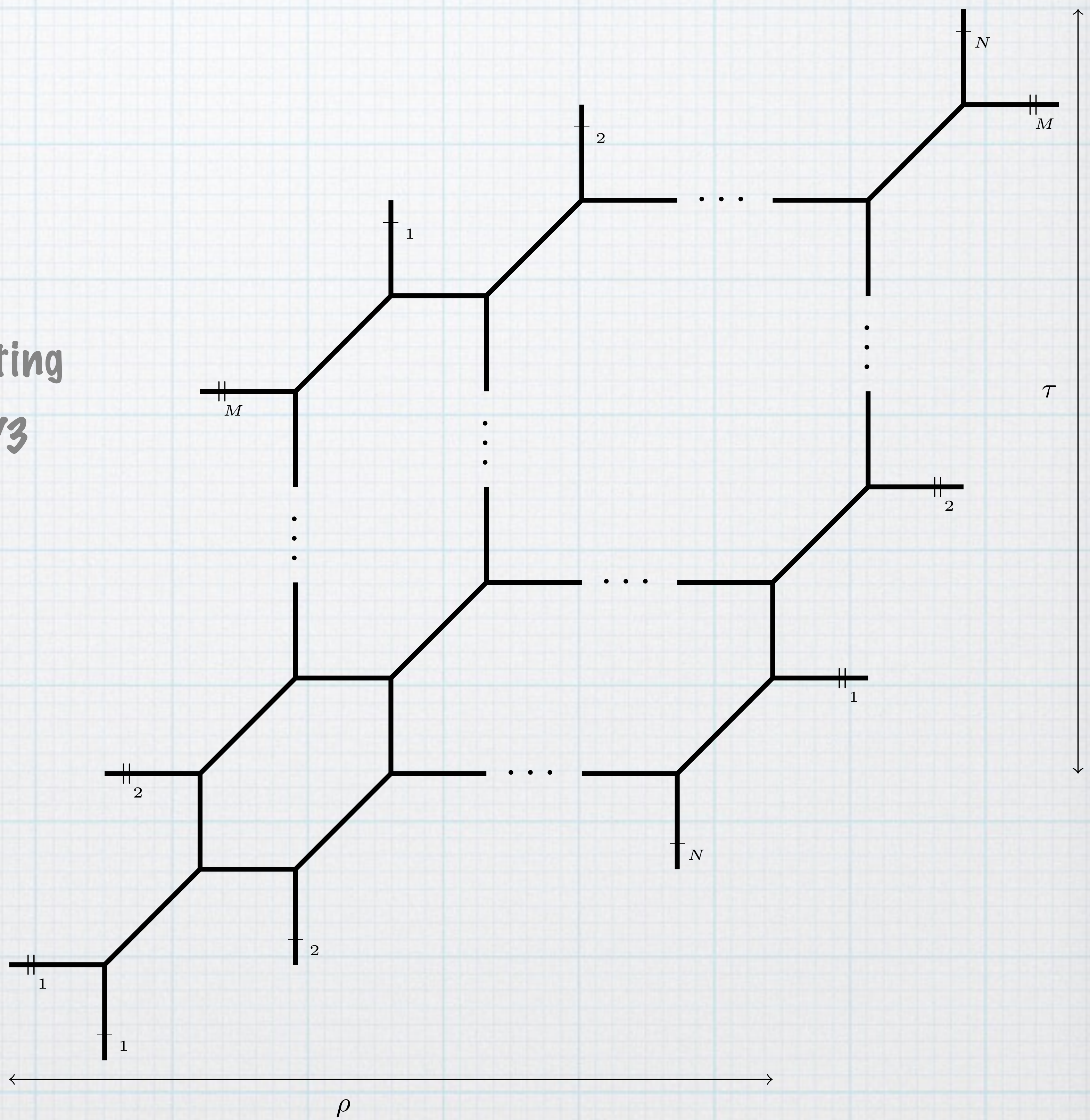

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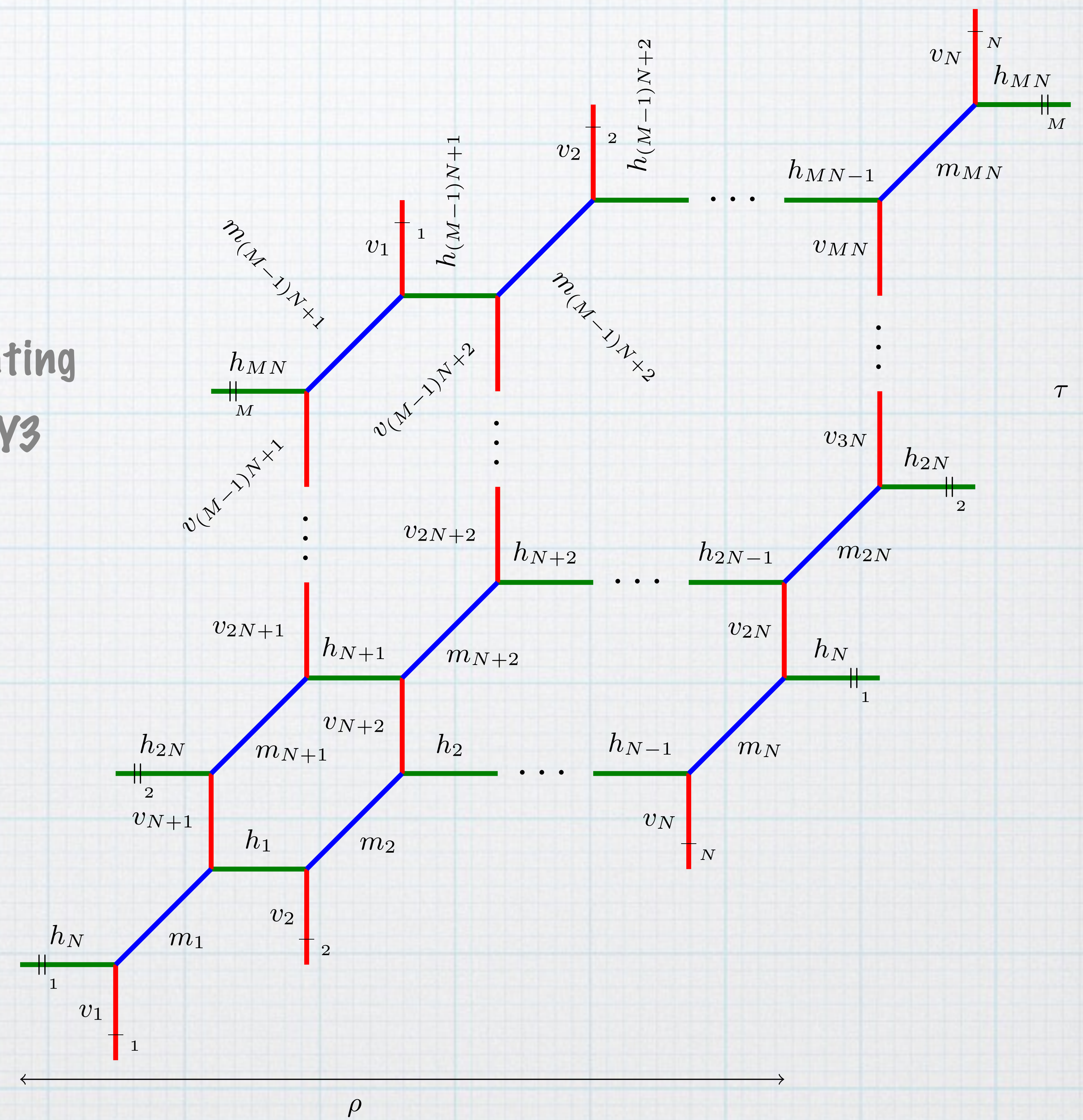

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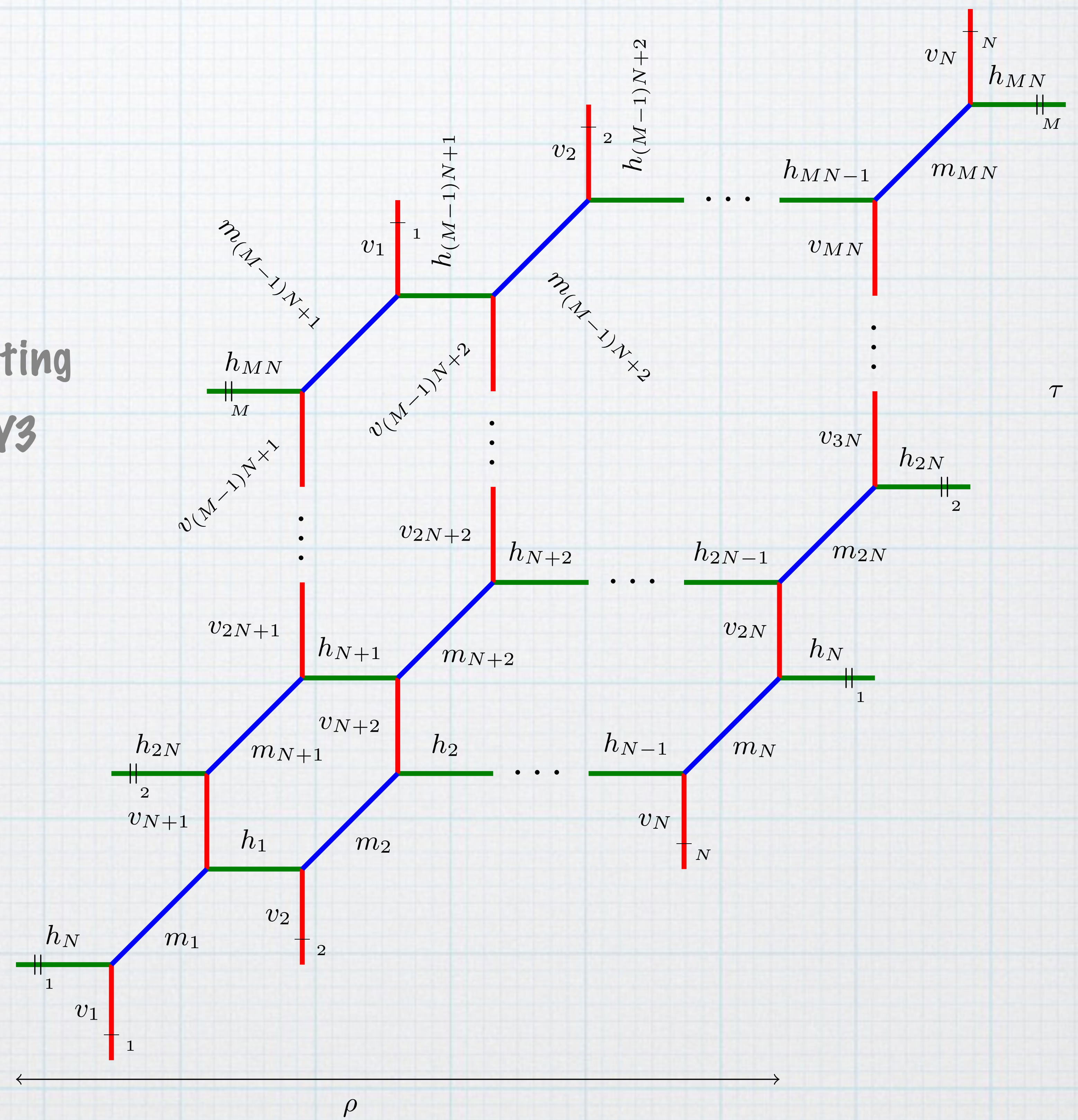

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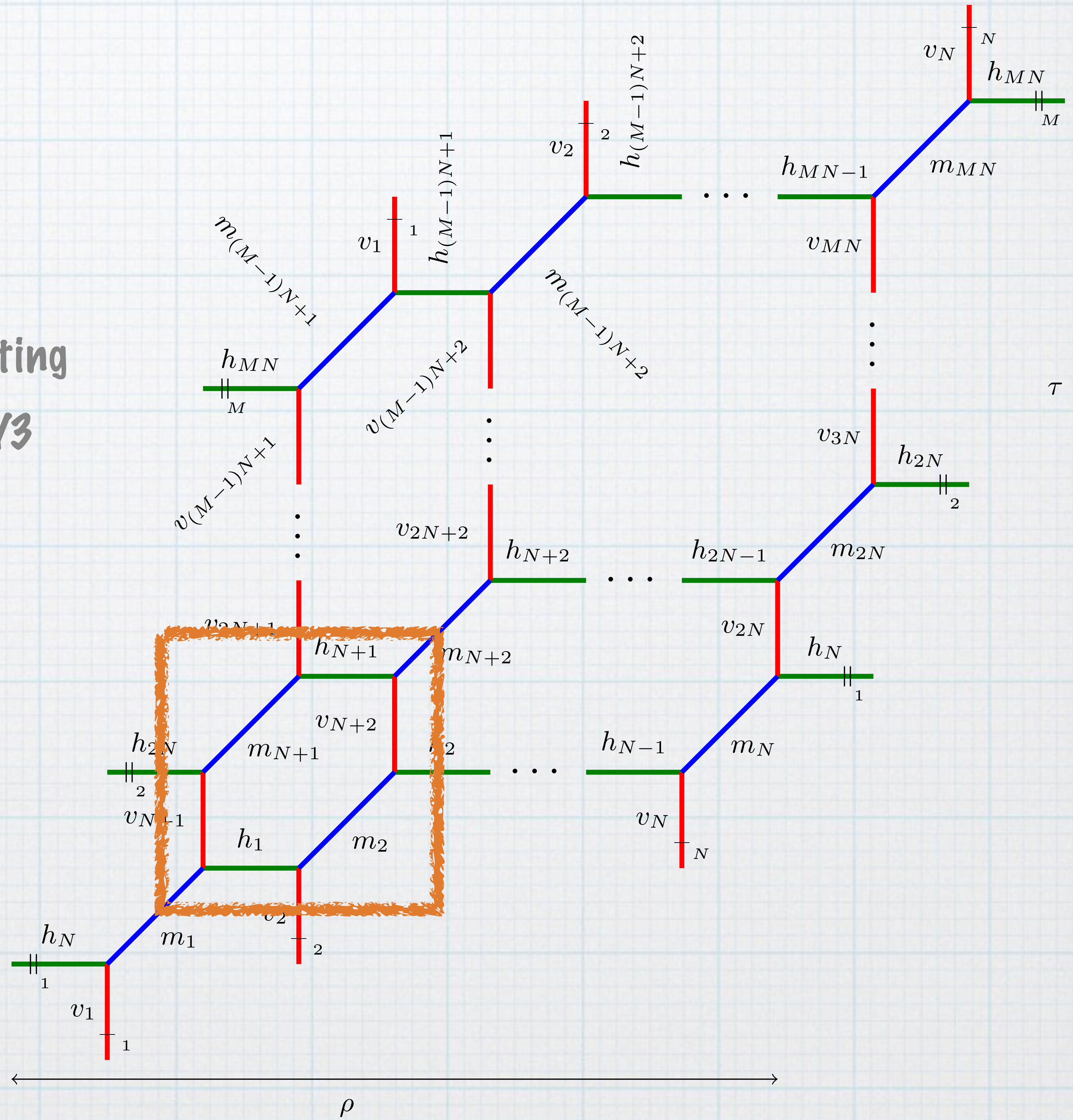

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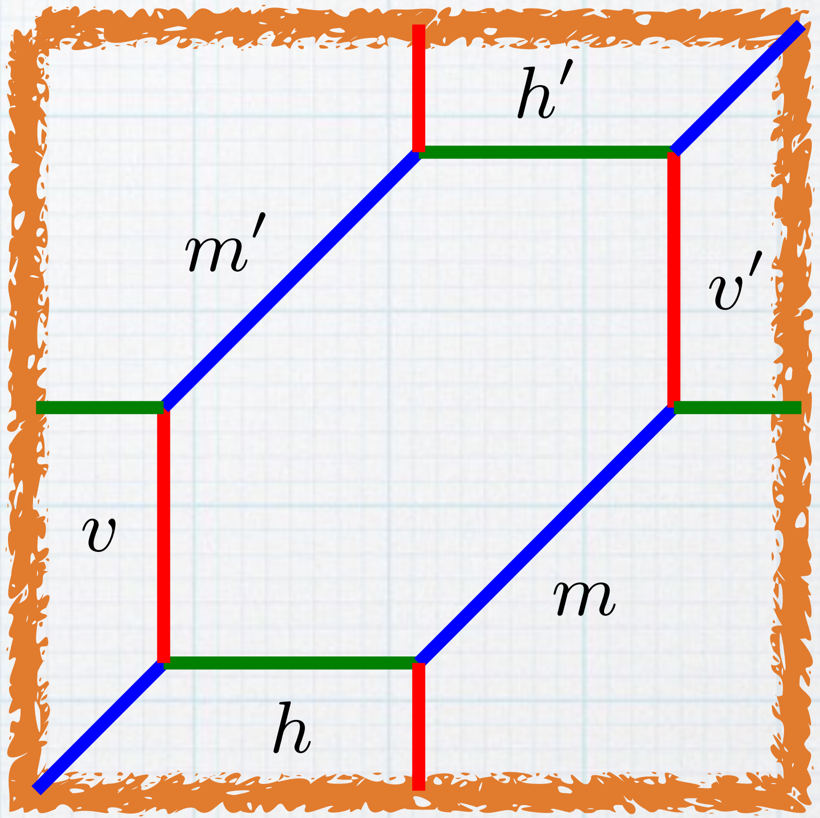

$$h + m = h' + m'$$

$$v + m' = m + v'$$

different possible choices for set of independent parameters

BPS States and Topological String

Free Energy: Counts number of BPS configurations, i.e. M2-branes wrapping holomorphic curves on the CY3 $X_{N,M}$.

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Compute the topological string partition function $\mathcal{Z}_{N,M}$ using the refined topological vertex

[Aganagic, Klemm, Marino, Vafa 2003]
[Iqbal, Kozçaz, Vafa 2007]

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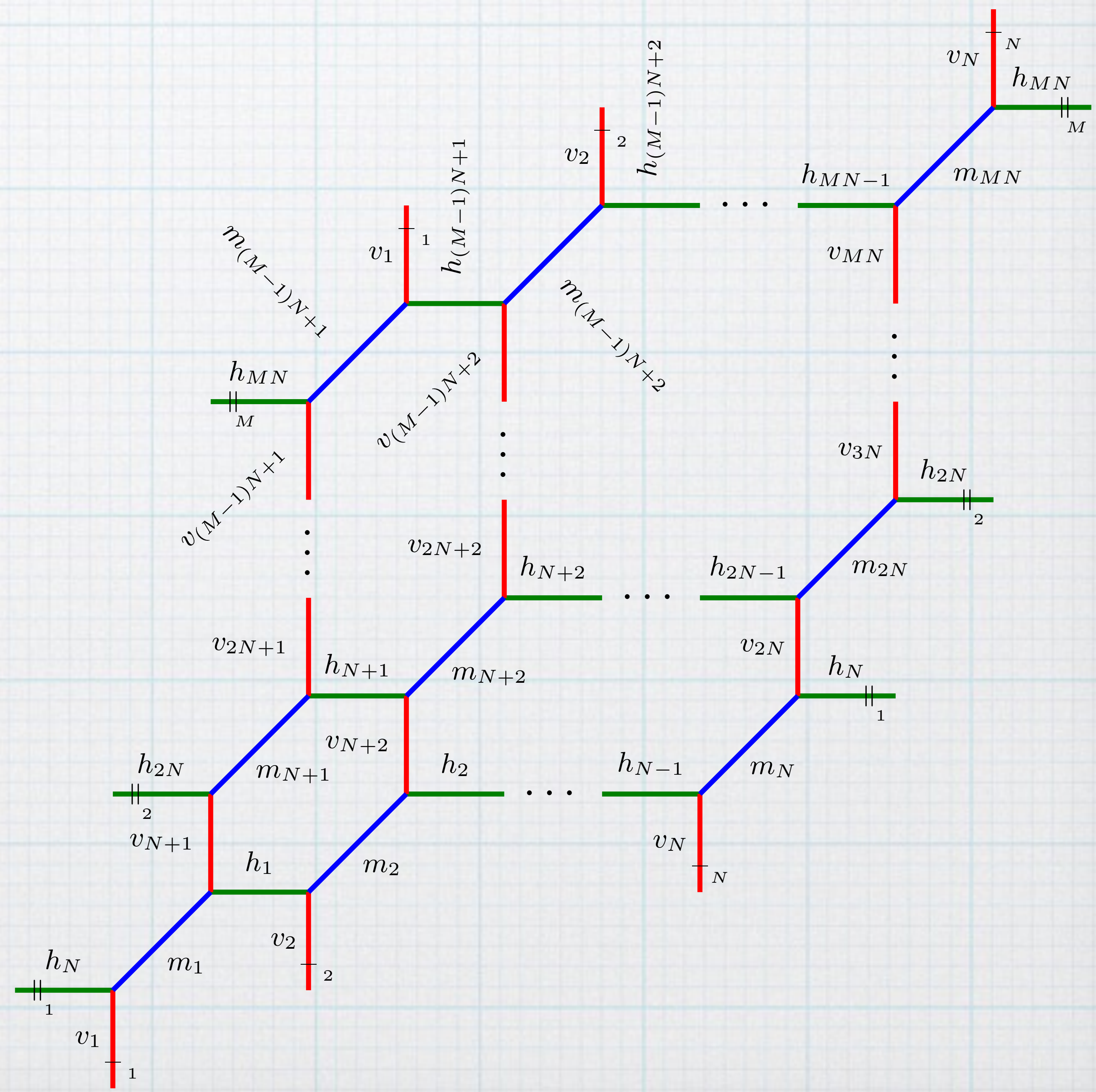

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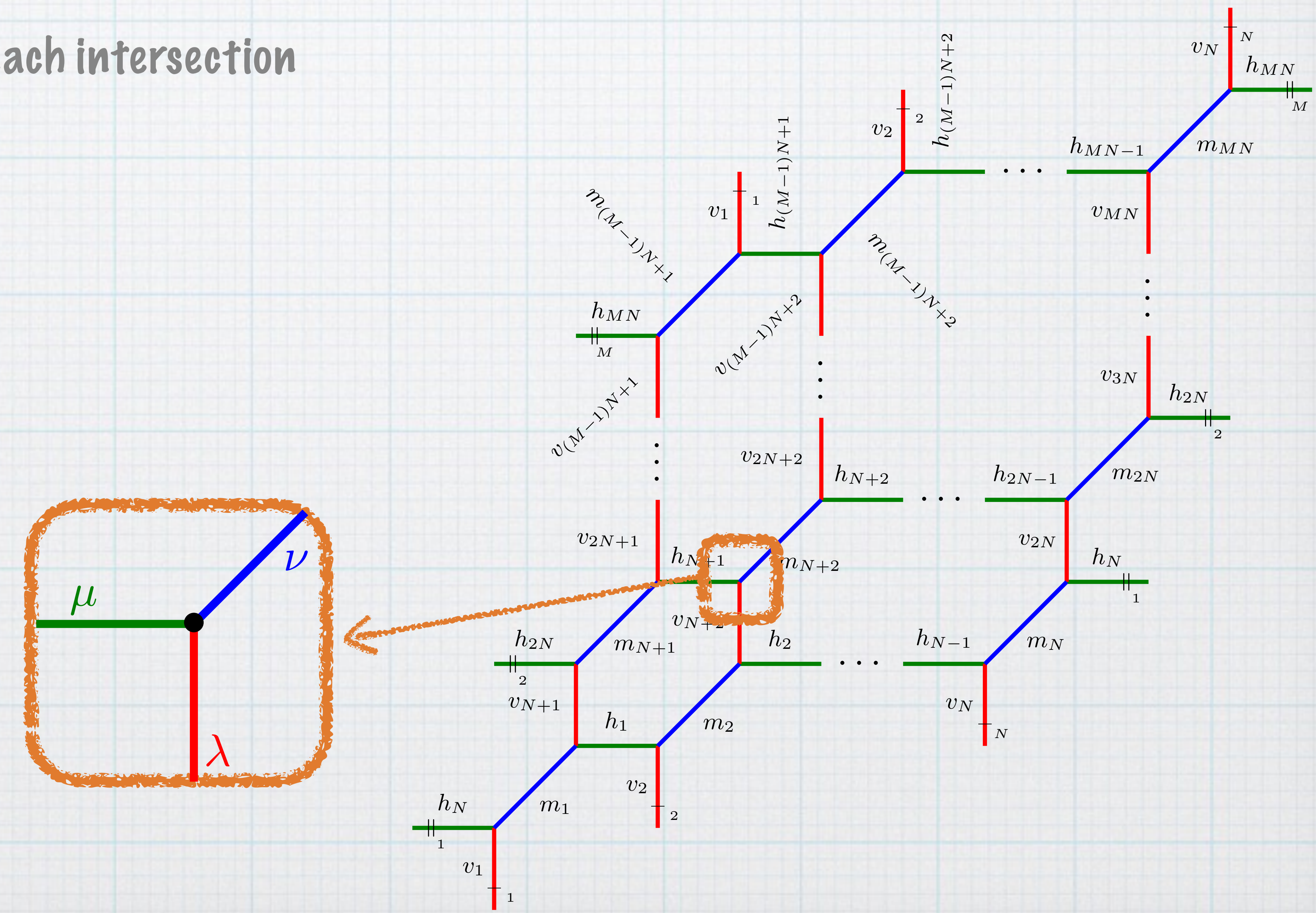
Compute the topological string partition function $\mathcal{Z}_{N,M}$ using the **refined topological vertex**

-) assign trivalent vertex to each intersection



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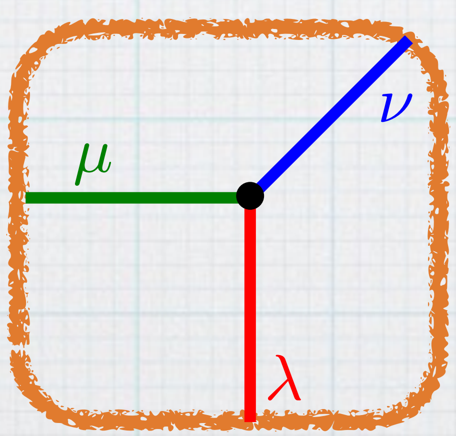
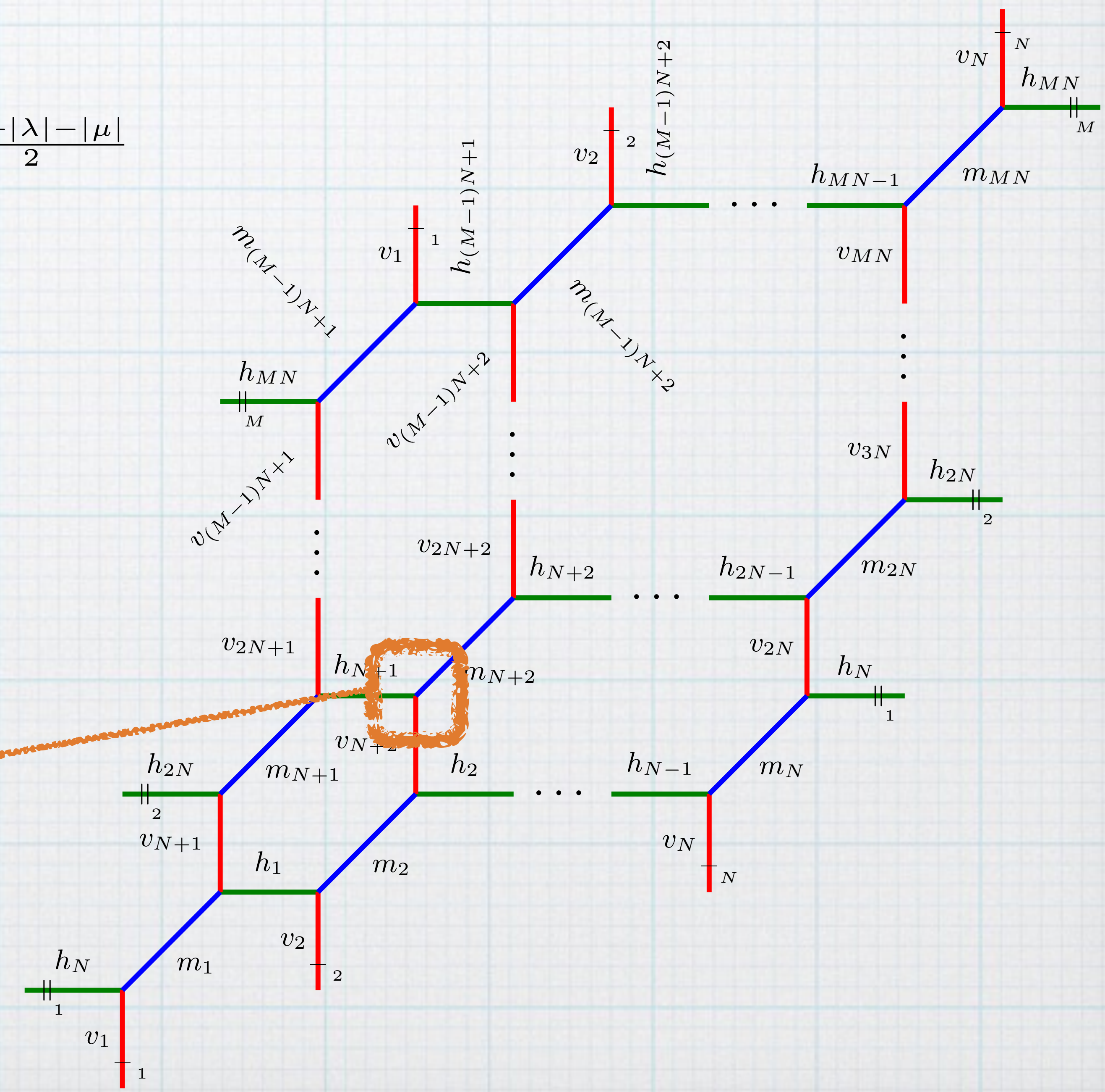
$$C_{\lambda\mu\nu} = q^{\frac{||\mu||^2}{2}}\, t^{-\frac{||\mu^t||^2}{2}}\, q^{\frac{||\nu||^2}{2}}\, \tilde{Z}_\nu(t,q) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}}$$

$$\times\; s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu})\, s_{\mu/\eta}(q^{-\rho}t^{-\nu^t})$$

$$\tilde{Z}_\nu(t,q) = \prod_{(i,j)\in\nu} \left(1 - t^{\nu_j^t - i + 1} q^{\nu_i - j}\right)^{-1},$$





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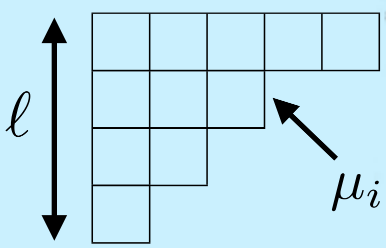
Notation:

$q = e^{2\pi i \epsilon_1}$ and $t = e^{-2\pi i \epsilon_2}$

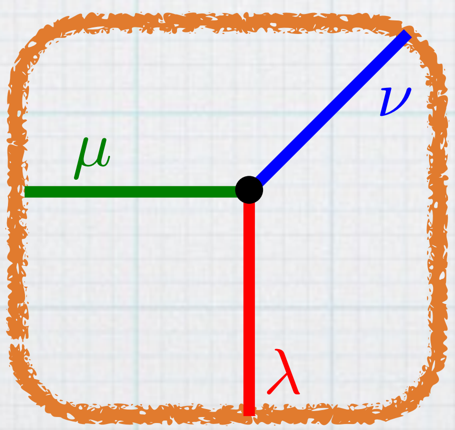
$\mu\,,\nu\,,\lambda$ integer partitions

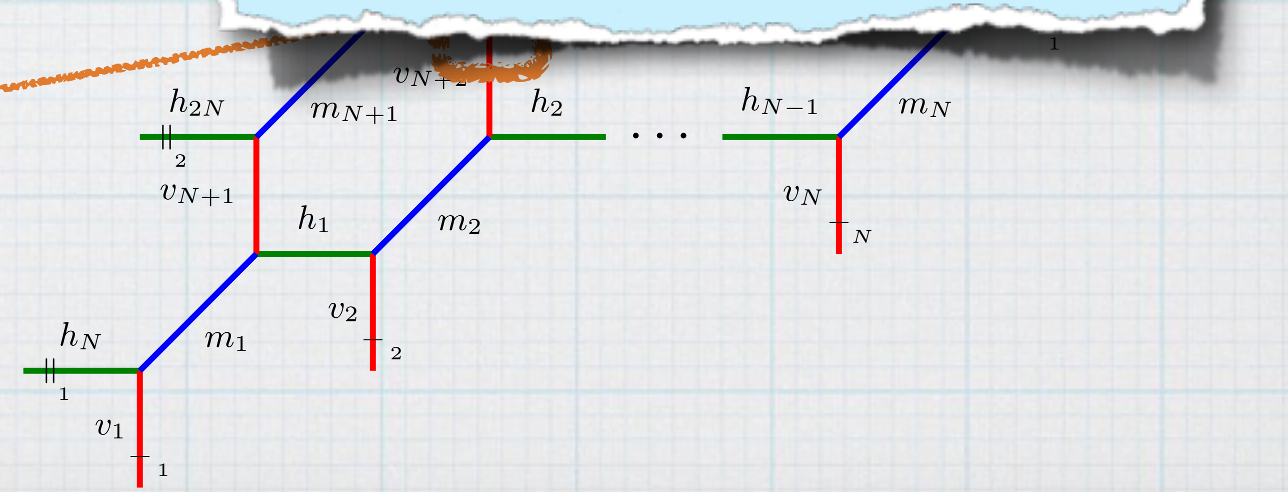
$$|\mu| = \sum_{i=1}^{\ell} \mu_i$$



$$||\mu||^2 = \sum_{i=1}^{\ell} \mu_i^2$$

$s_{\mu/\eta}$ skew Schur function





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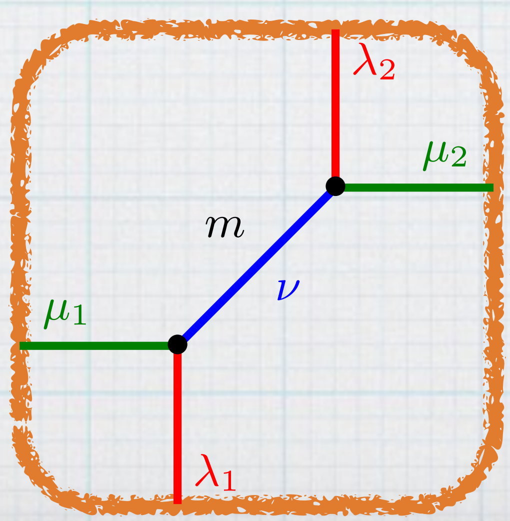
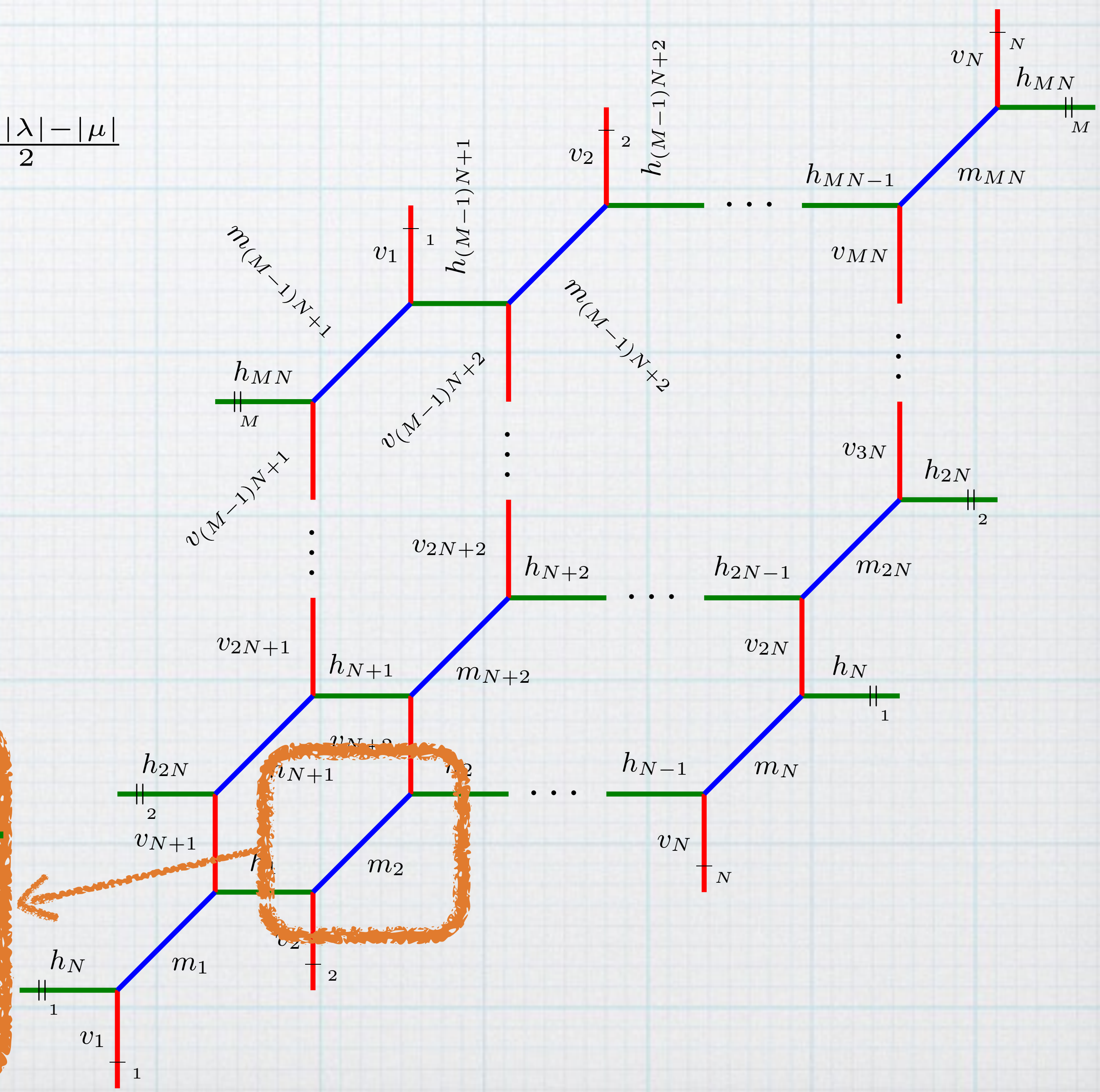
-) glue vertices according to web diagram





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[Haghighat, Kozcaz, Lockhart, Vafa 2013]
[SH, Iqbal 2013]

Compute the topological string partition function $\mathcal{Z}_{N,M}$ using the **refined topological vertex**

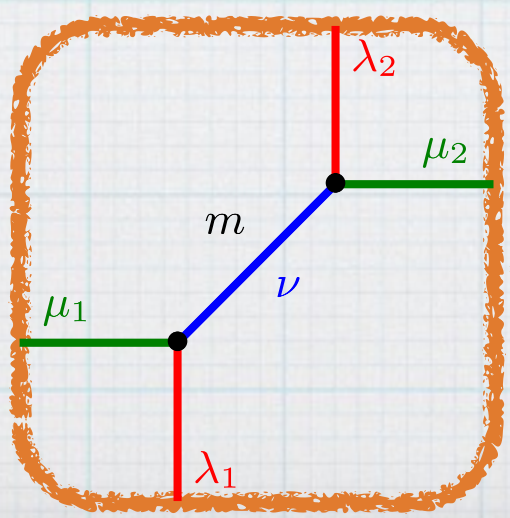
-) assign trivalent vertex to each intersection

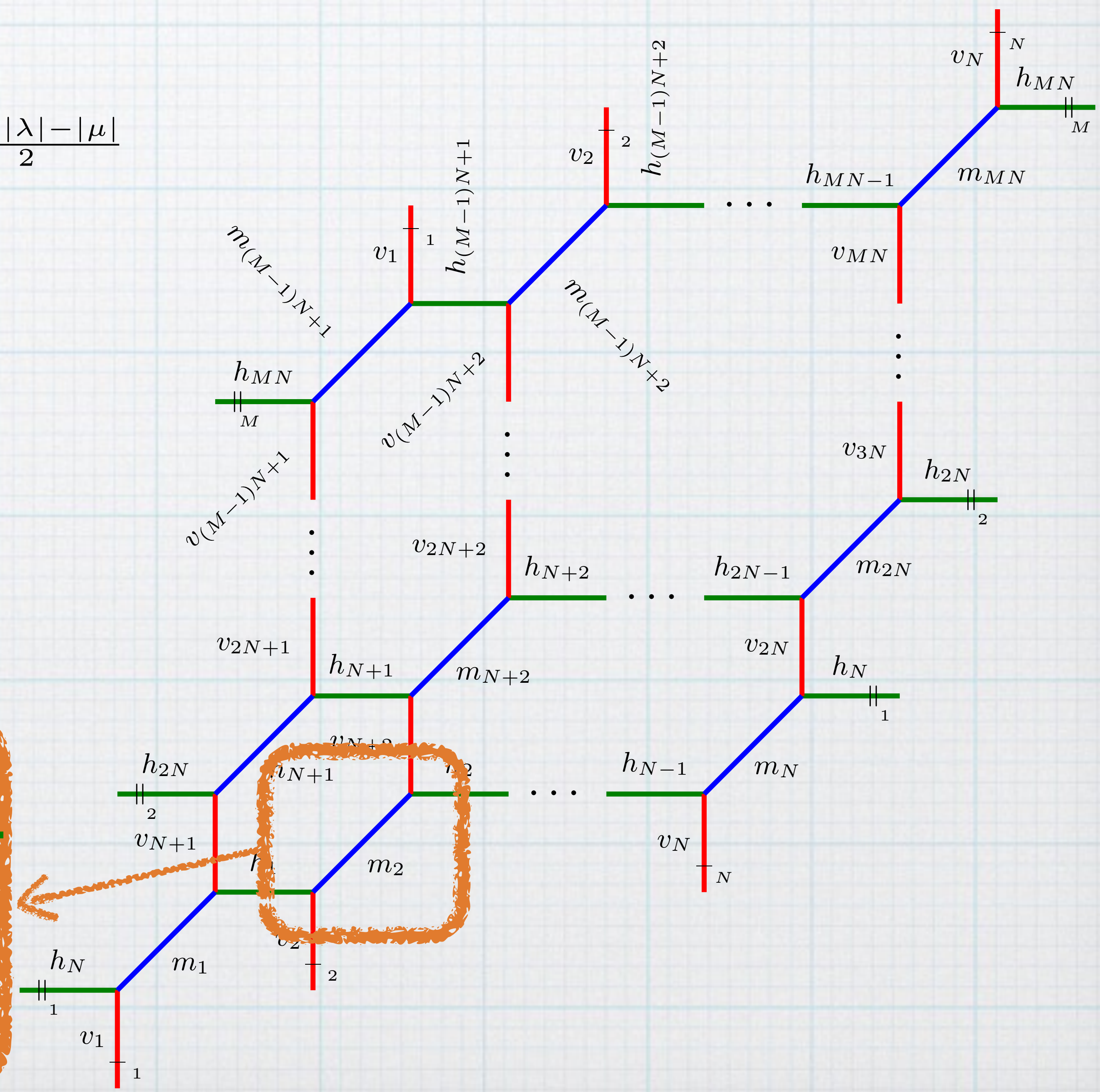
$$C_{\lambda\mu\nu} = q^{\frac{||\mu||^2}{2}}\, t^{-\frac{||\mu^t||^2}{2}}\, q^{\frac{||\nu||^2}{2}}\, \tilde{Z}_\nu(t,q) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} \times\, s_{\lambda^t/\eta}(t^{-\rho} q^{-\nu})\, s_{\mu/\eta}(q^{-\rho} t^{-\nu^t})$$

$$\tilde{Z}_\nu(t,q) = \prod_{(i,j)\in\nu} \left(1 - t^{\nu_j^t - i + 1} q^{\nu_i - j}\right)^{-1},$$

-) glue vertices according to web diagram

$$\sum_\nu (-e^{2\pi i m})^{|\nu|} C_{\mu_1\lambda_1\nu} C_{\mu_2^t\lambda_2^t\nu^t}$$





BPS States and Topological String

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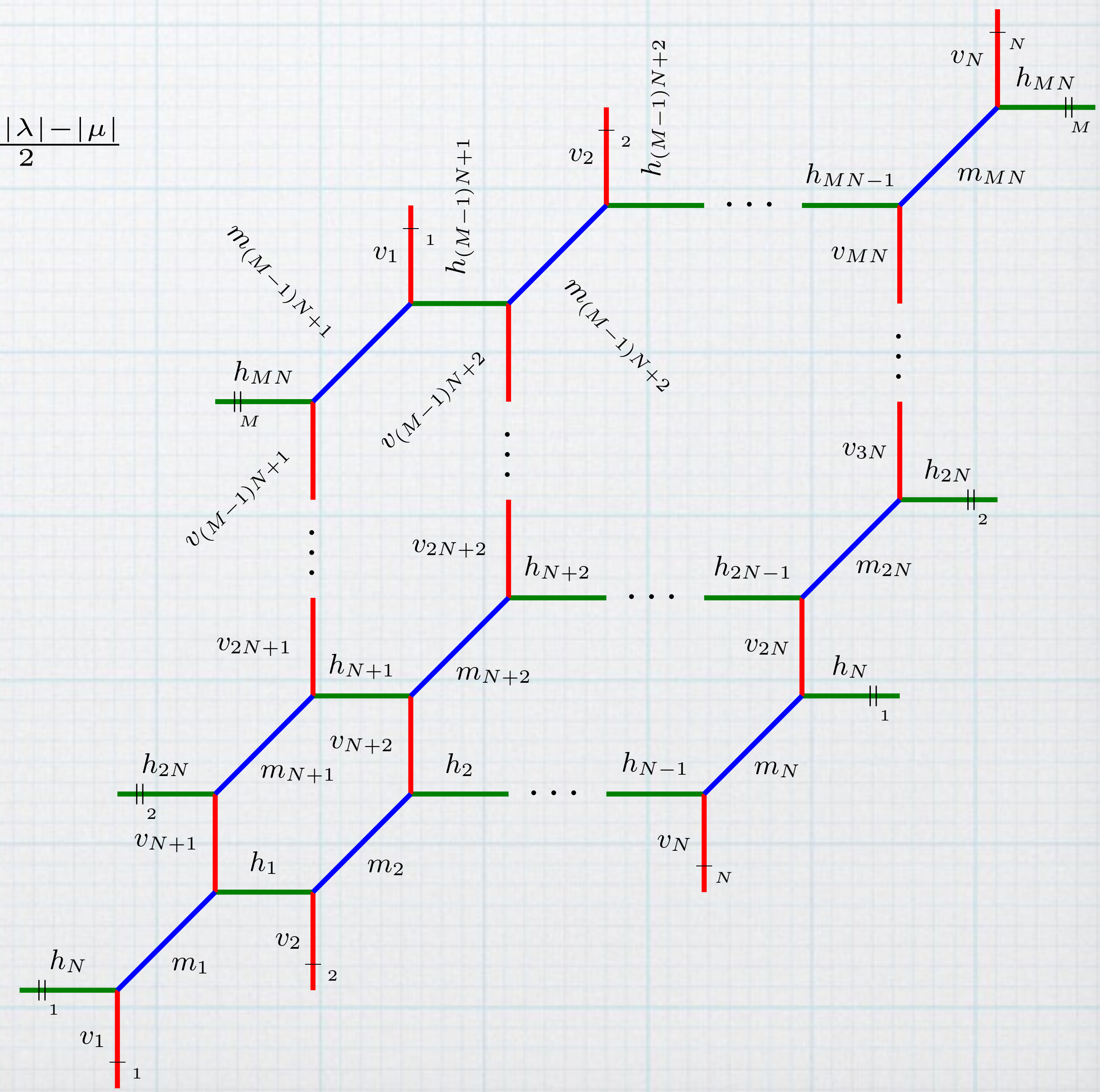
-) choose **preferred direction**



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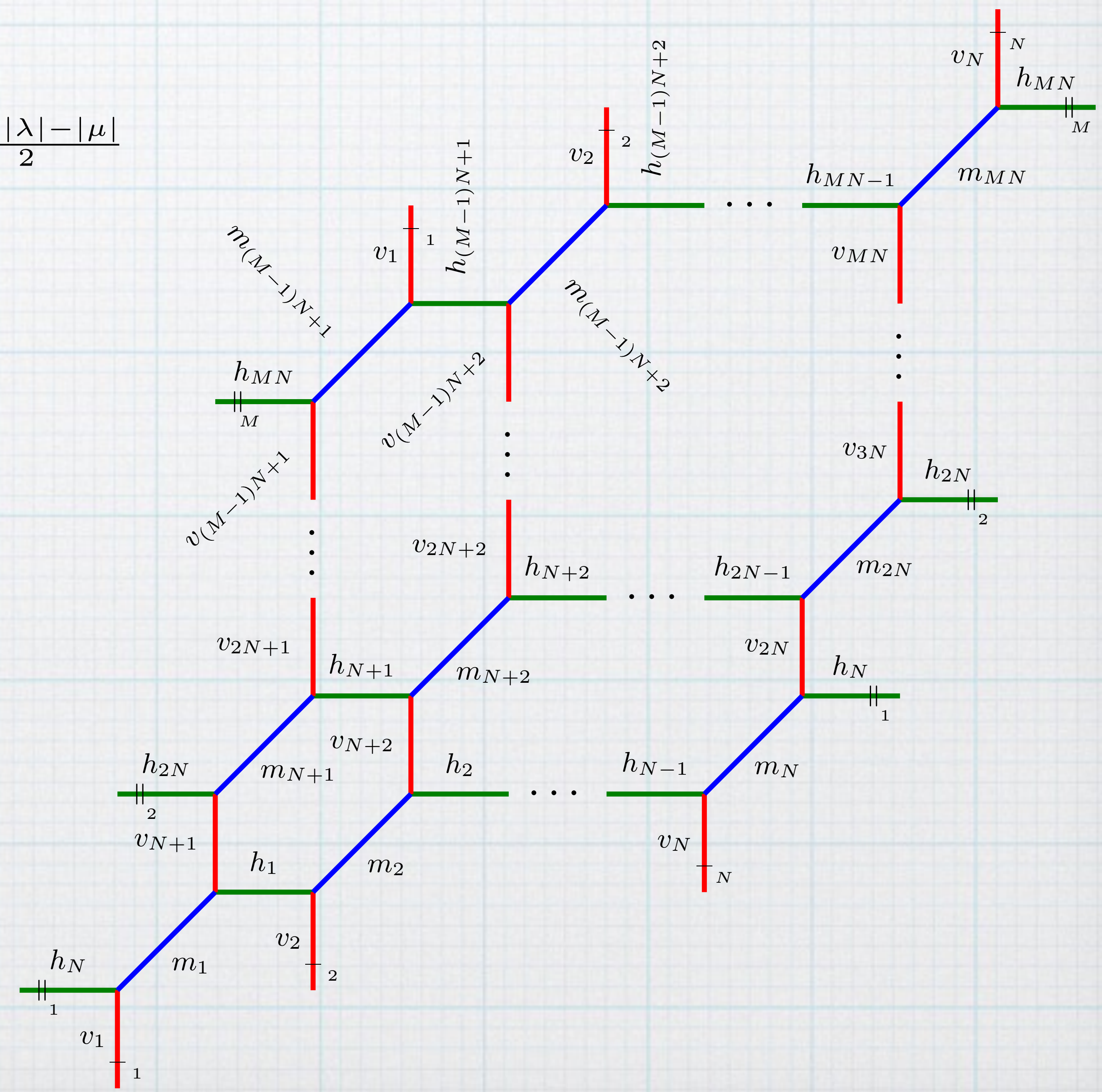
must be common to all vertices of diagram

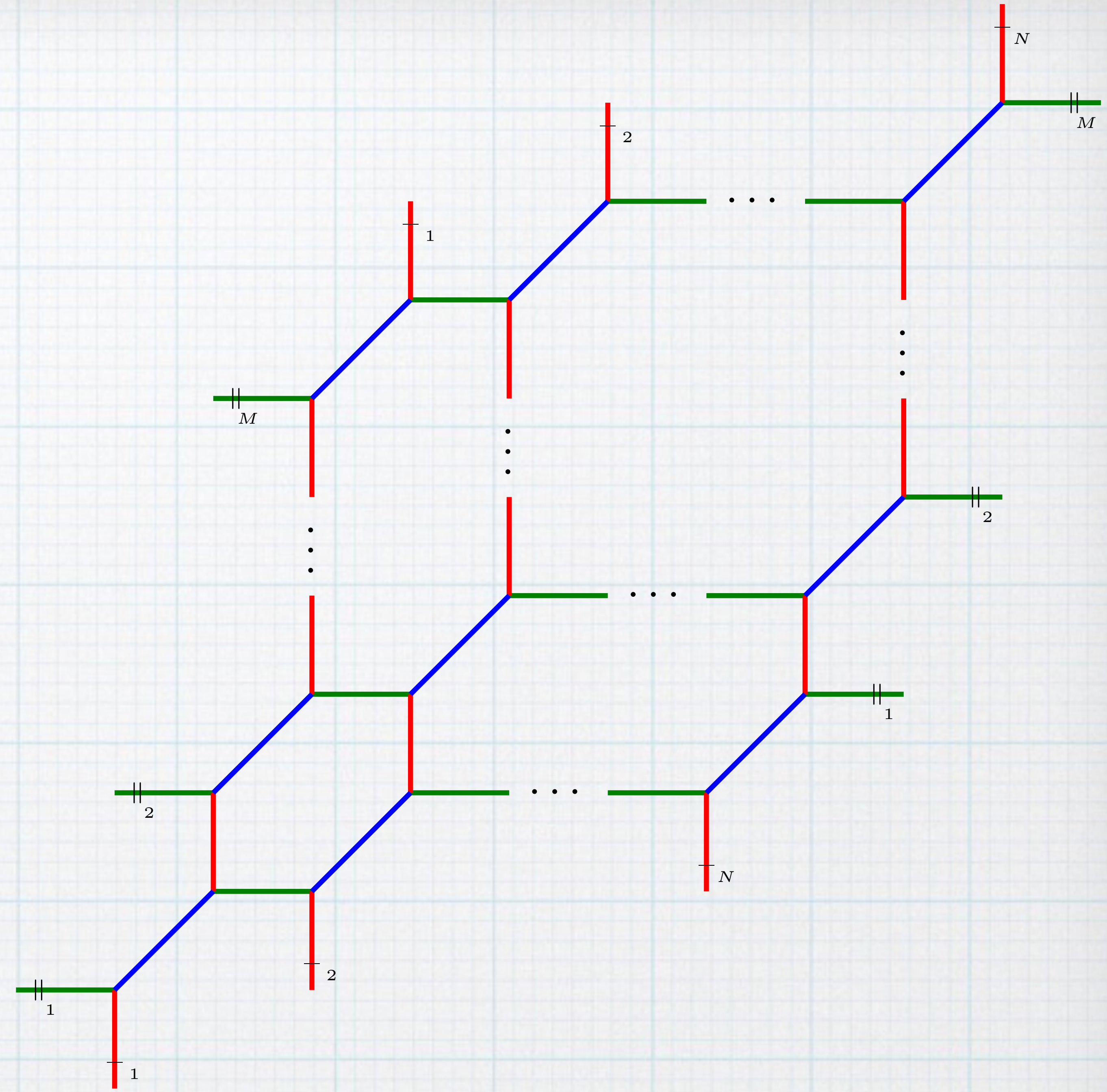


preferred direction

3 different choices for the preferred direction:

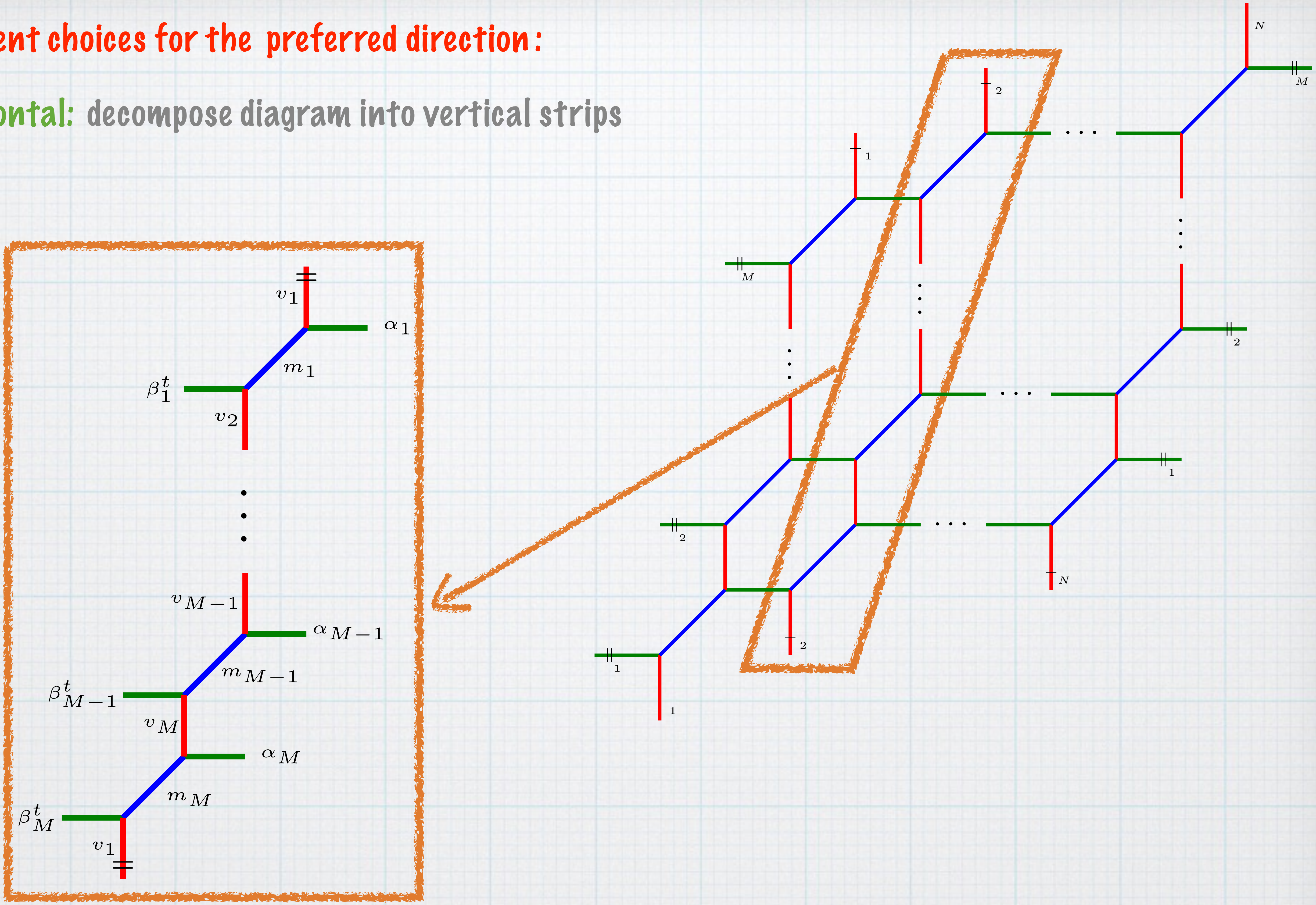
3 different choices for the preferred direction:

1) horizontal:



3 different choices for the preferred direction:

1) horizontal: decompose diagram into vertical strips



3 different choices for the preferred direction:

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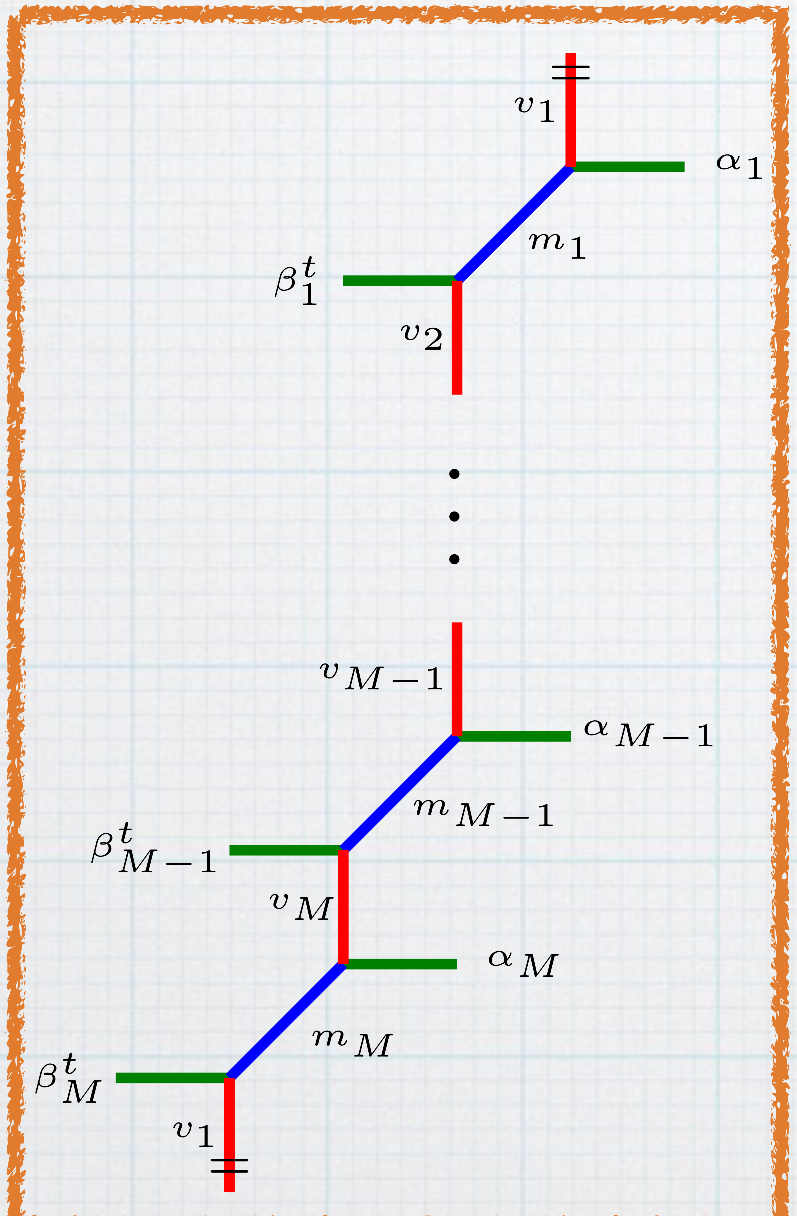
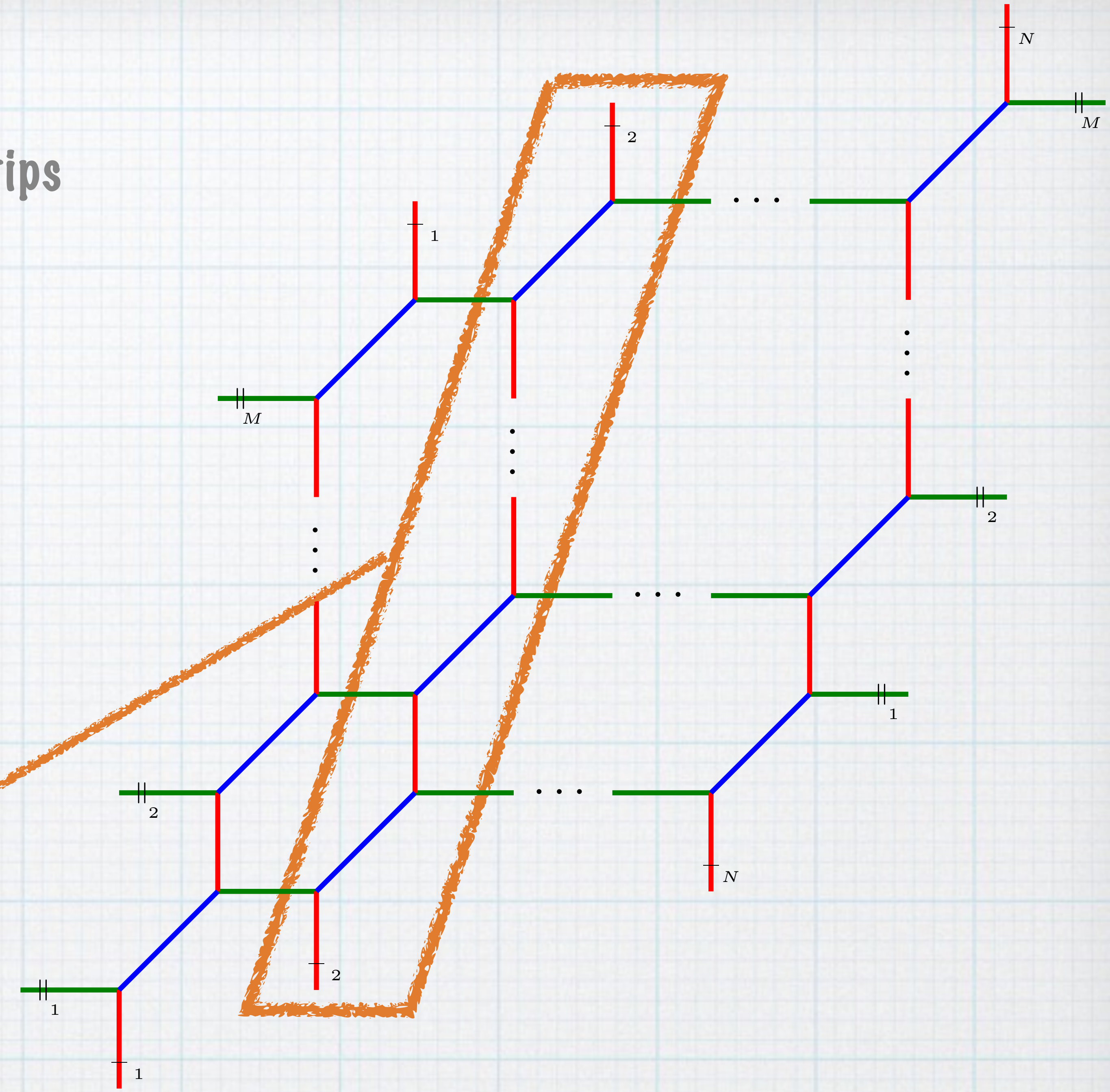
building block: $W^{\alpha_1\ldots\alpha_M}_{\beta_1\ldots\beta_M}(\{v\},\{m\})$





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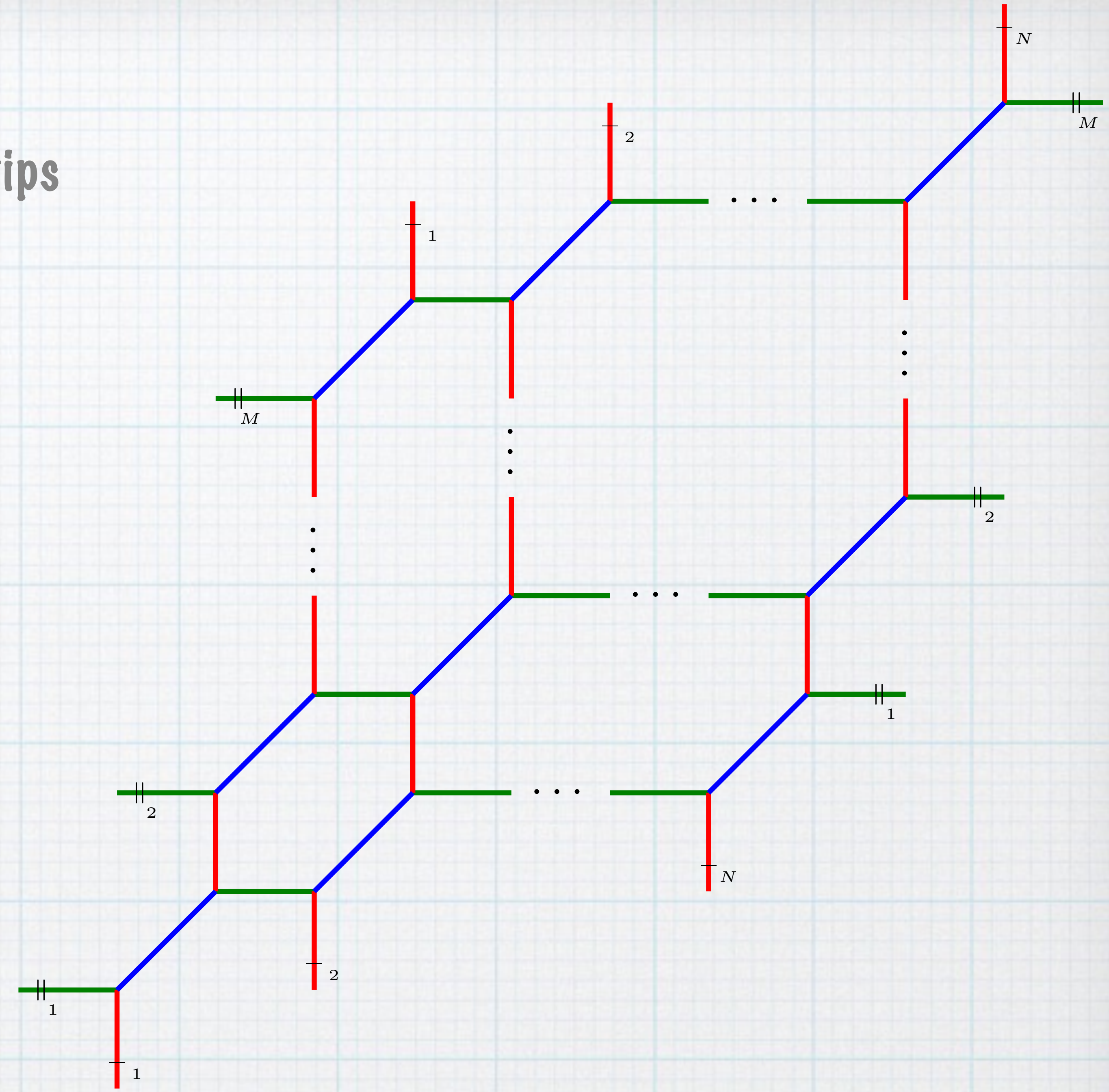

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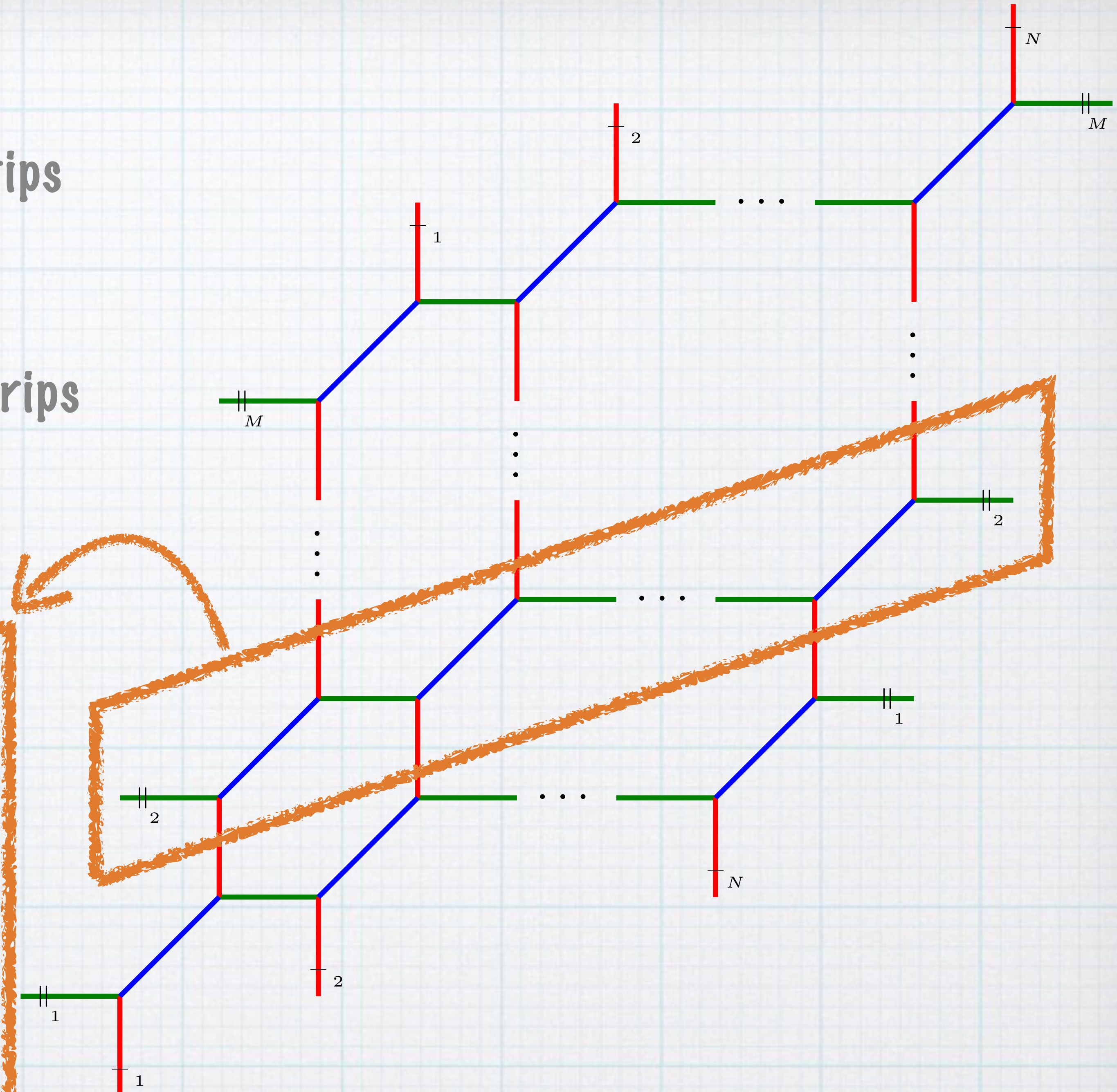

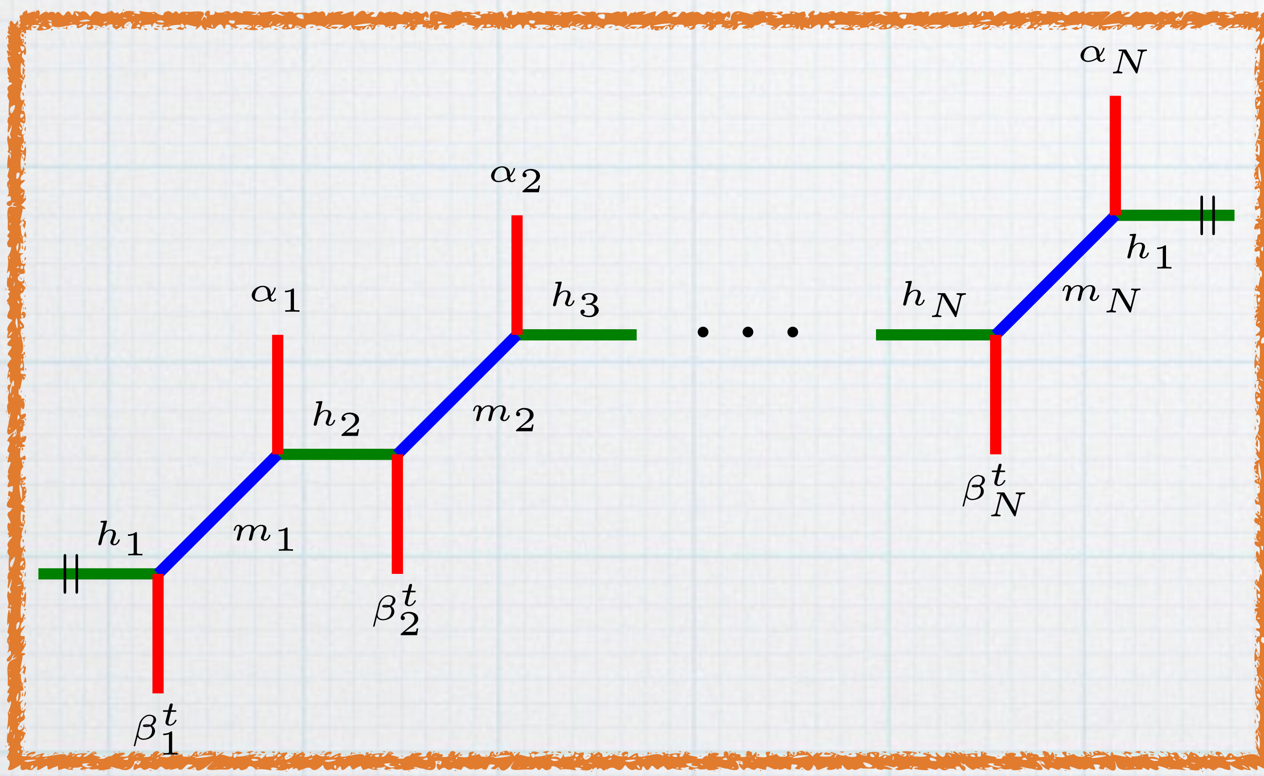

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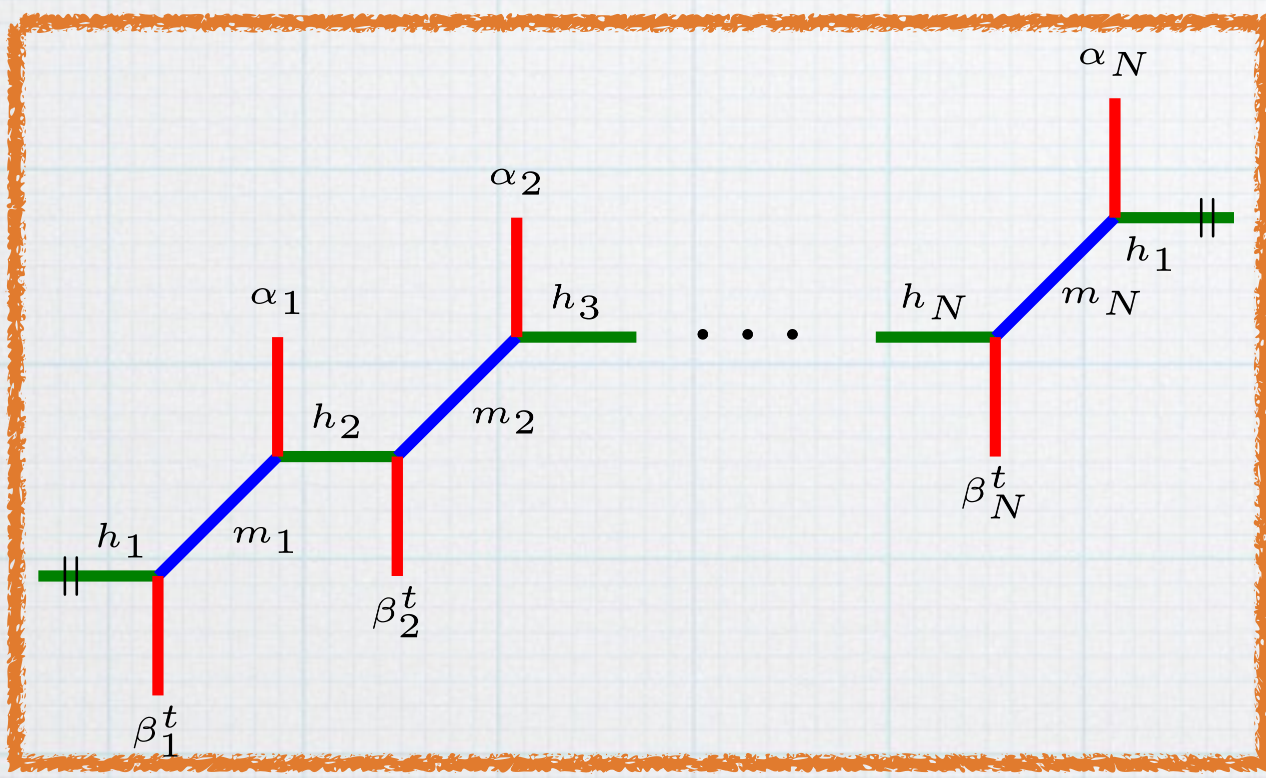

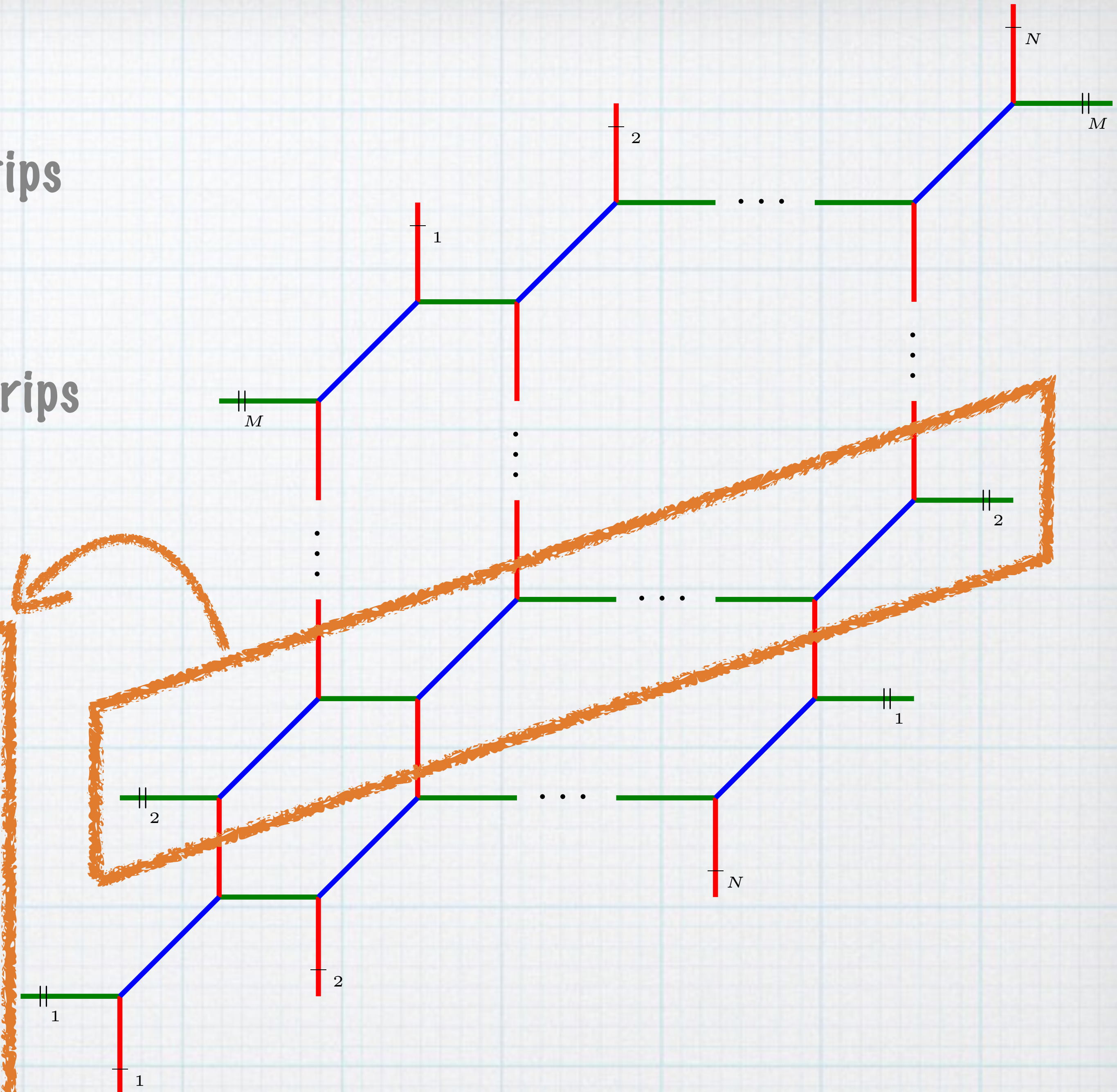

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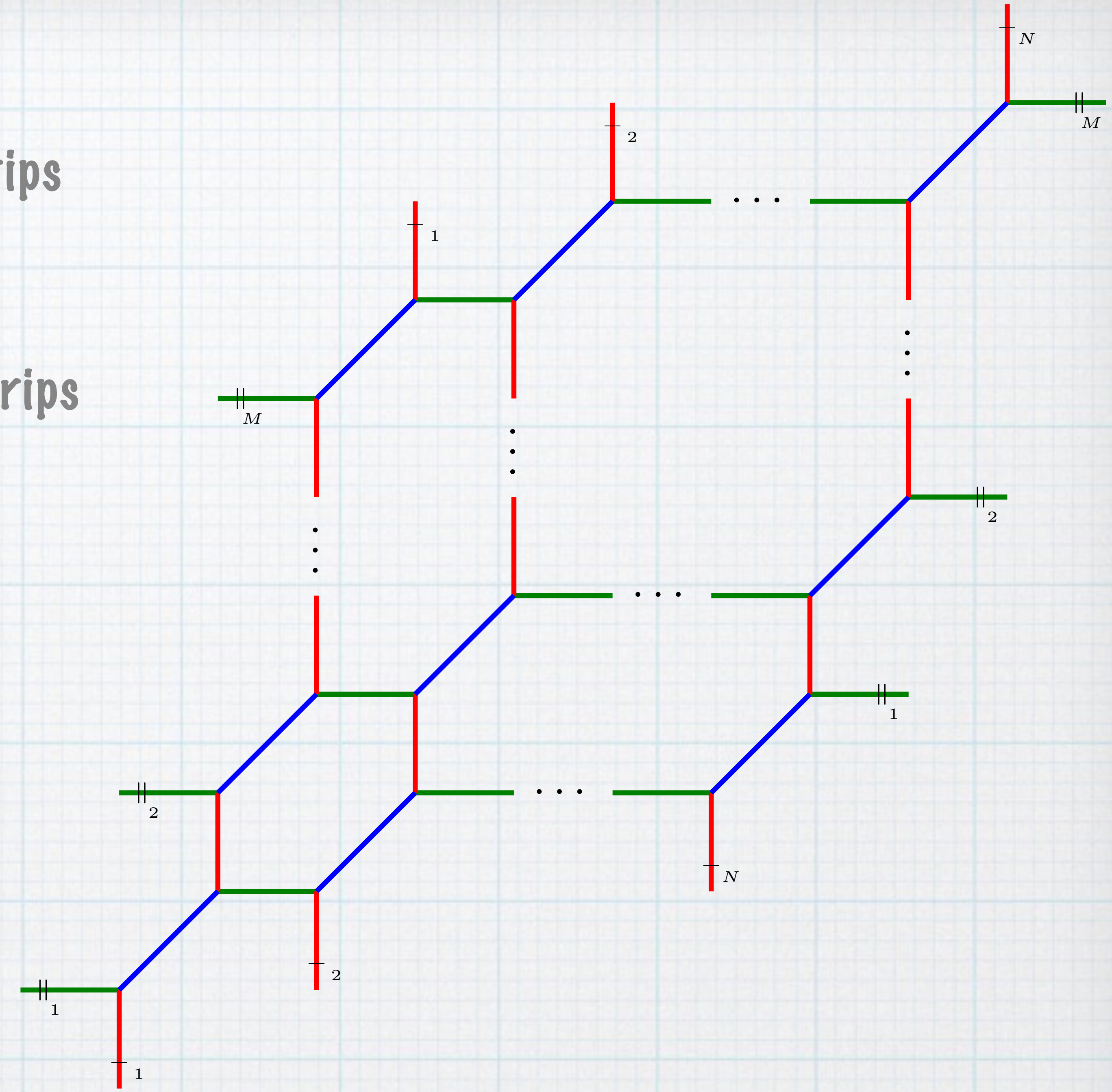

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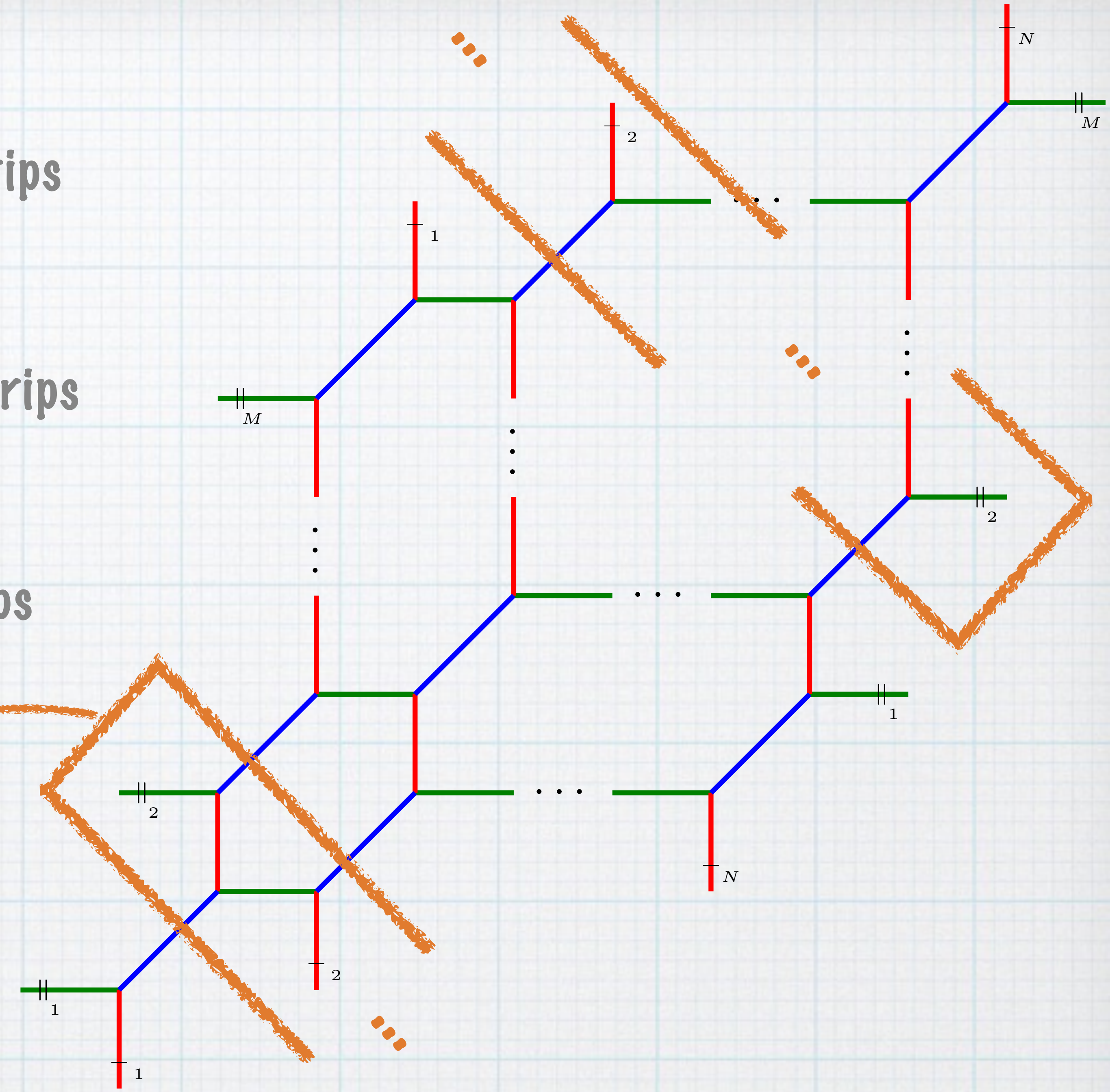
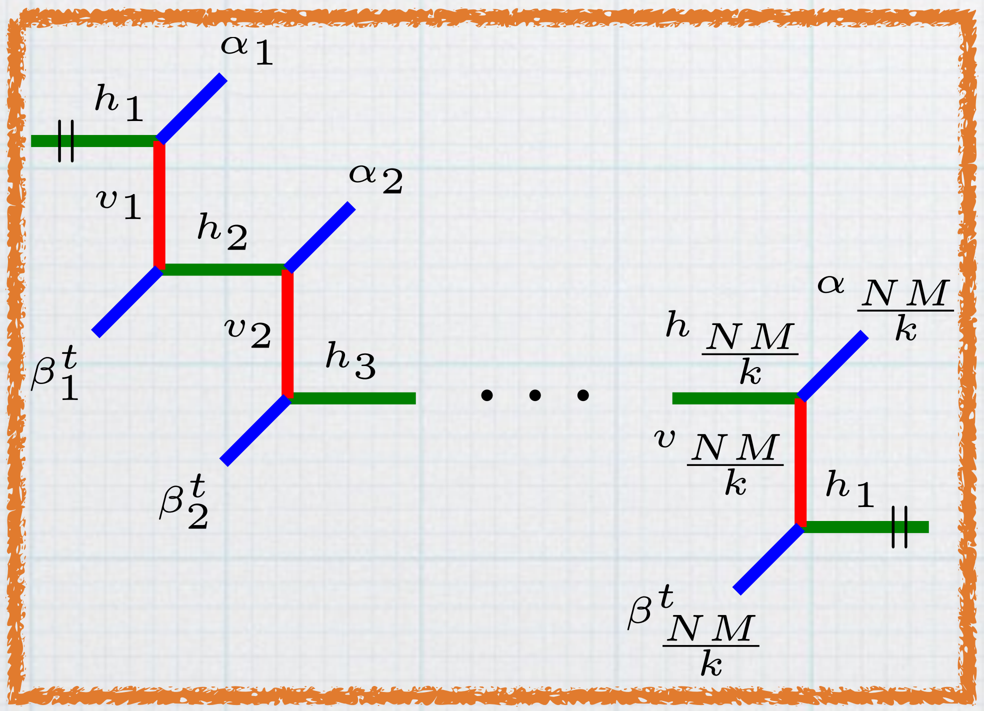
3) diagonal: decompose diagram into diagonal strips





where $k = \gcd(N, M)$

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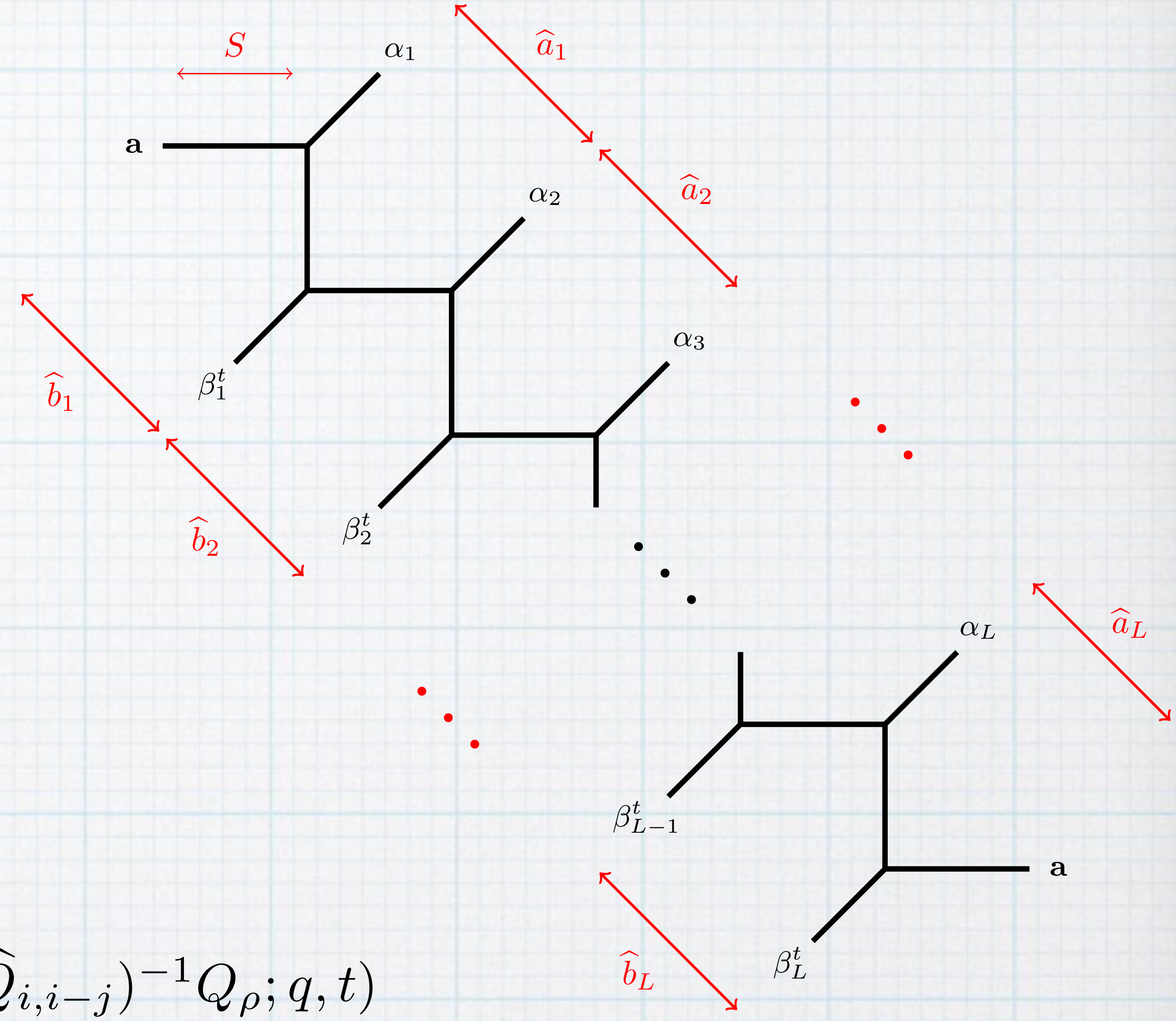

generic form of the building block

$$W^{\alpha_1\ldots\alpha_L}_{\beta_1\ldots\beta_L} = W_L(\emptyset)\cdot\hat{Z}\cdot\prod_{i,j=1}^{L}\frac{\mathcal{J}_{\alpha_i\beta_j}(\widehat{Q}_{i,i-j};q,t)\mathcal{J}_{\beta_j\alpha_i}((\widehat{Q}_{i,i-j})^{-1}Q_\rho;q,t)}{\mathcal{J}_{\alpha_i\alpha_j}(\overline{Q}_{i,i-j}\sqrt{q/t};q,t)\mathcal{J}_{\beta_j\beta_i}(\dot{Q}_{i,j-i}\sqrt{t/q};q,t)}$$

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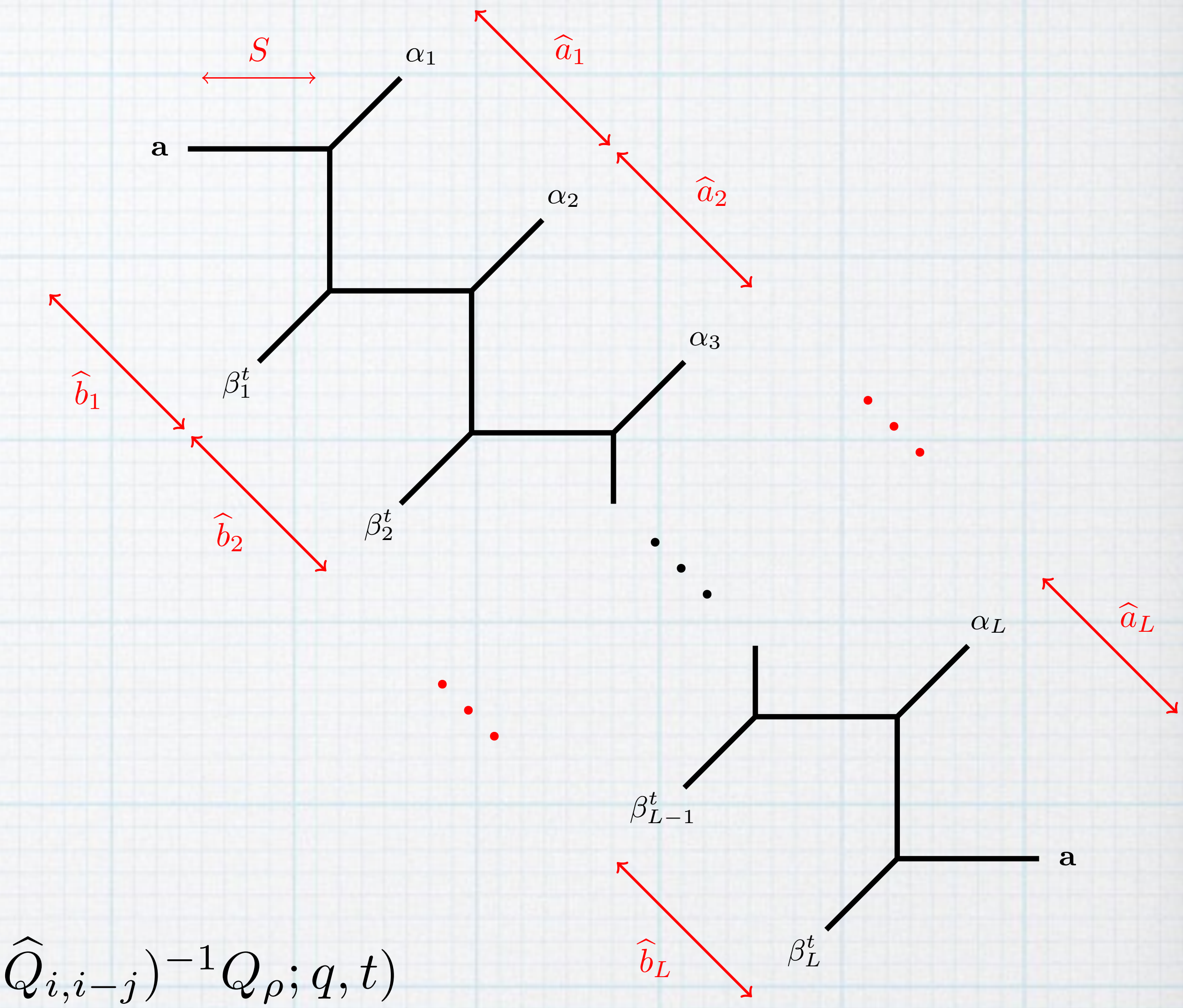

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generic form of the building block

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with

$$W_L(\emptyset)=\prod_{i,j=1}^{L}\prod_{k,r,s=1}^{\infty}\frac{(1-\widehat{Q}_{i,j}\,Q_\rho^{k-1}q^{r-\frac{1}{2}}t^{s-\frac{1}{2}})(1-\widehat{Q}_{i,j}^{-1}Q_\rho^{k}q^{s-\frac{1}{2}}t^{r-\frac{1}{2}})}{(1-\overline{Q}_{i,j}Q_\rho^{k-1}q^{r}t^{s-1})(1-\dot{Q}_{i,j}Q_\rho^{k-1}q^{s-1}t^{r})}\,,$$

$$\hat{Z}=\prod_{i=1}^{L}t^{\frac{||\alpha_k||^2}{2}}q^{\frac{||\alpha_k^t||^2}{2}}\tilde{Z}_{\alpha_k}(q,t)\tilde{Z}_{\alpha_k^t}(t,q)\,,\qquad \tilde{Z}_\nu(t,q)=\prod_{(i,j)\in\nu}\left(1-t^{\nu_j^t-i+1}q^{\nu_i-j}\right)^{-1}$$

$$\mathcal{J}_{\mu\nu}(x;t,q)=\prod_{k=1}^{\infty}J_{\mu\nu}(Q_\rho^{k-1}x;t,q)\,,$$

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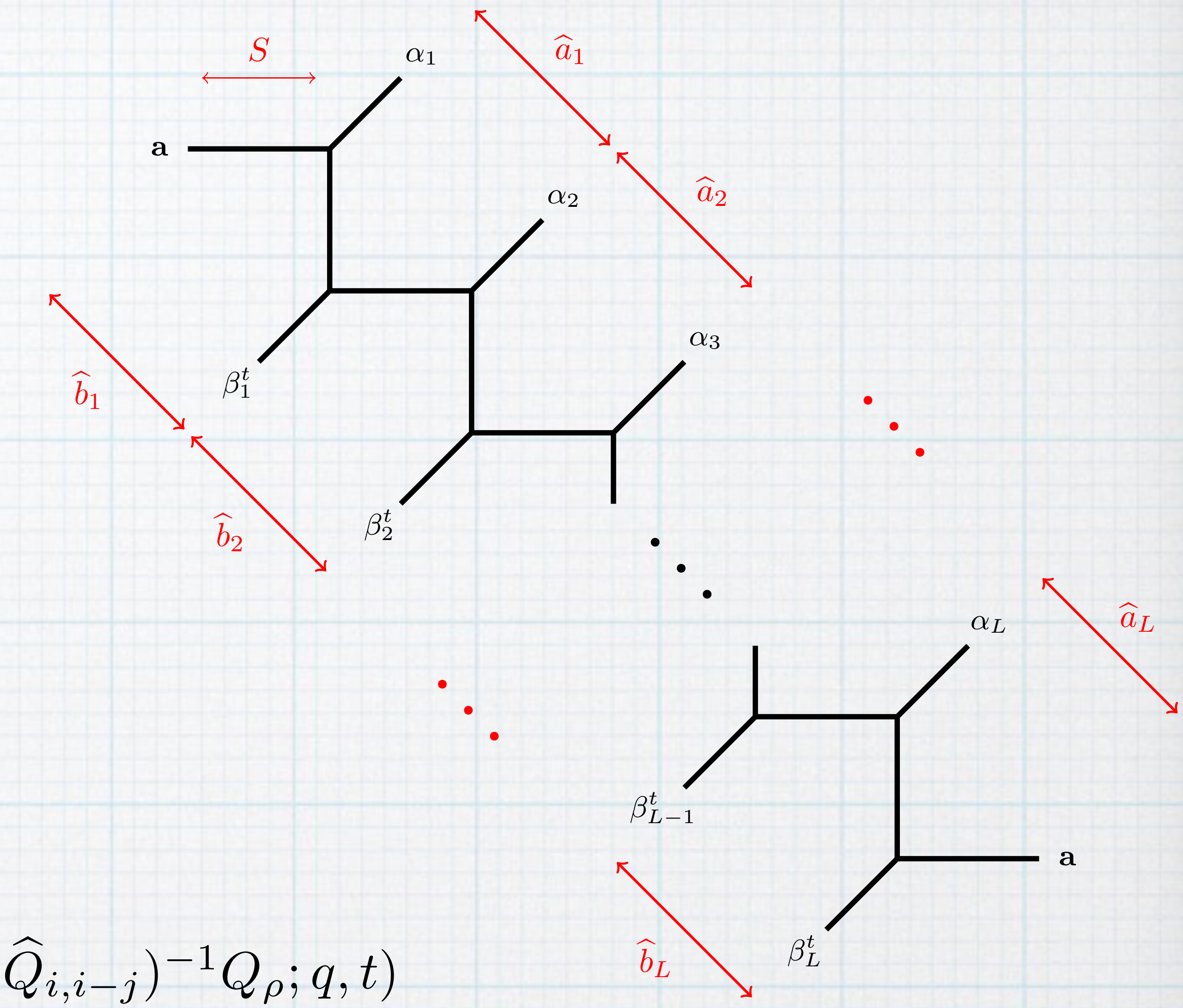
3 different choices for the preferred direction:



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Notation:

$$\widehat{Q}_{i,j}=Q_S\prod_{r=1}^{i}(Q_{a_r}Q_{b_r}^{-1})\prod_{k=1}^{j-1}Q_{a_{i-k}}\,,$$

$$\overline{Q}_{i,j}=\begin{cases}1 & \text{if } j=L\\ \prod_{k=1}^{j}Q_{a_{i-k}} & \text{if } j\neq L\end{cases}$$

$$\dot{Q}_{i,j}=\prod_{k=1}^{j}Q_{b_{i+k}}$$

and $Q_S=e^{-S}$

$Q_{a_i}=e^{-\widehat{a}_i}$

$Q_{b_i}=e^{-\widehat{b}_i}$

3 different choices for the preferred direction:

1) horizontal: decompose diagram into vertical strips

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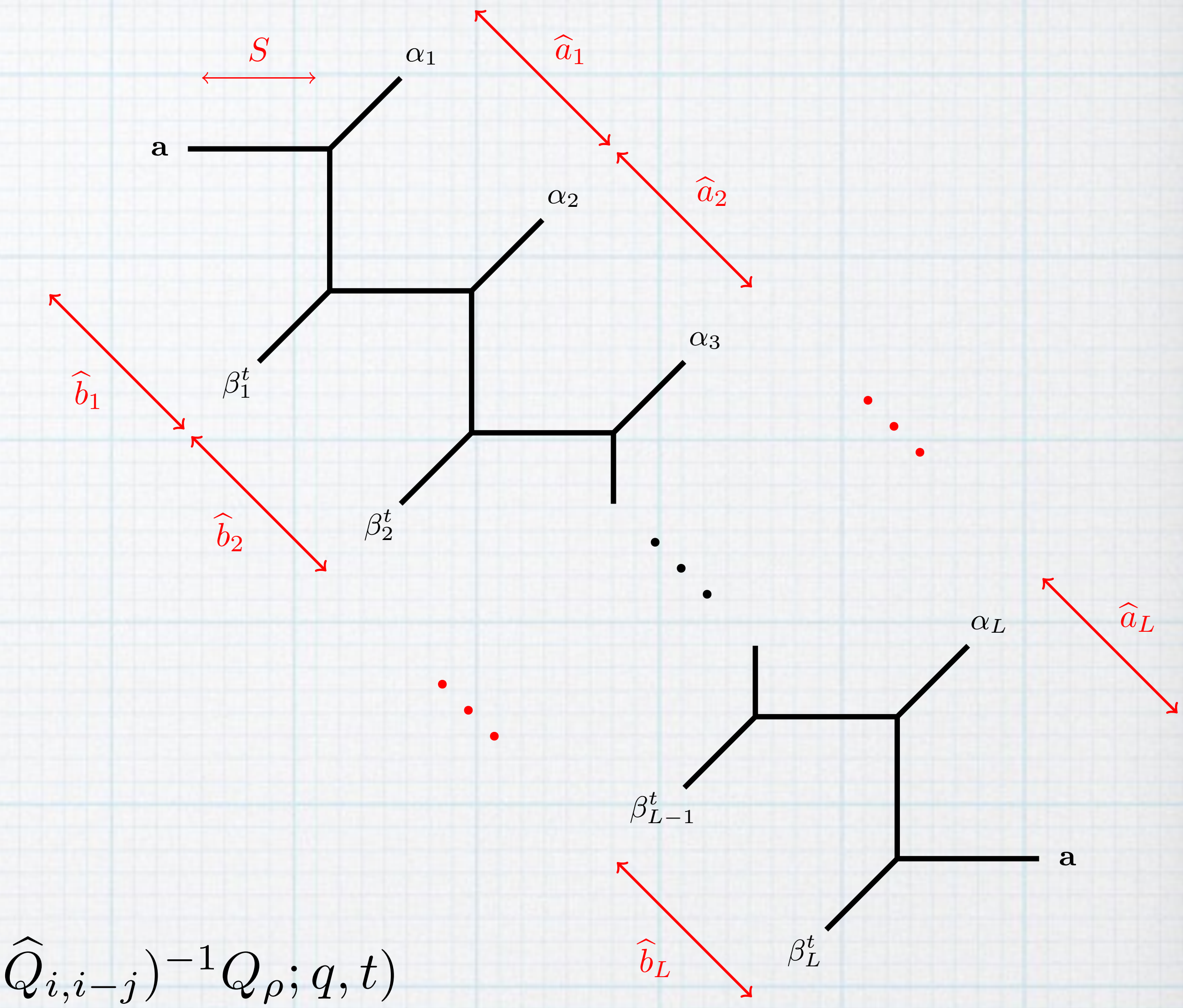
3) diagonal: decompose diagram into diagonal strips

building block: $W^{\alpha_1\ldots\alpha_{\frac{NM}{k}}}_{\beta_1\ldots\beta_{\frac{NM}{k}}}(\{h\},\{v\})$



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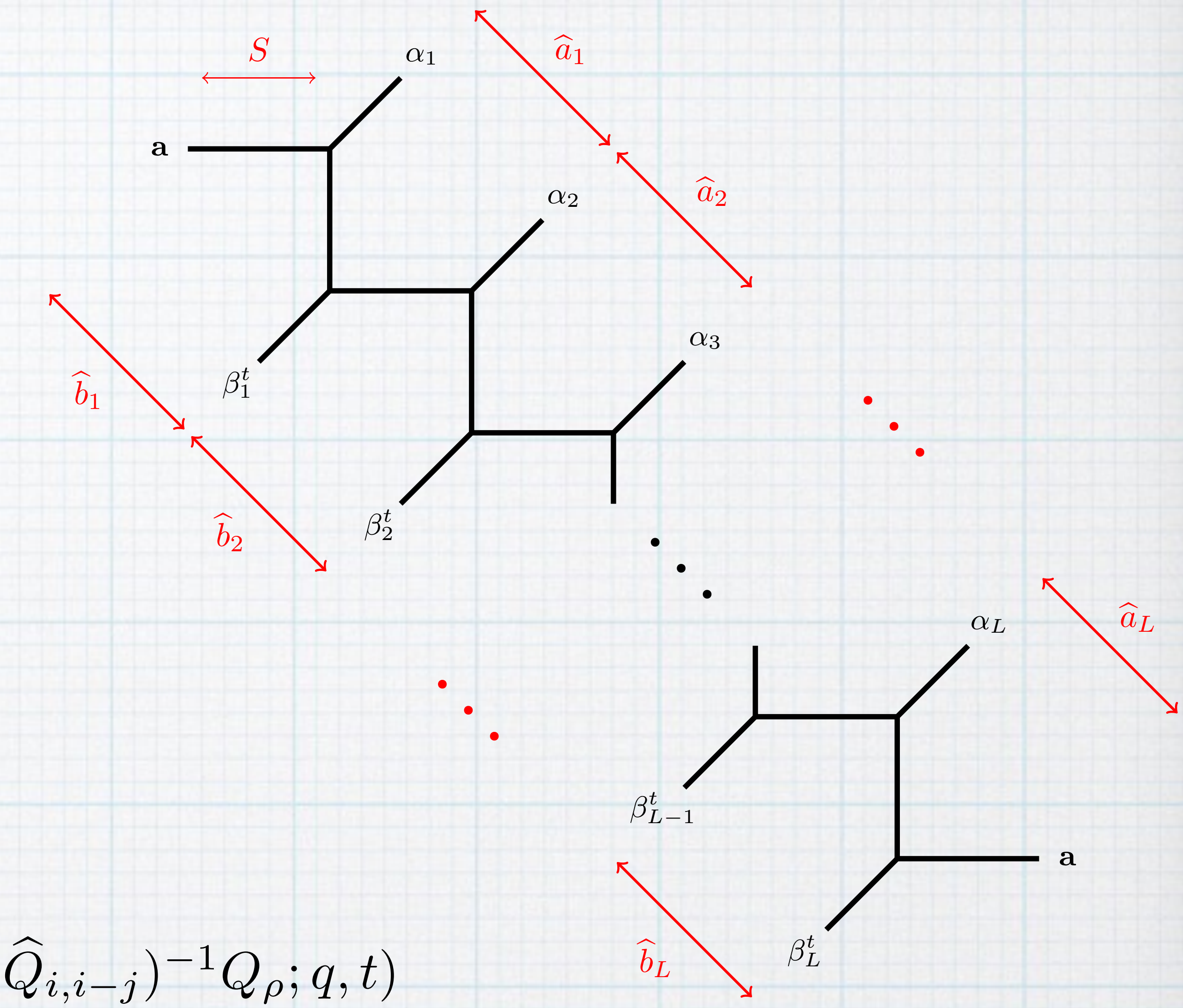

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suitable for **all three** expansions upon identifying:

	horizontal	vertical	diagonal
$\widehat{a}_i$	$v_{i+1}+m_i$	h_i+m_i	v_i+h_{i+1}
$\widehat{b}_i$	v_i+m_i	h_i+m_{i-1}	h_i+v_i
S	v_1	m_N	h_1
L	M	N	$\frac{NM}{k}$

Newton Polygons

Alternative view on the three gauge theories: Newton polygons as dual of web diagrams

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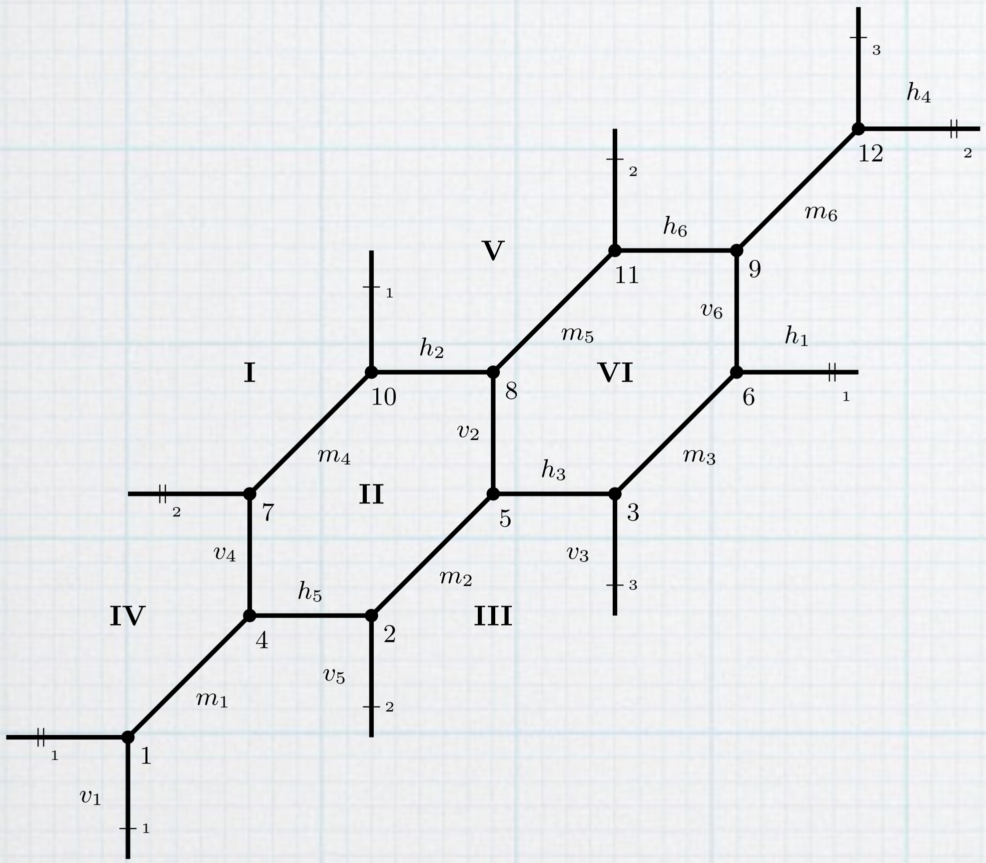
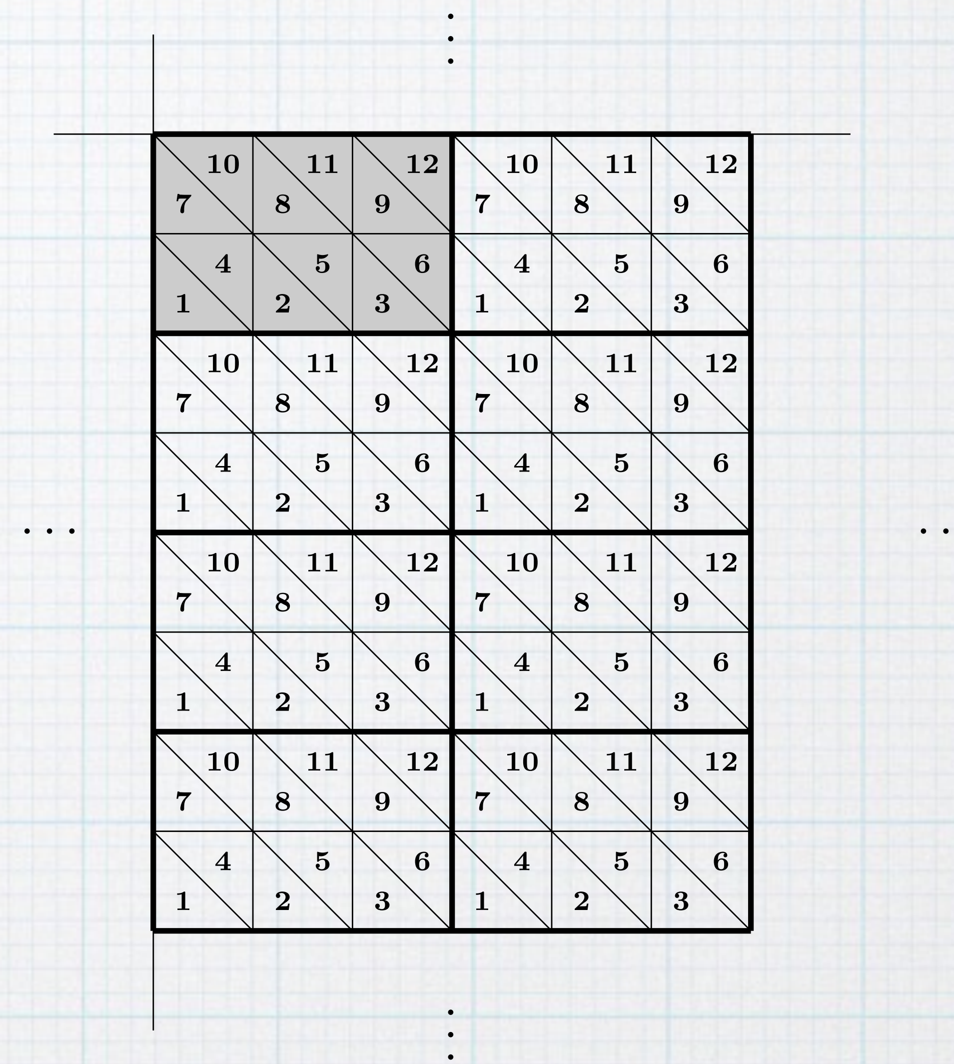
Alternative view on the three gauge theories: Newton polygons as dual of web diagrams

Example: $(N, M) = (3, 2)$



dual of web diagram



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Alternative view on the three gauge theories: Newton polygons as dual of web diagrams

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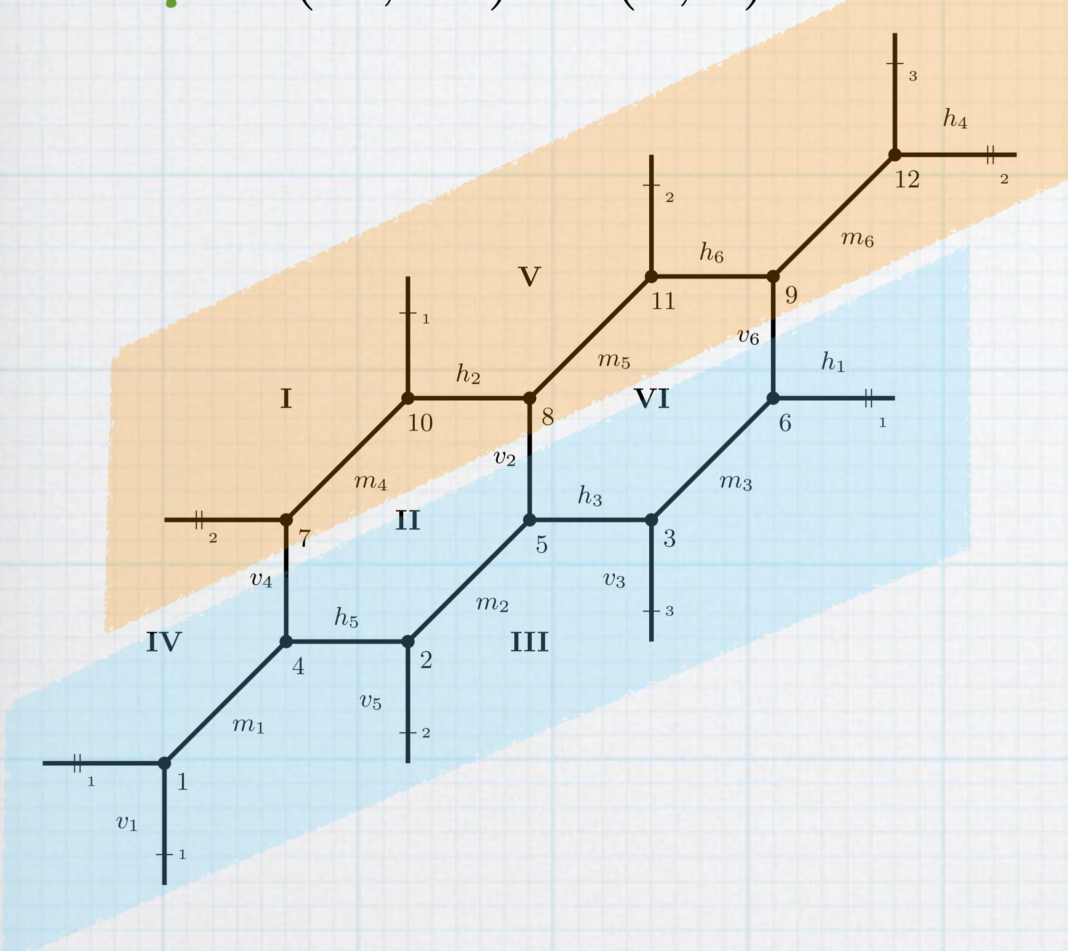

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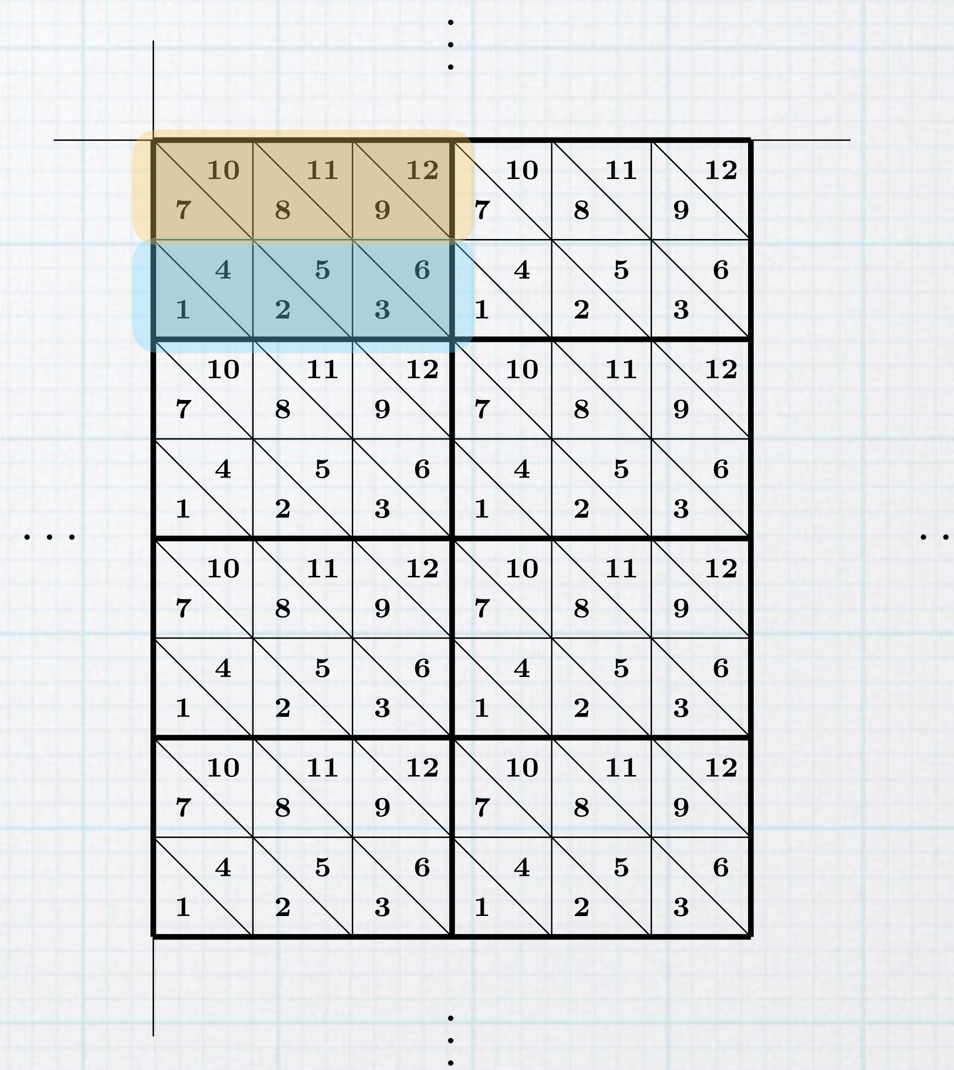

-) decomposition into two **horizontal** strips $W^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}$

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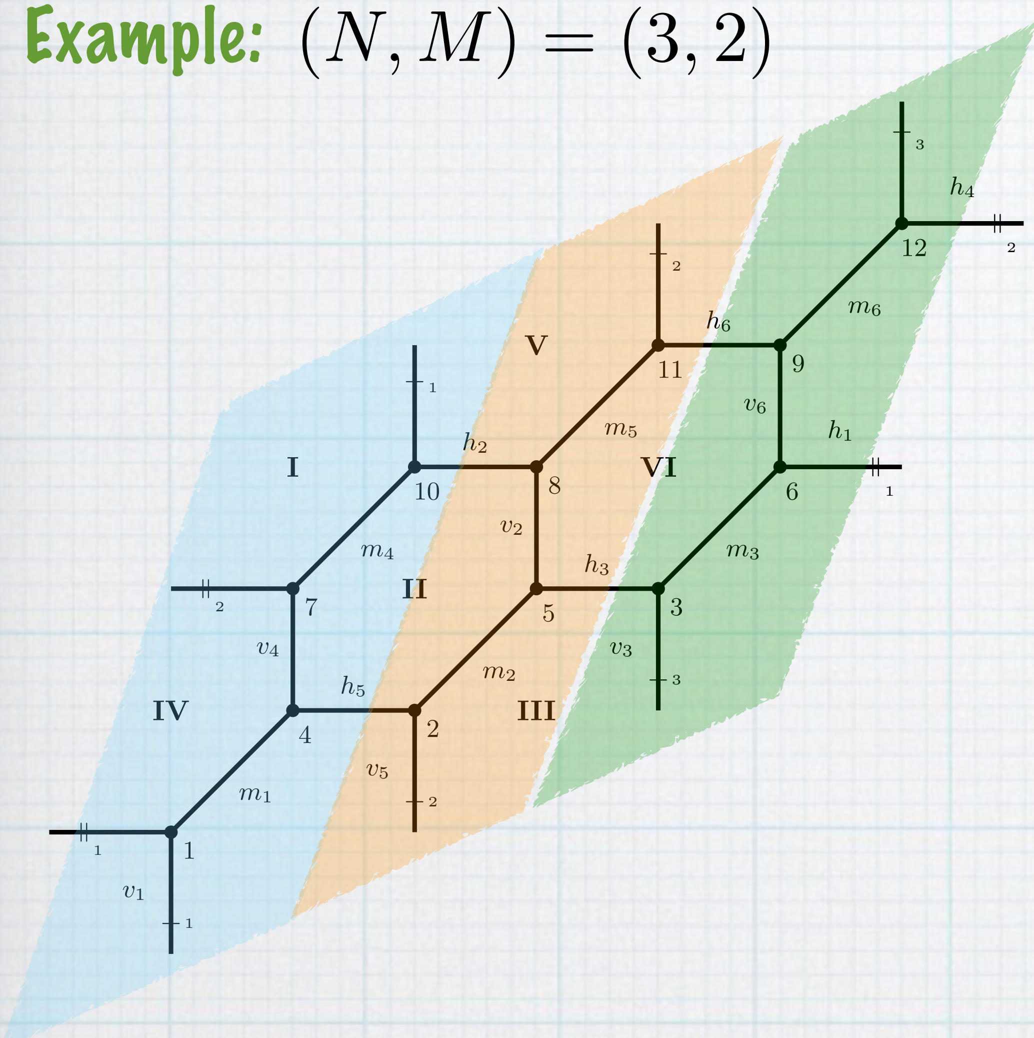
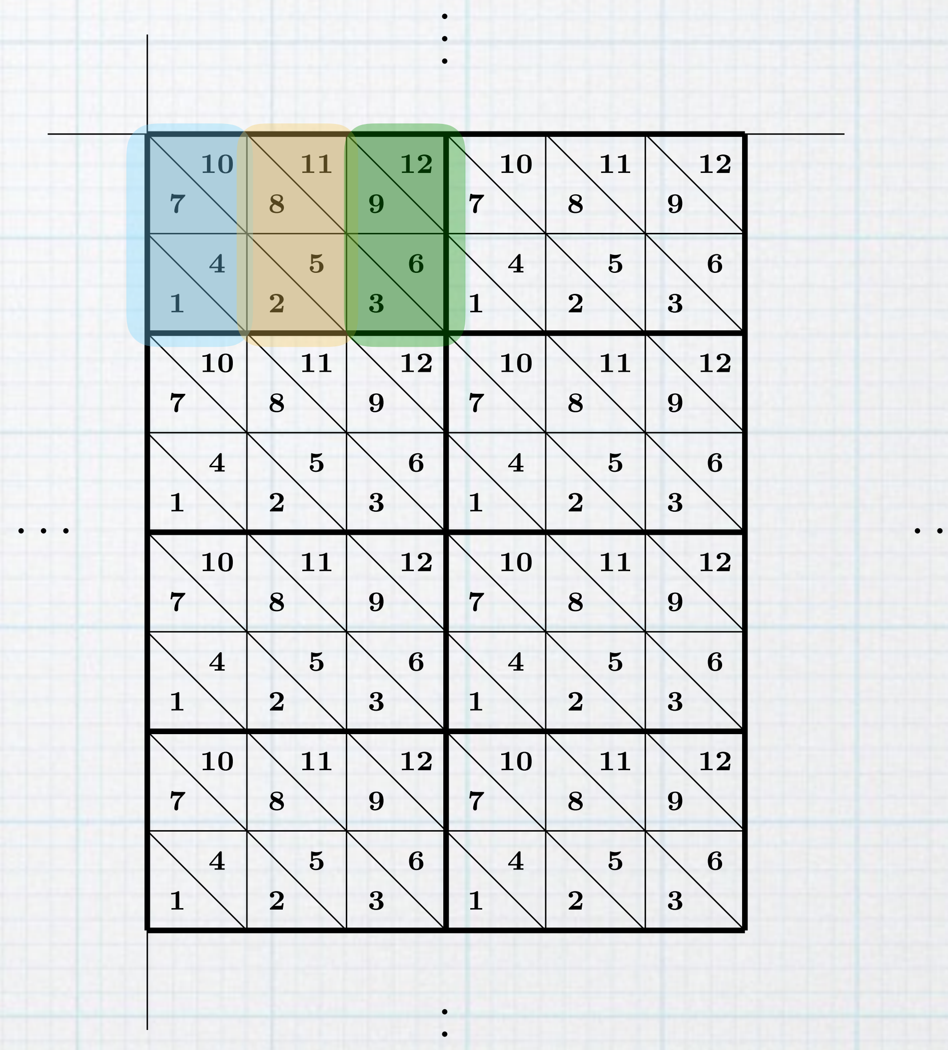


dual of web diagram



-) decomposition into two **horizontal** strips $W^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}$

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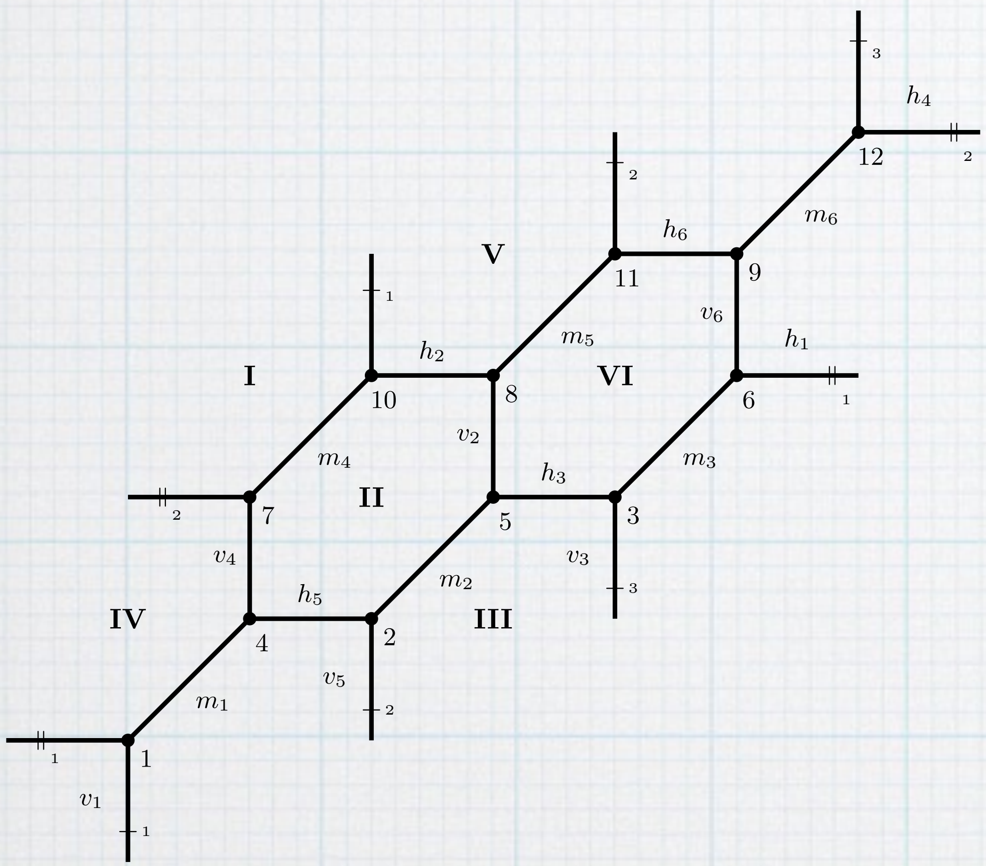

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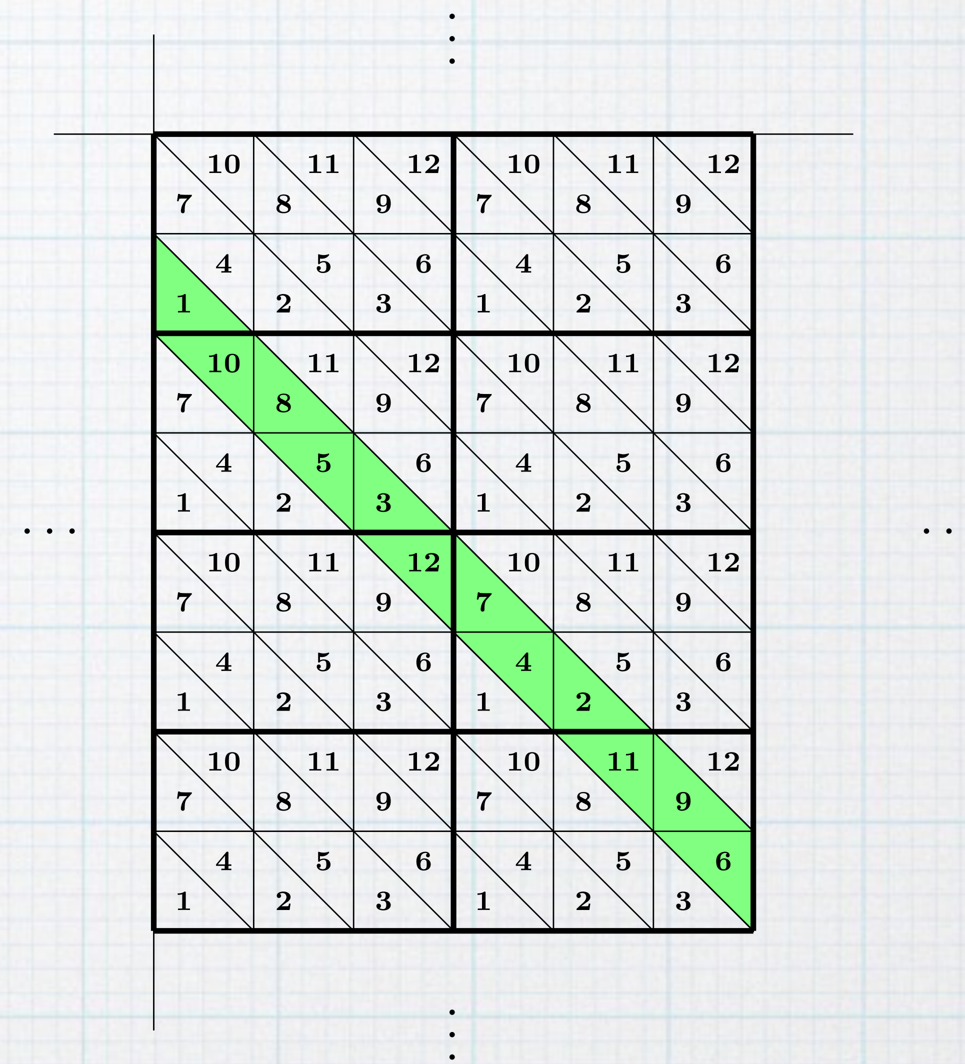


- decomposition into two **horizontal** strips $W^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}$

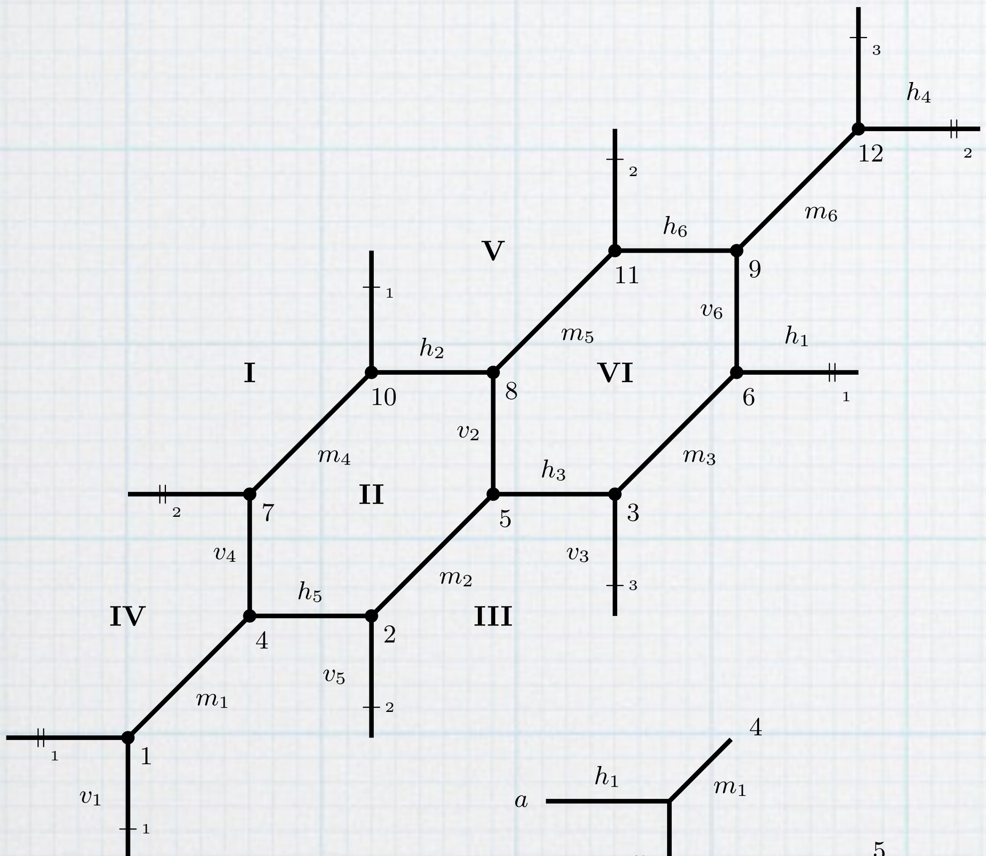
- decomposition into three **vertical** strips $W^{\alpha_1\alpha_2}_{\beta_1\beta_2}$

- for **diagonal** decomposition: choose different fundamental domain
  single strip $W^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}_{\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6}$

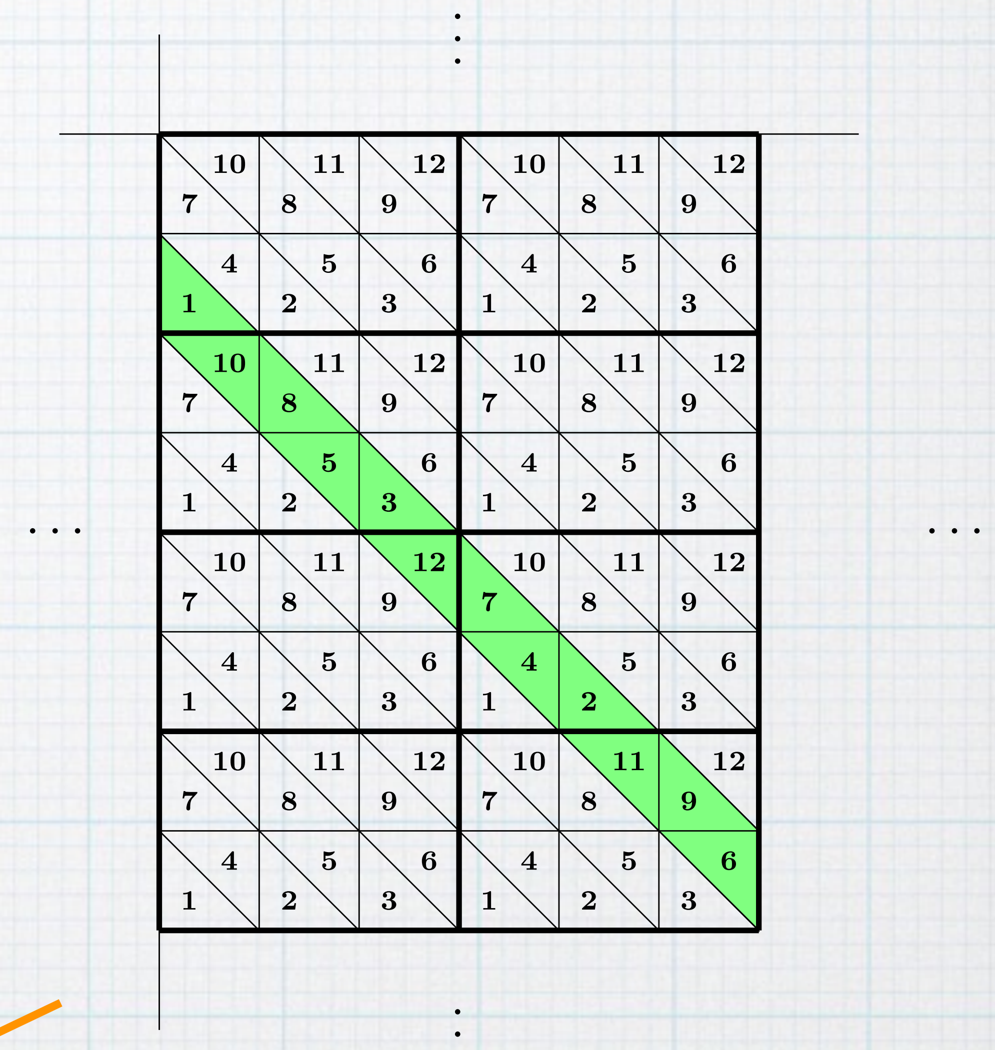
Newton Polygons

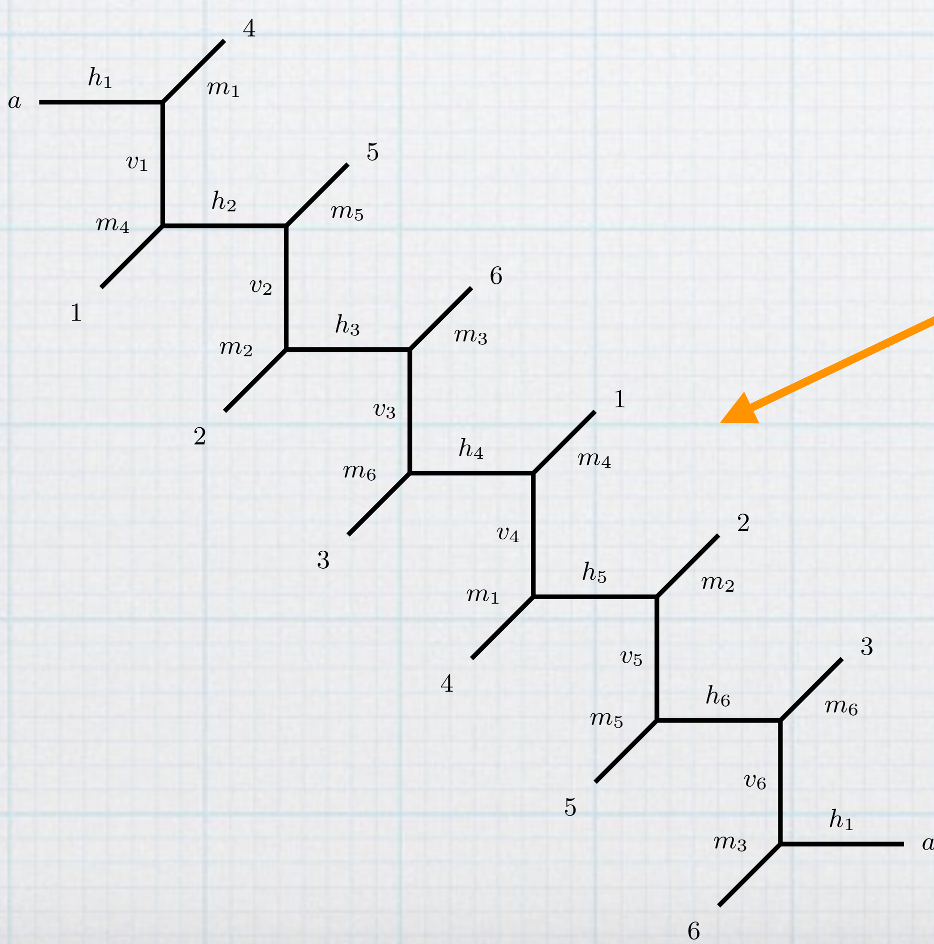
Alternative view on the three gauge theories: Newton polygons as dual of web diagrams

Example: $(N, M) = (3, 2)$



dual of web diagram





presentation of the web diagram associated with alternative fundamental domain

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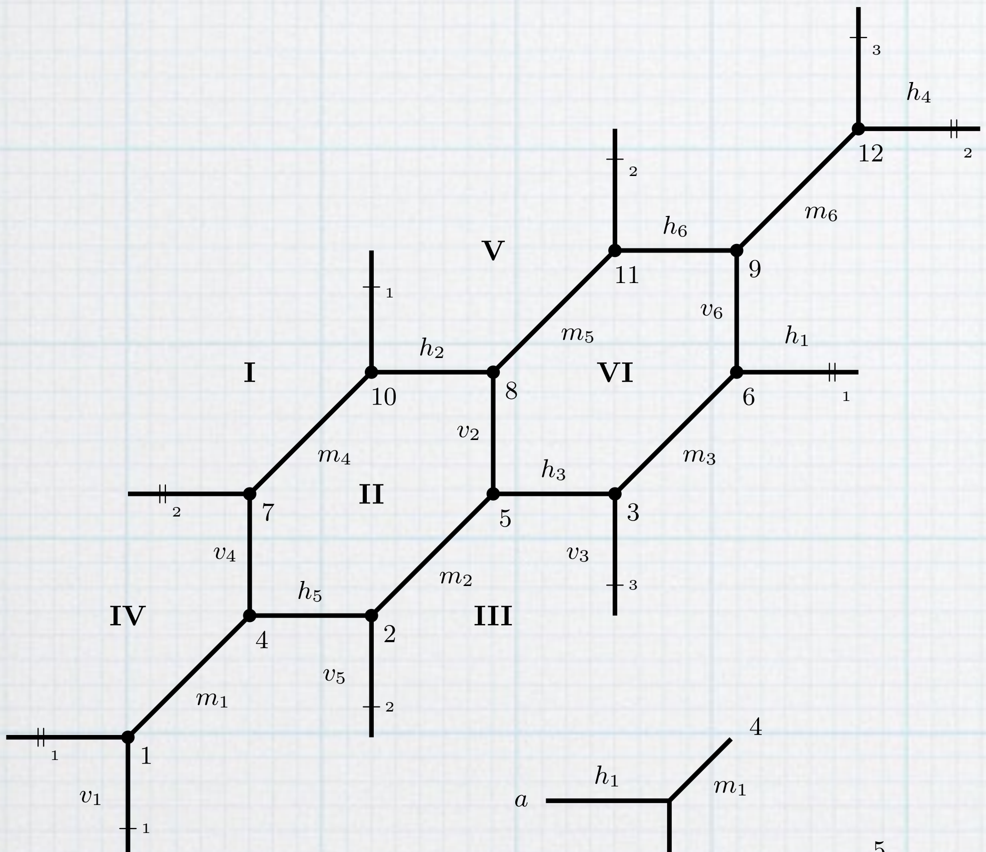

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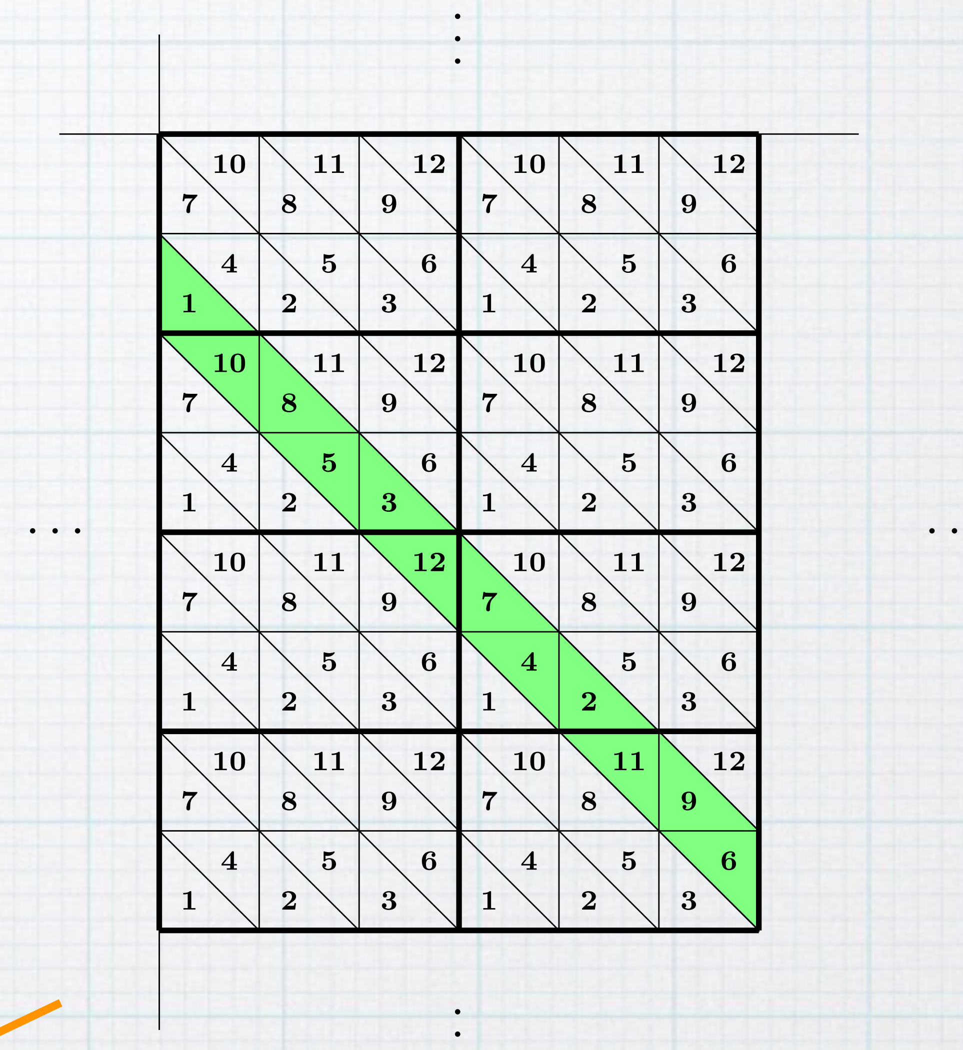

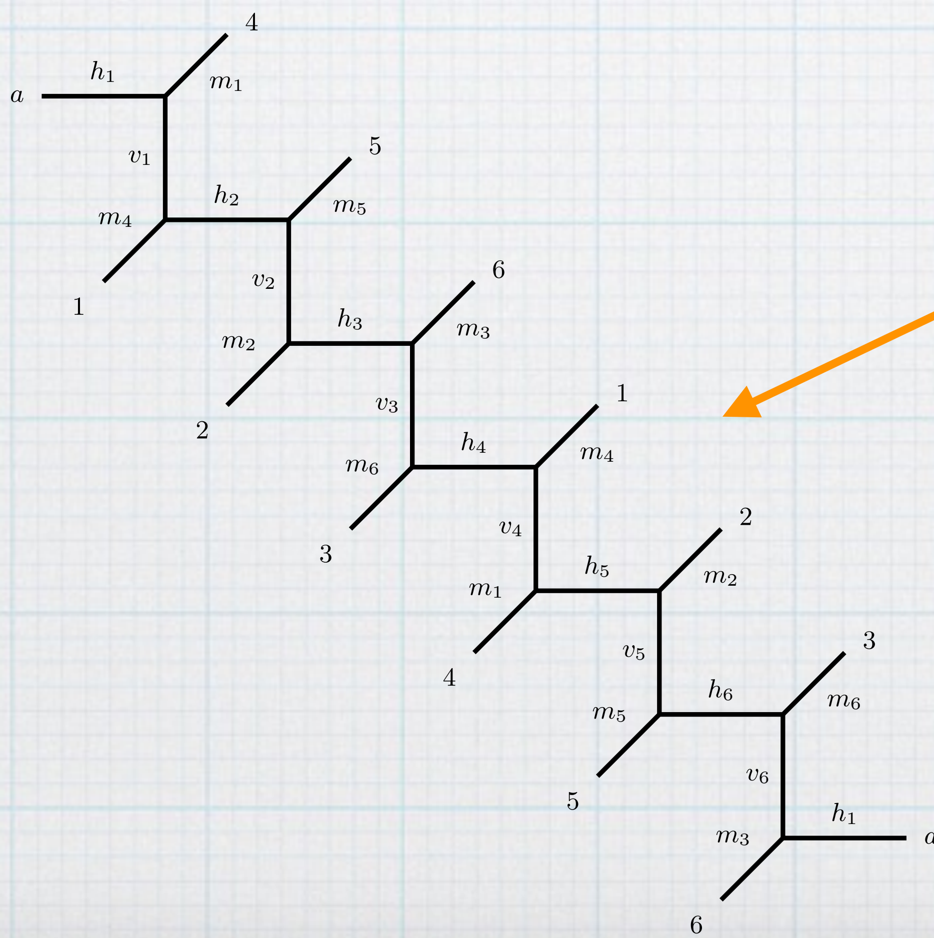

presentation of the web diagram associated with alternative fundamental domain

-) all fundamental domains equivalent

-) lead to same partition function

Topological Partition Function

The full partition function is obtained by gluing together the building blocks $W^{\alpha_1\ldots\alpha_M}_{\beta_1\ldots\beta_M}$

$$\mathcal{Z}_{N,M} = \sum_{\alpha} \Big(\prod_{i=1,j=1}^{M,N} e^{-u_{ij}\,|\alpha^i_j|} \Big) \prod_{j=1}^{N} W^{\alpha^1_j \cdots \alpha^M_j}_{\alpha^1_{j+1} \cdots \alpha^M_{j+1}}$$

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parameters used to glue the strips together

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parameters used to glue the strips together

Different choices of preferred direction afford different (but equivalent) expansions:

$$\begin{aligned}
\mathcal{Z}_{N,M}(\{h\},\{v\},\{m\},\epsilon_{1,2}) &= Z_p(\{v\},\{m\}) \sum_{\vec{k}} e^{-\vec{k}\cdot\mathbf{h}}\, Z_{\vec{k}}(\{v\},\{m\}) = Z^{(N,M)}_{\text{hor}} \\
&= Z_p(\{h\},\{m\}) \sum_{\vec{k}} e^{-\vec{k}\cdot\mathbf{v}}\, Z_{\vec{k}}(\{h\},\{m\}) = Z^{(N,M)}_{\text{vert}} \\
&= Z_p(\{h\},\{v\}) \sum_{\vec{k}} e^{-\vec{k}\cdot\mathbf{m}}\, Z_{\vec{k}}(\{h\},\{v\}) = Z^{(N,M)}_{\text{diag}}
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common normalisation factor (perturbative partition function)

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Compare different series expansions with instanton partition functions of quiver gauge theories.

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Need to choose **independent Kähler** parameters of $X_{N,M}$

Bases of independent Kähler parameters

For each of the expansion we can choose a suitable set of $NM+2$ independent Kähler parameters:

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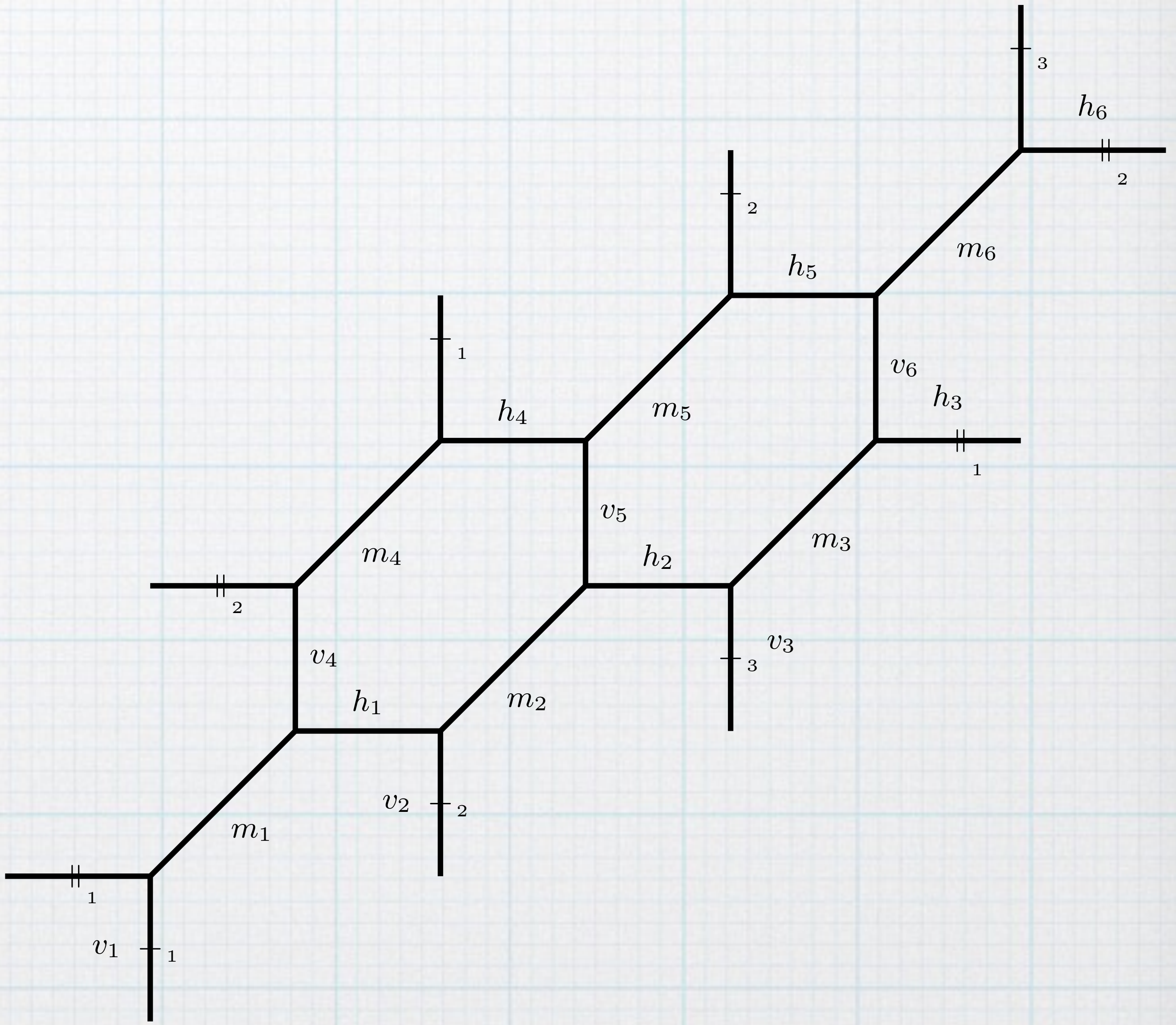

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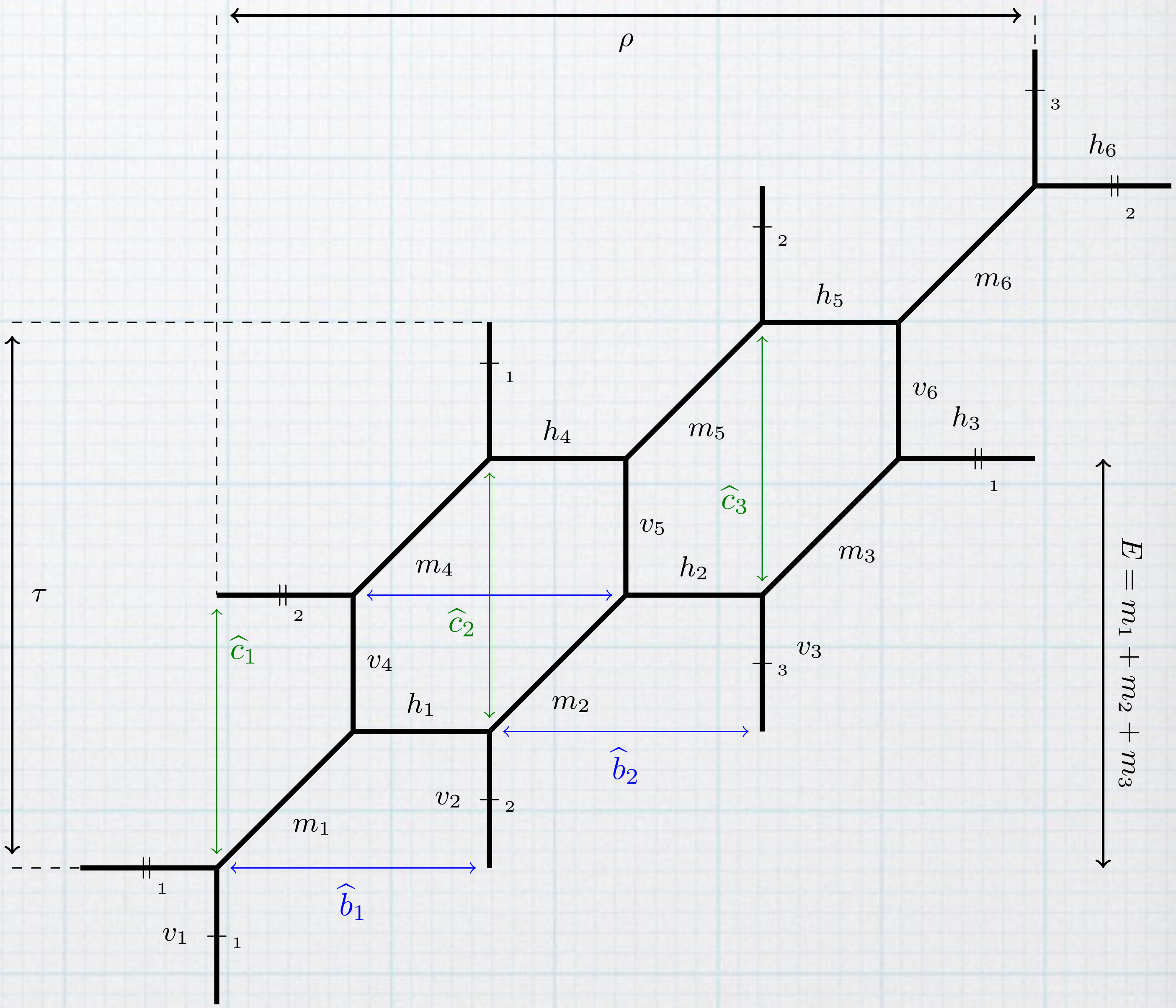

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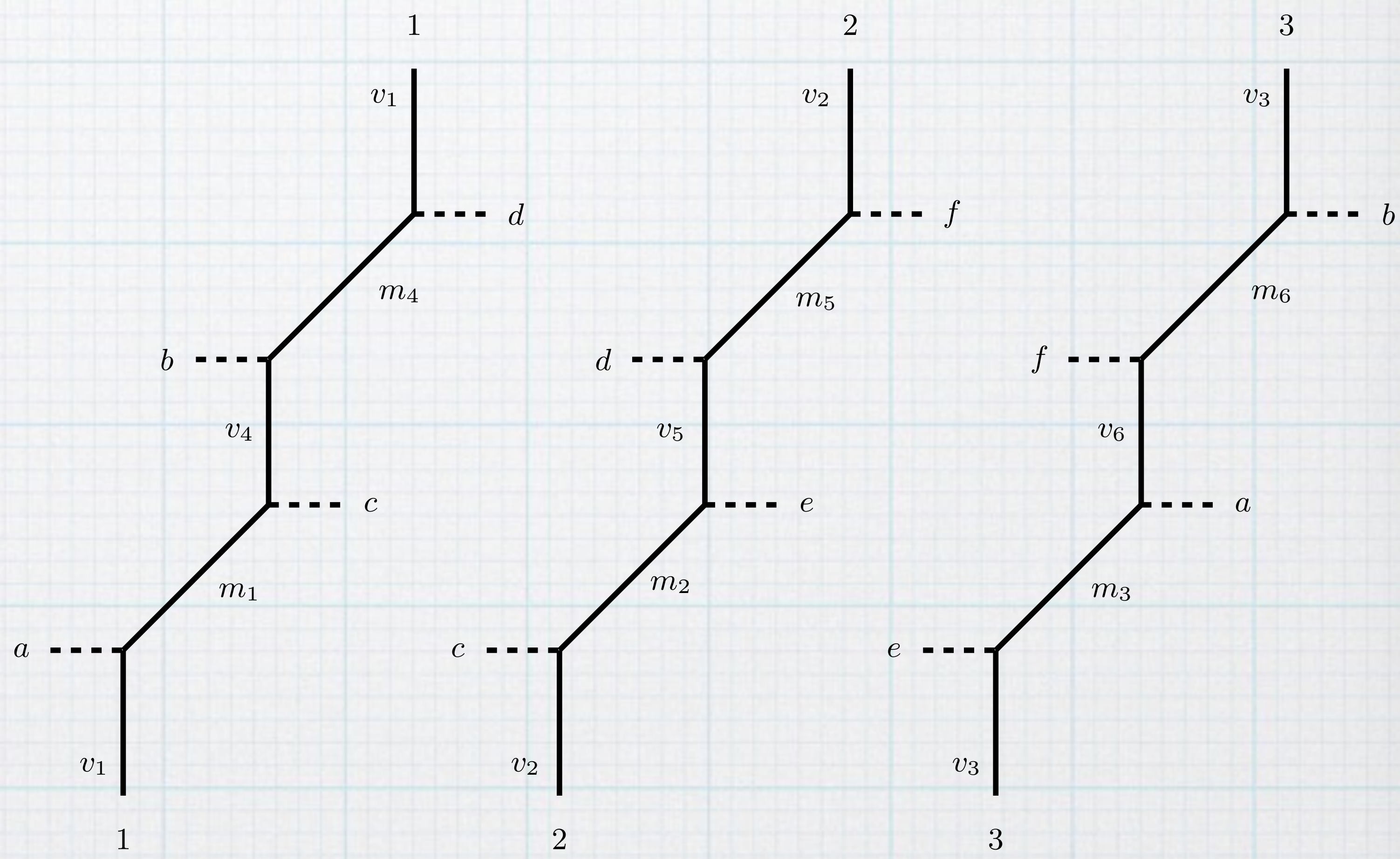

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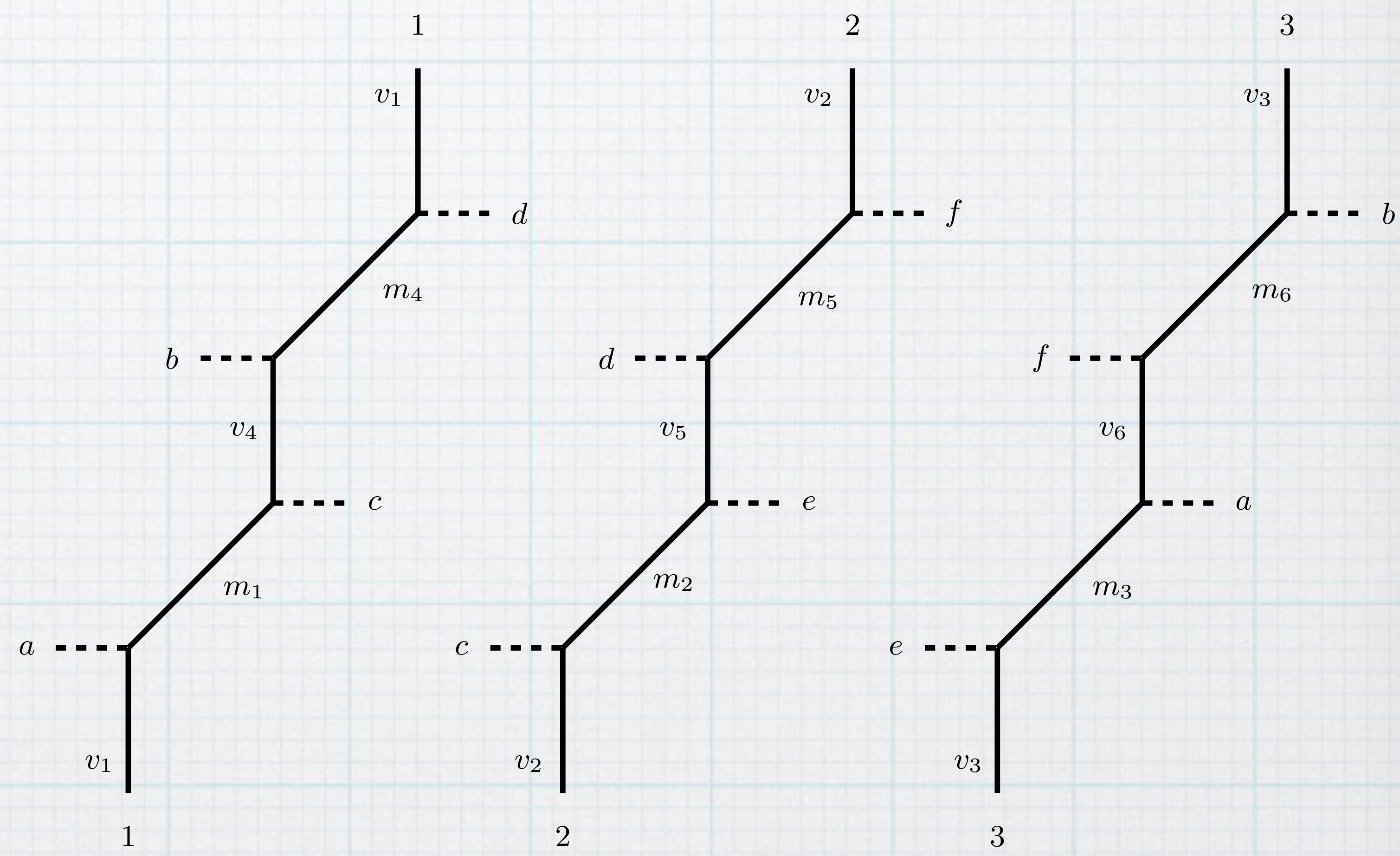
gauge theory: $U(2)\times U(2)\times U(2)$



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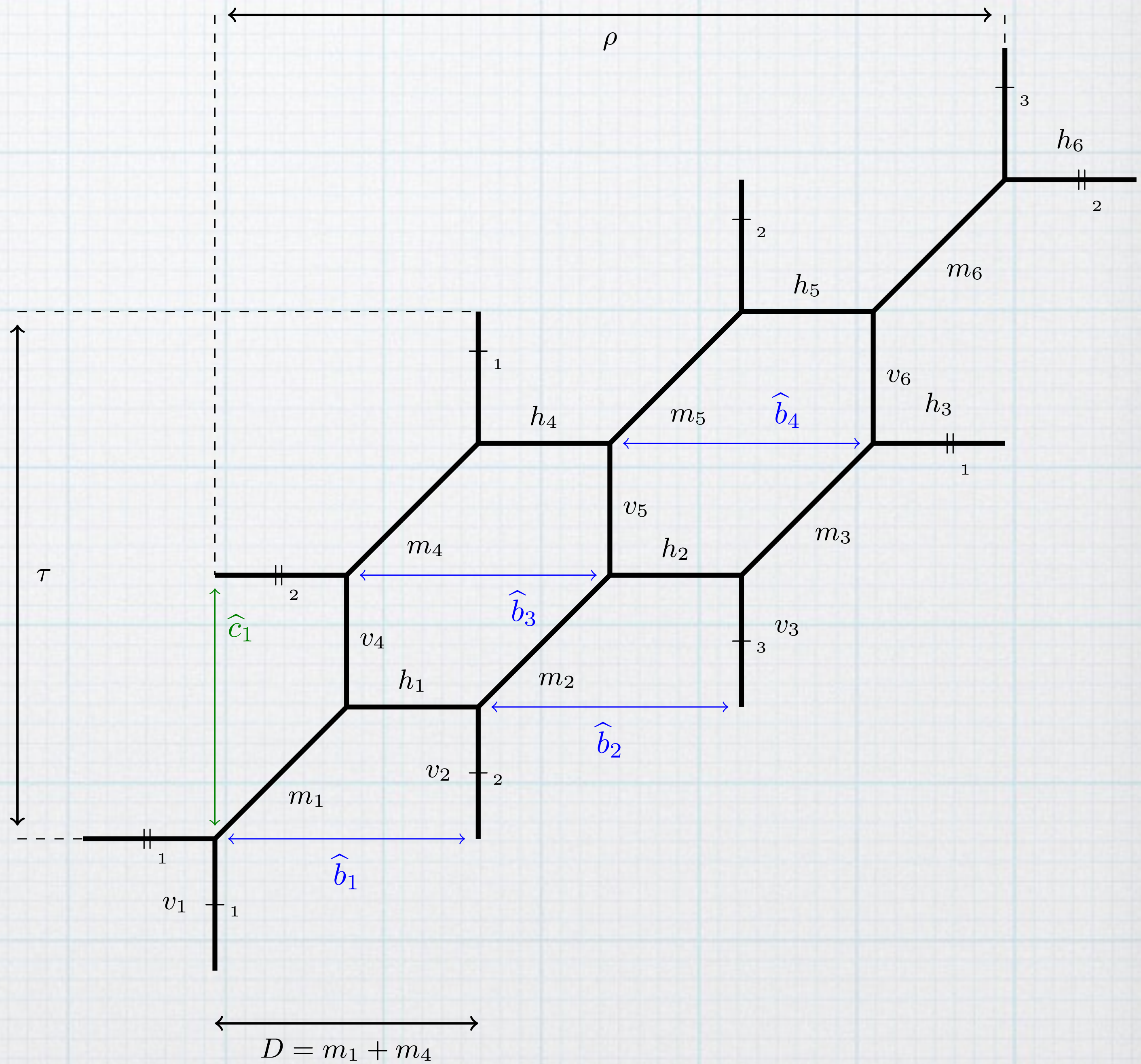

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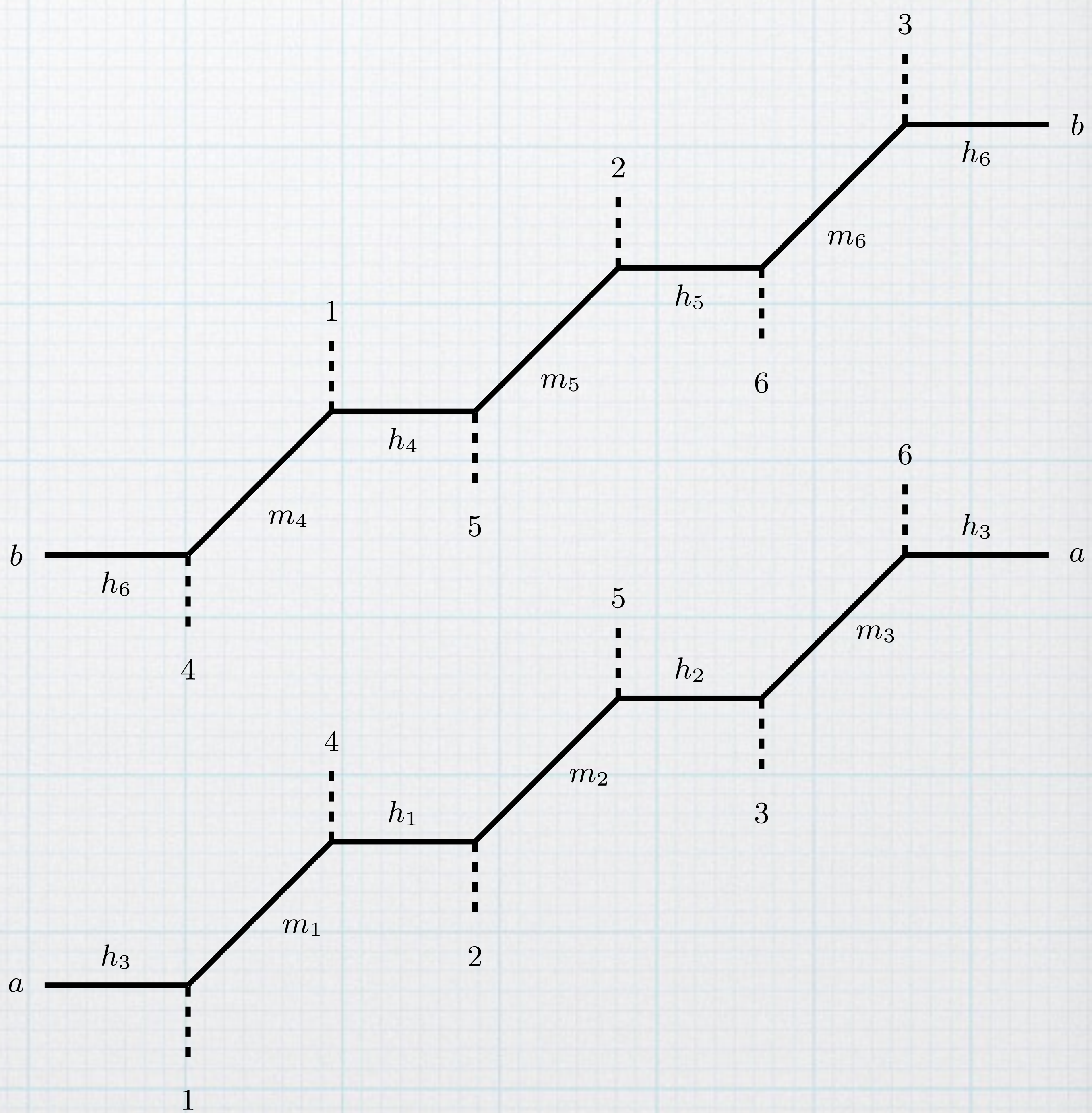

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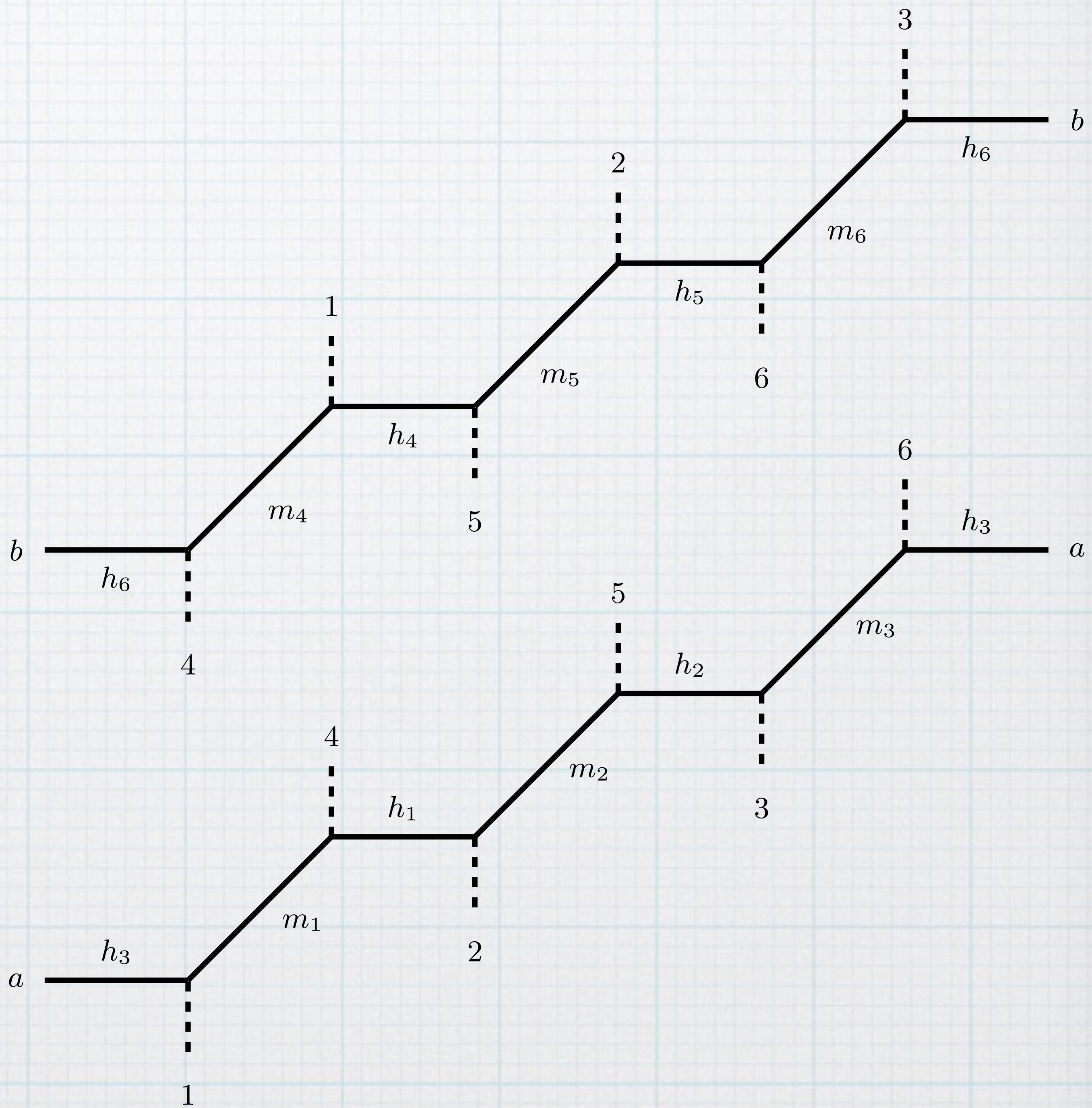

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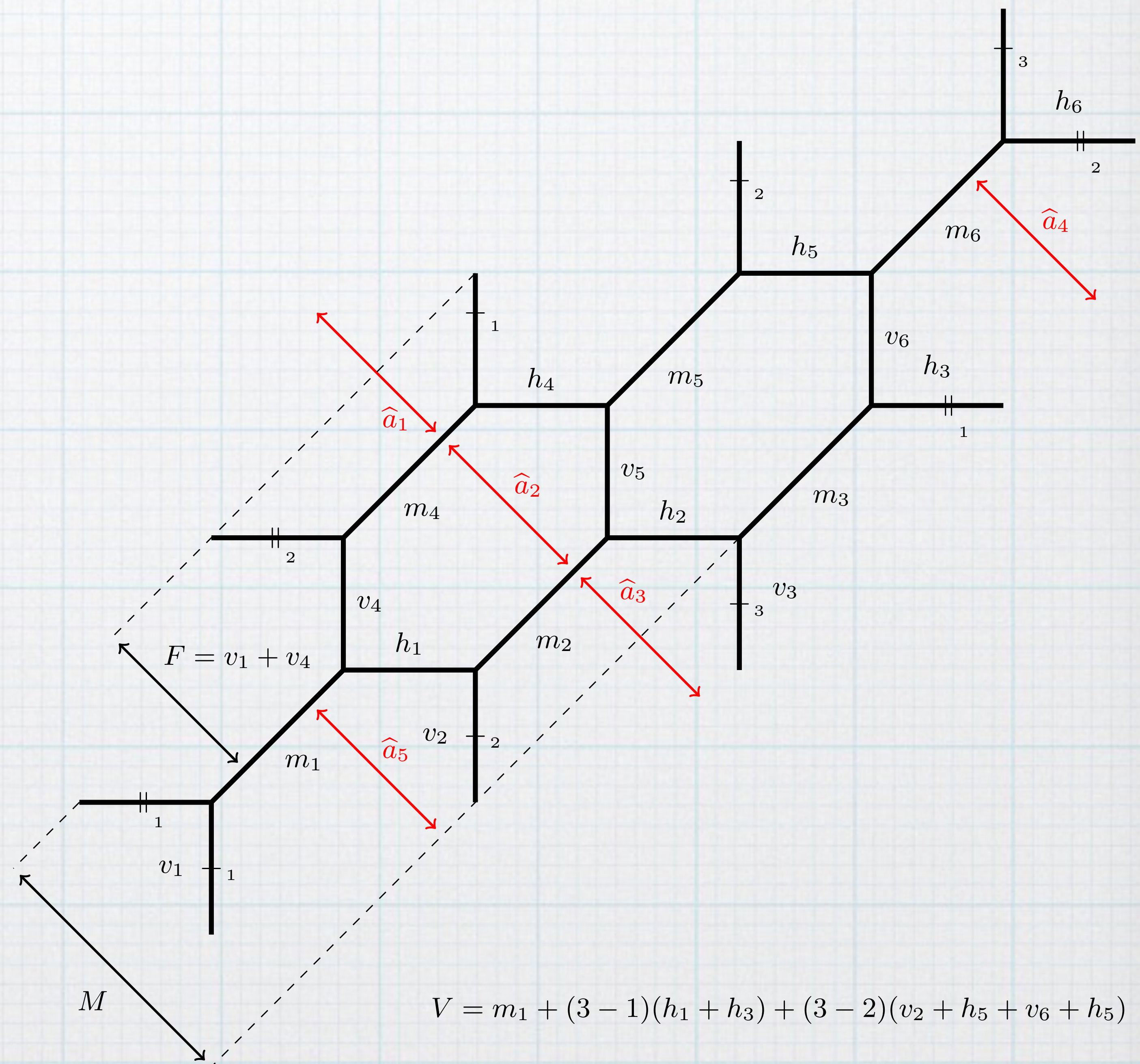

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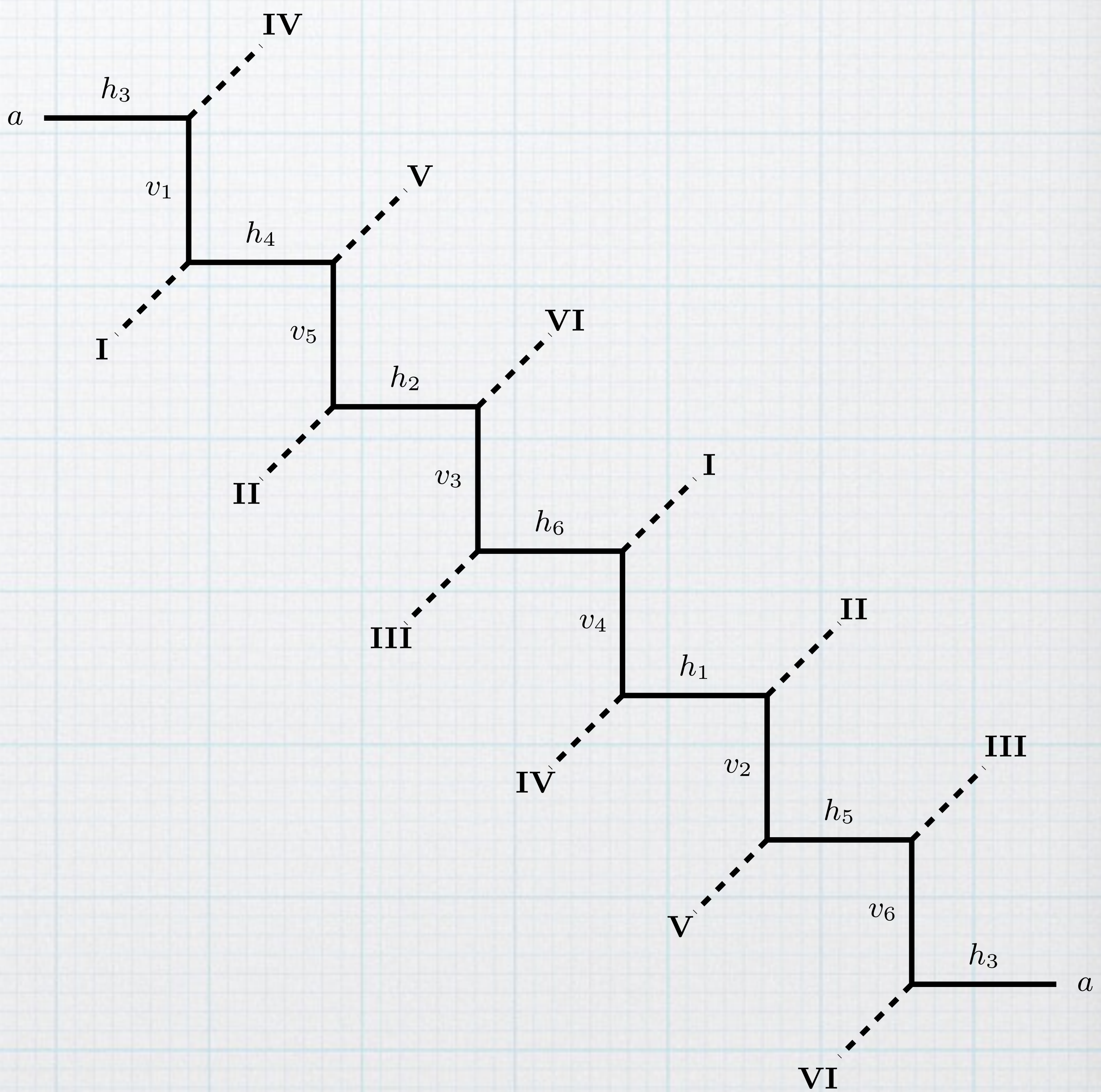

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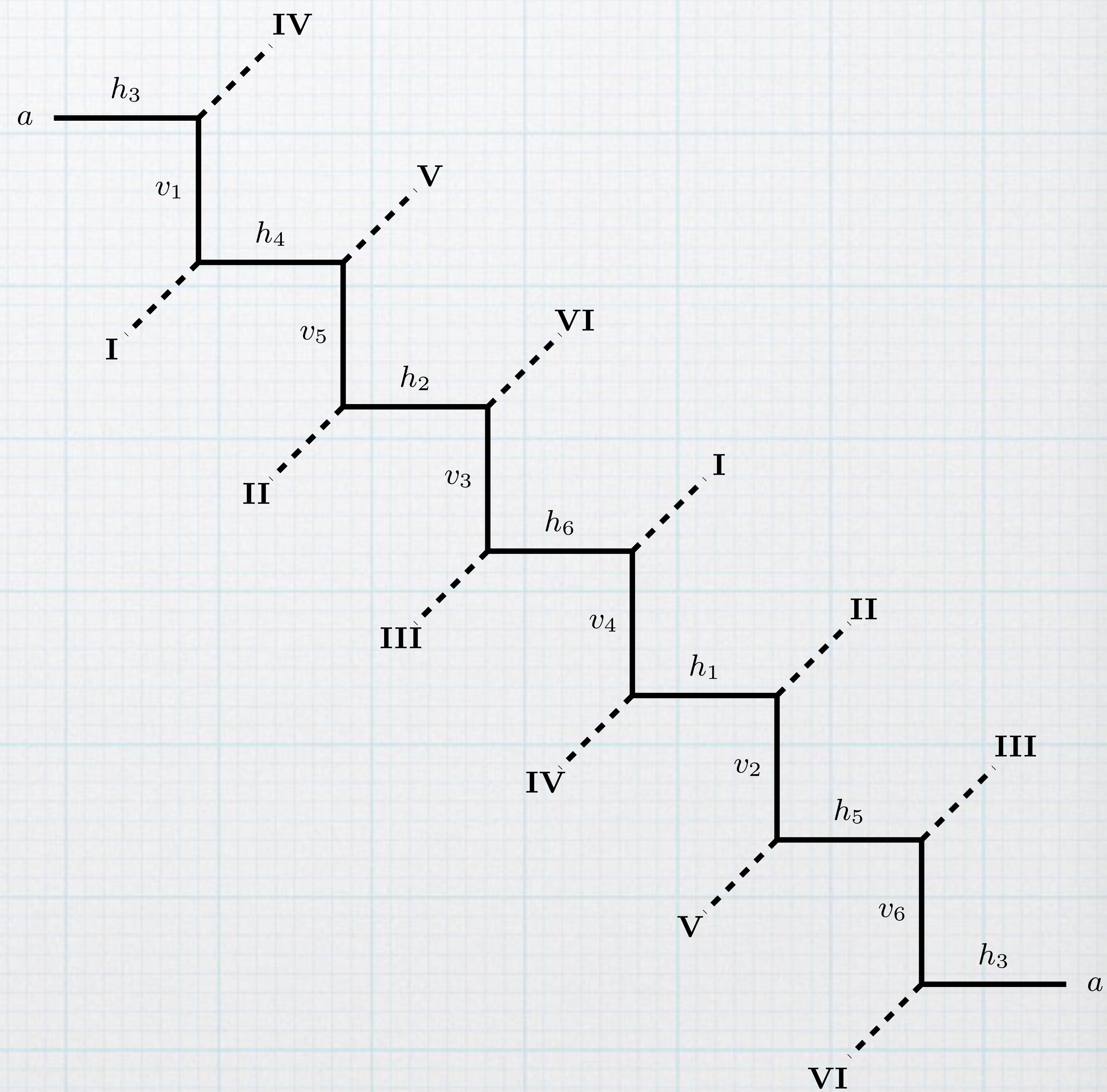

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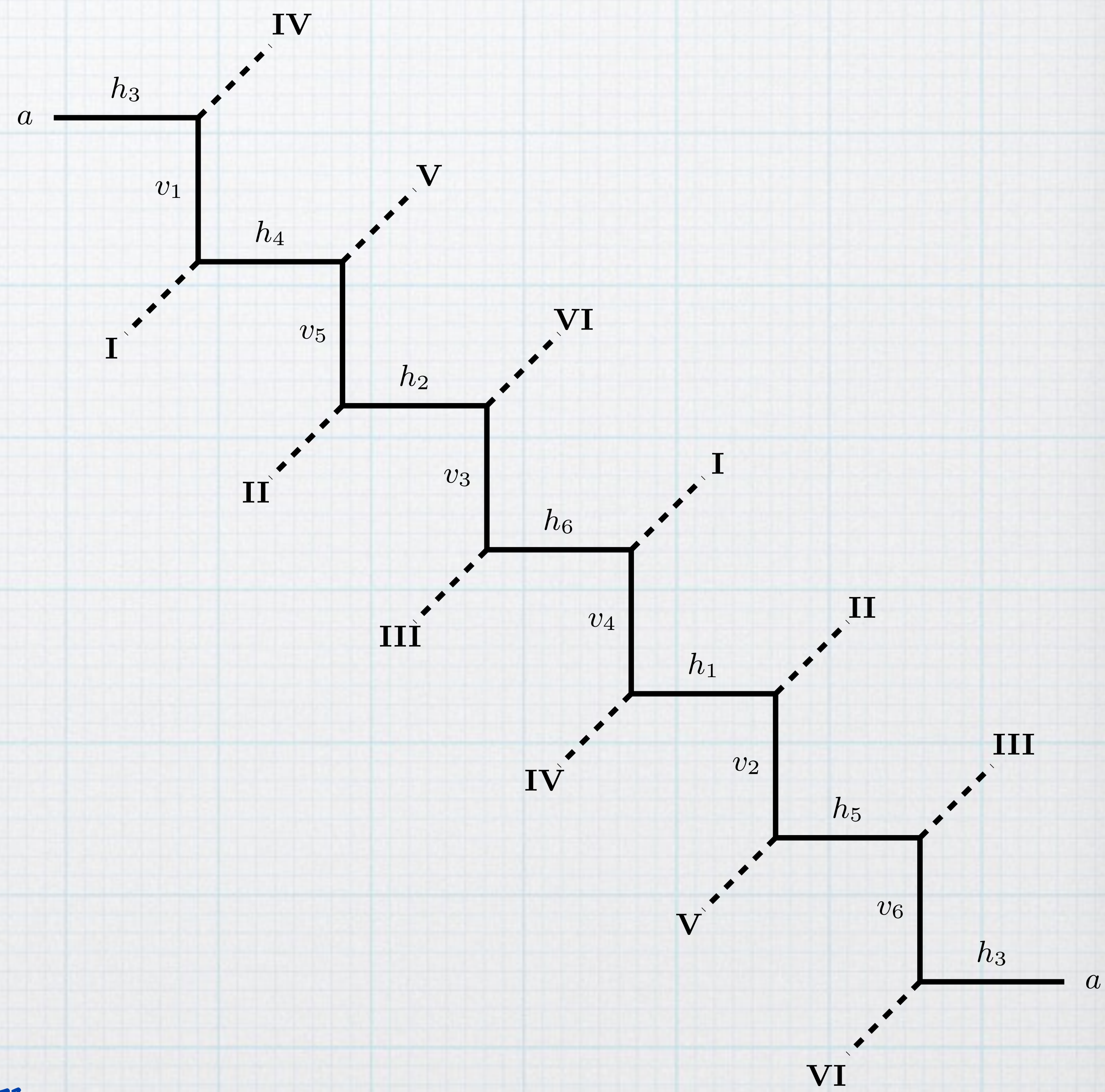
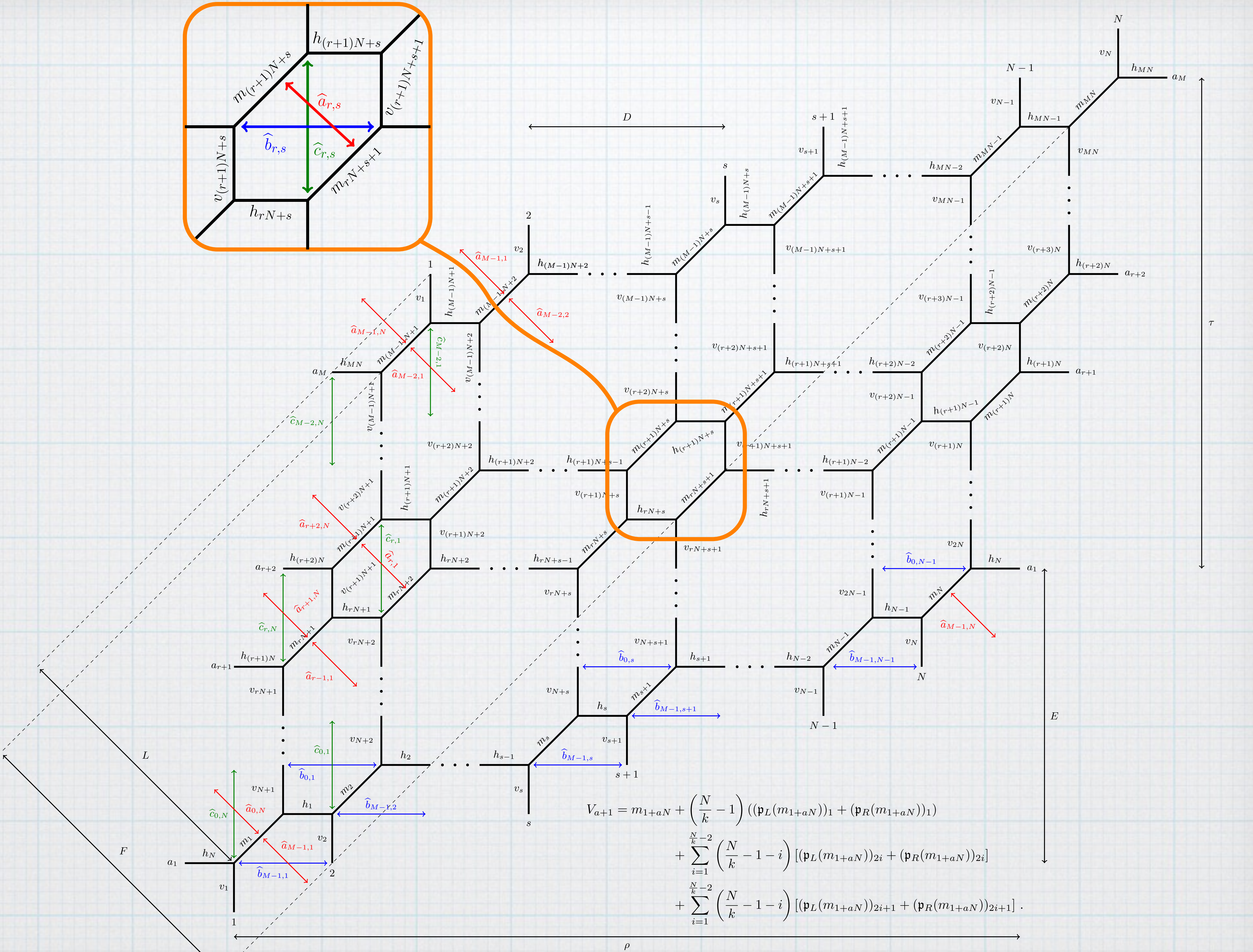


Similar sets of independent Kähler parameters proposed for generic (N,M) [Bastian, SH, Iqbal, Rey 2017]



$$V_{a+1} = m_{1+aN} + \left(\frac{N}{k} - 1\right)\left((\mathfrak{p}_L(m_{1+aN}))_1 + (\mathfrak{p}_R(m_{1+aN}))_1\right)$$
$$+ \sum_{i=1}^{\frac{N}{k}-2}\left(\frac{N}{k} - 1 - i\right)\left[(\mathfrak{p}_L(m_{1+aN}))_{2i} + (\mathfrak{p}_R(m_{1+aN}))_{2i}\right]$$
$$+ \sum_{i=1}^{\frac{N}{k}-2}\left(\frac{N}{k} - 1 - i\right)\left[(\mathfrak{p}_L(m_{1+aN}))_{2i+1} + (\mathfrak{p}_R(m_{1+aN}))_{2i+1}\right] .$$

5d Quiver Gauge Theory Interpretation

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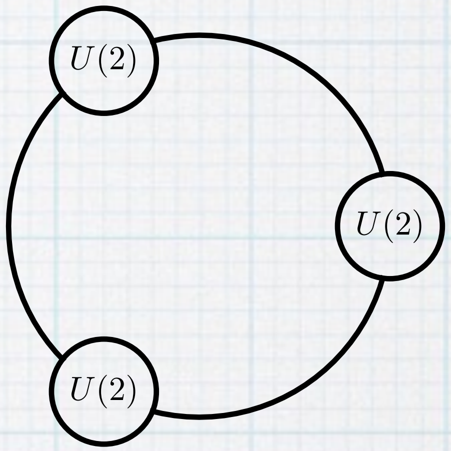
1) **horizontal:** $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



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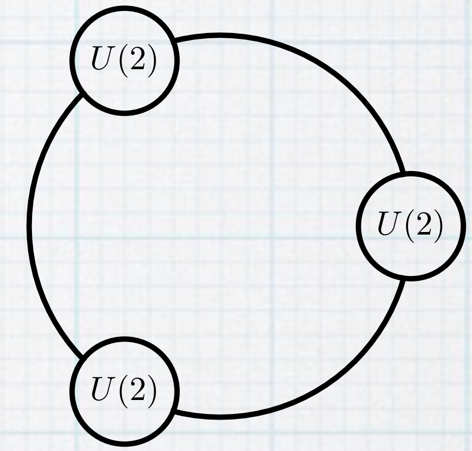
* $Z_{\text{hor}}^{(3,2)}$ series expansion in $e^{2\pi i(\rho - \widehat{b}_1 - \widehat{b}_2)}$, $e^{2\pi i \widehat{b}_1}$ and $e^{2\pi i \widehat{b}_2}$ related to the **instanton parameters**



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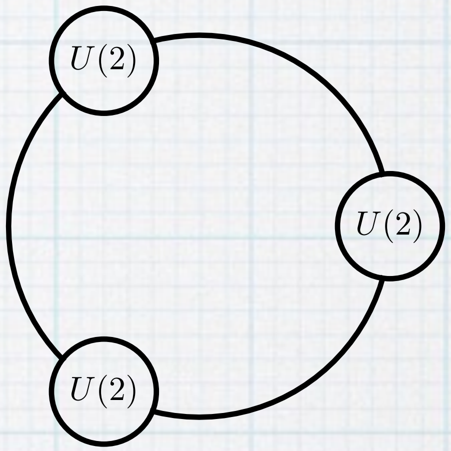

* $Z^{(3,2)}_{\text{hor}}$ series expansion in $e^{2\pi i(\rho - \widehat{b}_1 - \widehat{b}_2)}$, $e^{2\pi i \widehat{b}_1}$ and $e^{2\pi i \widehat{b}_2}$ related to the **instanton parameters**
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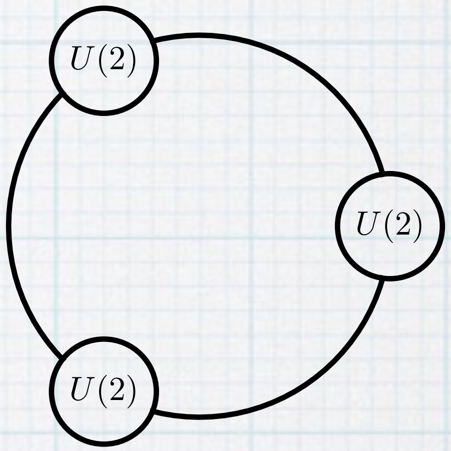

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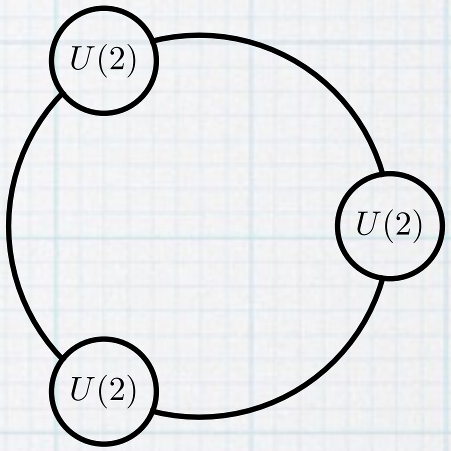

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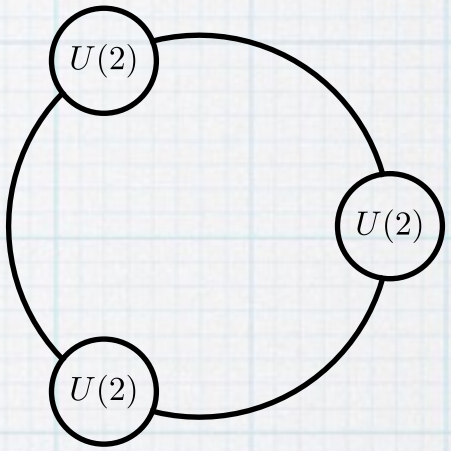
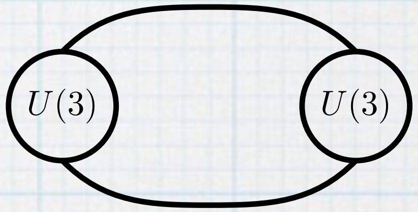
5d Quiver Gauge Theory Interpretation

1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



- $Z^{(3,2)}_{\text{hor}}$ series expansion in $e^{2\pi i(\rho - \widehat{b}_1 - \widehat{b}_2)}$, $e^{2\pi i \widehat{b}_1}$ and $e^{2\pi i \widehat{b}_2}$ related to the **instanton parameters**
- $\widehat{c}_{1,2,3}$ interpreted as simple, positive **roots** of three copies of $\mathfrak{a}_1$
- τ interpreted as (common) **imaginary root** extending $\mathfrak{a}_1$ to $\widehat{\mathfrak{a}}_1$

2) vertical: $(\tau, \widehat{c}_1; \widehat{b}_1, \widehat{b}_2, \widehat{b}_3, \widehat{b}_4; \rho, D)$: $U(3) \times U(3)$ quiver gauge theory



5d Quiver Gauge Theory Interpretation

1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory

* $Z_{\text{hor}}^{(3,2)}$ series expansion in $e^{2\pi i(\rho - \widehat{b}_1 - \widehat{b}_2)}$, $e^{2\pi i \widehat{b}_1}$ and $e^{2\pi i \widehat{b}_2}$ related to the **instanton parameters**

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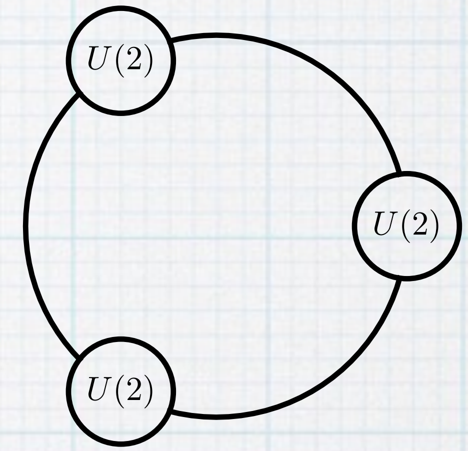
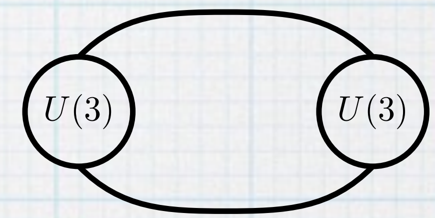
* τ interpreted as (common) **imaginary root** extending $\mathfrak{a}_1$ to $\widehat{\mathfrak{a}}_1$



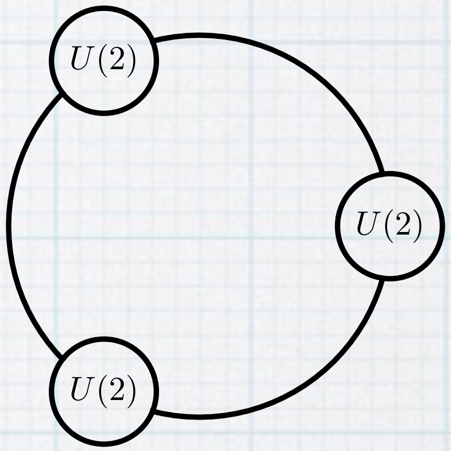
2) vertical: $(\tau, \widehat{c}_1; \widehat{b}_1, \widehat{b}_2, \widehat{b}_3, \widehat{b}_4; \rho, D)$: $U(3) \times U(3)$ quiver gauge theory

* $Z_{\text{vert}}^{(3,2)}$ series expansion in $e^{2\pi i(\tau - \widehat{c}_1)}$ and $e^{2\pi i \widehat{c}_1}$ related to the **instanton parameters**



5d Quiver Gauge Theory Interpretation

1) horizontal: $(\rho,\widehat{b}_1,\widehat{b}_2;\widehat{c}_1,\widehat{c}_2,\widehat{c}_3;\tau,E)$: $U(2)\times U(2)\times U(2)$ quiver gauge theory



* $Z^{(3,2)}_{\text{hor}}$ series expansion in $e^{2\pi i(\rho-\widehat{b}_1-\widehat{b}_2)}$, $e^{2\pi i\widehat{b}_1}$ and $e^{2\pi i\widehat{b}_2}$ related to the **instanton parameters**
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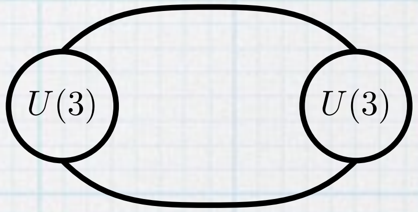
* τ interpreted as (common) **imaginary root** extending $\mathfrak{a}_1$ to $\widehat{\mathfrak{a}}_1$

2) vertical: $(\tau,\widehat{c}_1;\widehat{b}_1,\widehat{b}_2,\widehat{b}_3,\widehat{b}_4;\rho,D)$: $U(3)\times U(3)$ quiver gauge theory



* $Z^{(3,2)}_{\text{vert}}$ series expansion in $e^{2\pi i(\tau-\widehat{c}_1)}$ and $e^{2\pi i\widehat{c}_1}$ related to the **instanton parameters**
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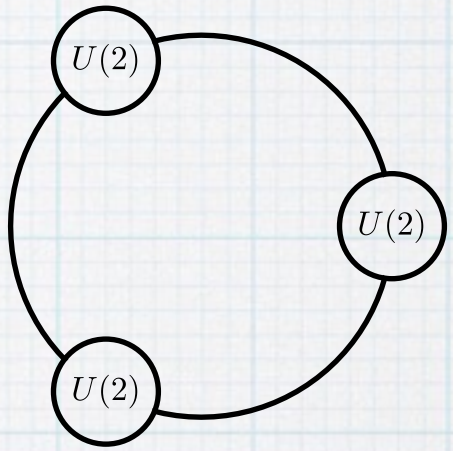
5d Quiver Gauge Theory Interpretation

1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



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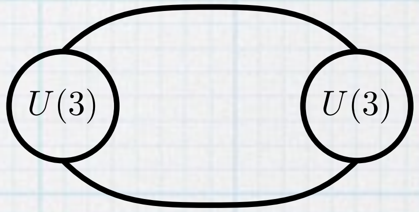

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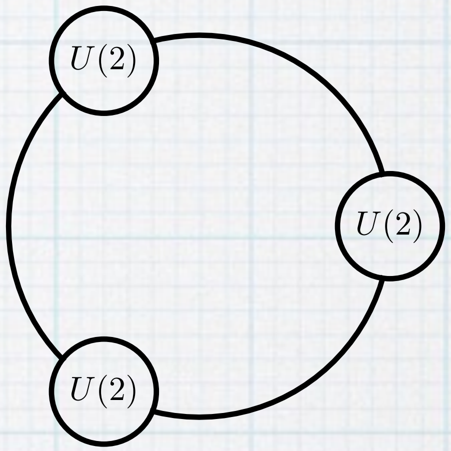
5d Quiver Gauge Theory Interpretation

1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



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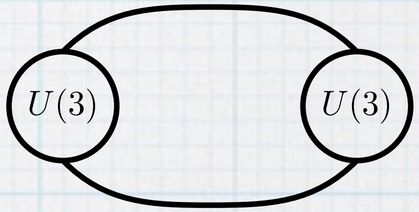

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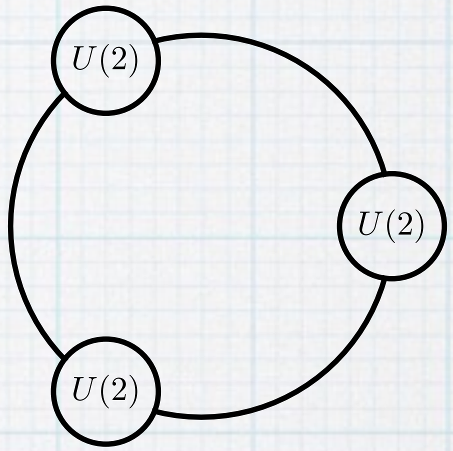
3) diagonal: $(V; \widehat{a}_1, \widehat{a}_2, \widehat{a}_3, \widehat{a}_4, \widehat{a}_5; M, F)$ gauge theory with gauge group $U(6)$

5d Quiver Gauge Theory Interpretation

1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



* $Z^{(3,2)}_{\text{hor}}$ series expansion in $e^{2\pi i(\rho - \widehat{b}_1 - \widehat{b}_2)}$, $e^{2\pi i \widehat{b}_1}$ and $e^{2\pi i \widehat{b}_2}$ related to the **instanton parameters**
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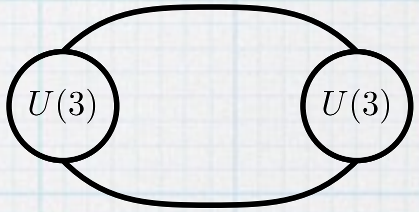

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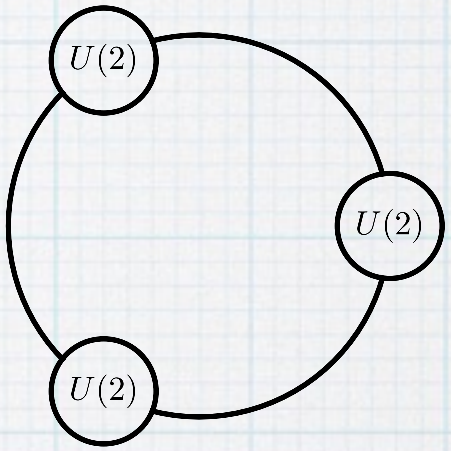
* $Z^{(3,2)}_{\text{diag}}$ can be written as a series expansion in $e^{2\pi i V}$ related to the **instanton parameters**

5d Quiver Gauge Theory Interpretation

1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



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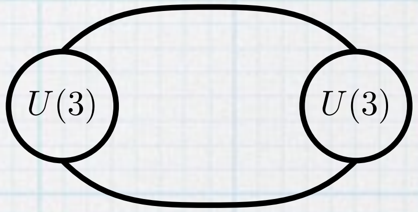

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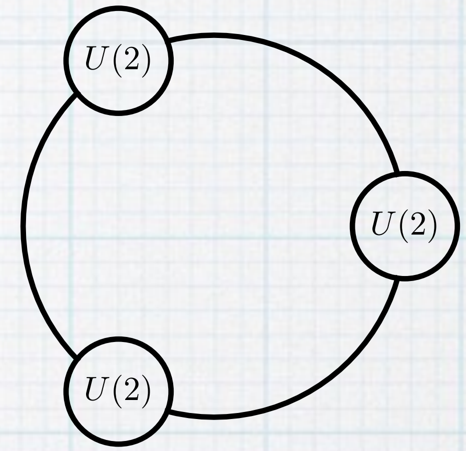
1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



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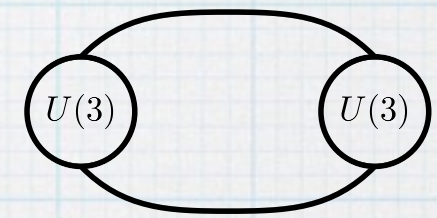

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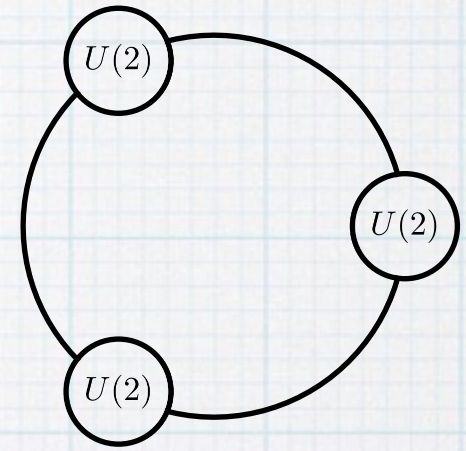
1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



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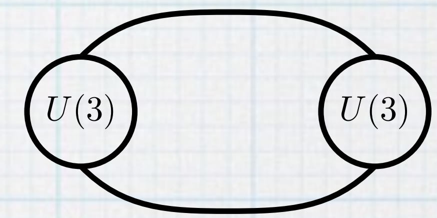

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Horizontal and vertical gauge theory interpretation well known in the literature

[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa 2013]
[Haghighat, Kozcaz, Lockhart, Vafa 2013]
[SH, Iqbal 2013]

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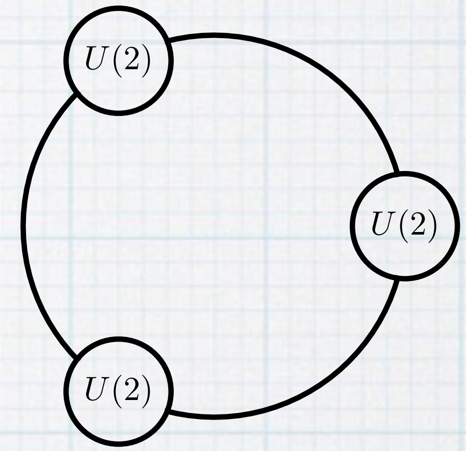
1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



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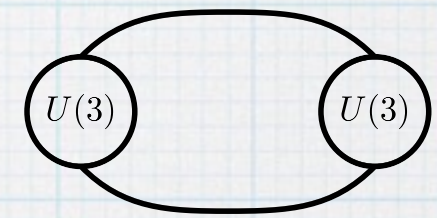

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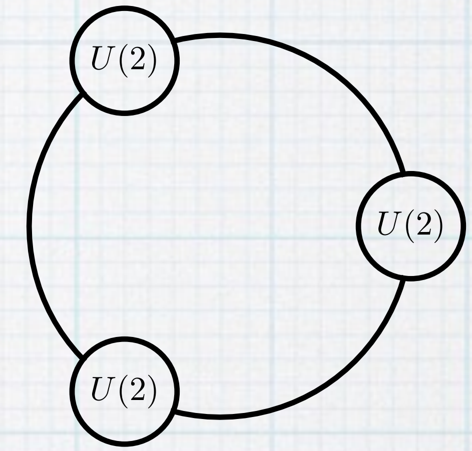
Diagonal expansion leads to novel gauge theory associated with $X_{N,M}$

5d Quiver Gauge Theory Interpretation

1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



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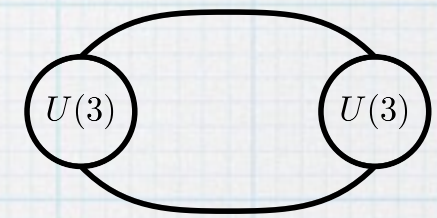

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Horizontal and vertical gauge theory interpretation well known in the literature

Diagonal expansion leads to novel gauge theory associated with $X_{N,M}$

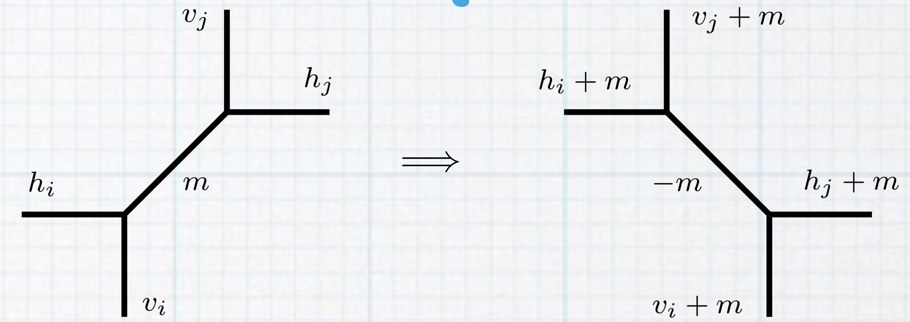
Triality!

[Bastian, SH, Iqbal, Rey 2017]

Flop Transitions and Duality

Flop Transitions and Duality

Flop transition for any two curves in the diagram:



Flop Transitions and Duality

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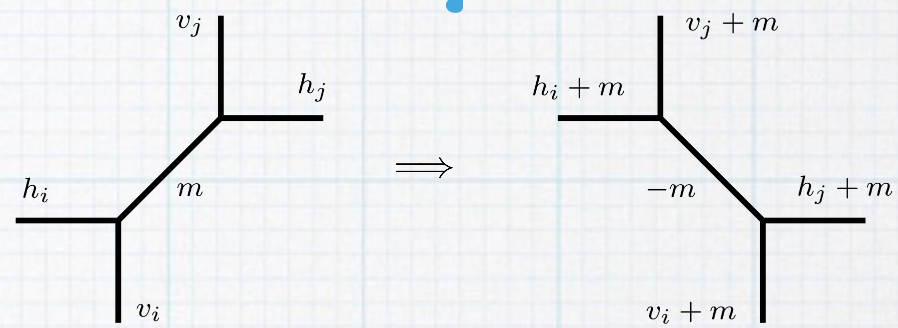

Series of flop and SL(2,Z) transformations for $X_{3,2} \sim X_{6,1}$ [SH, Iqbal, Rey 2016]

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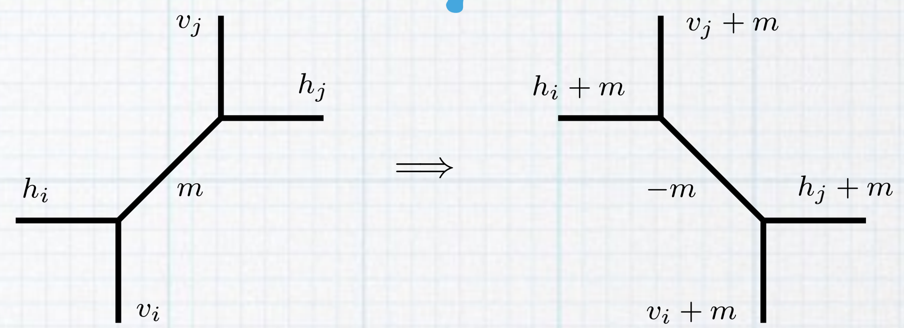
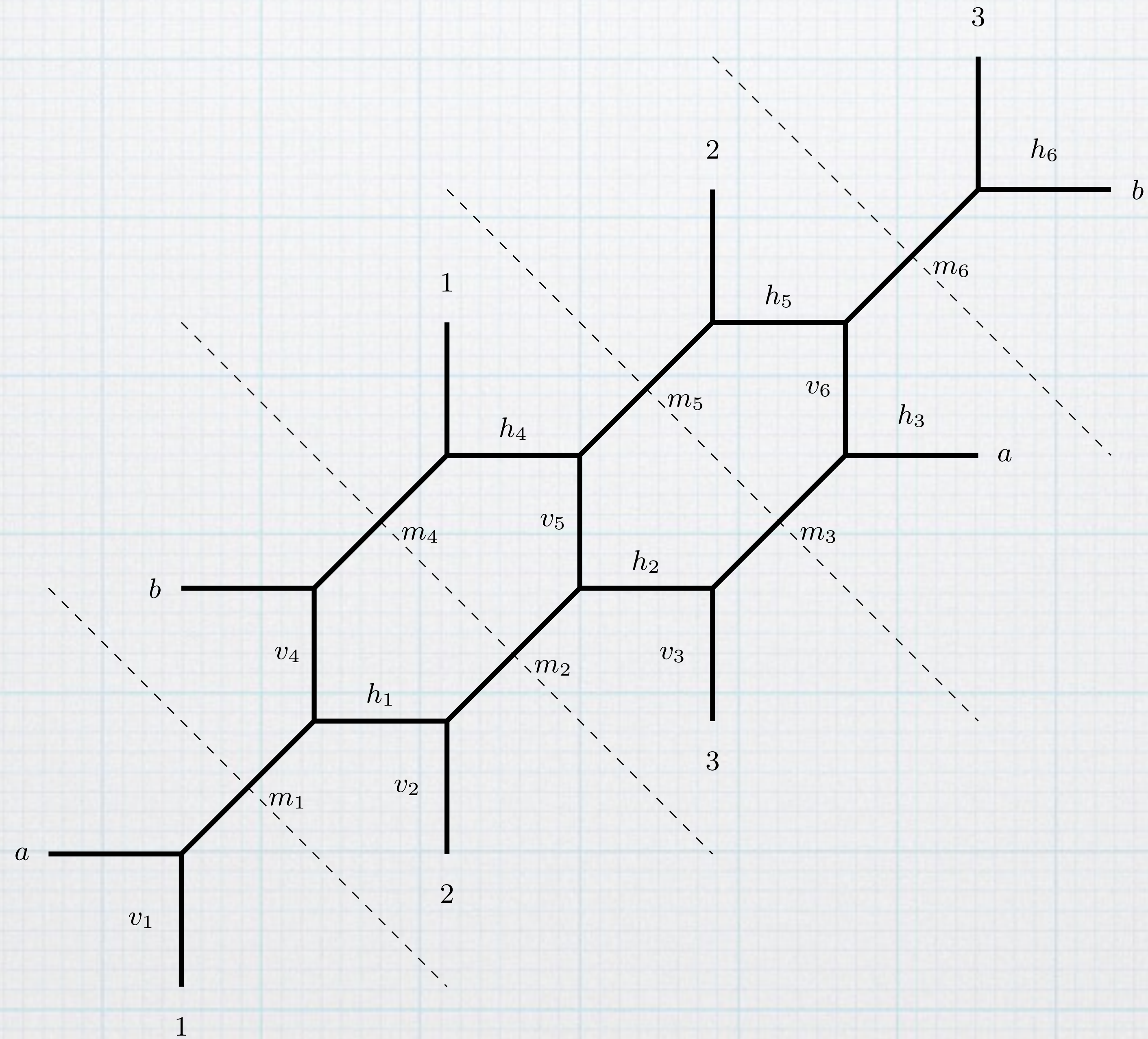


Series of flop and SL(2,Z) transformations for $X_{3,2} \sim X_{6,1}$ [SH, Iqbal, Rey 2016]

Cut diagram along dashed lines and re-glue



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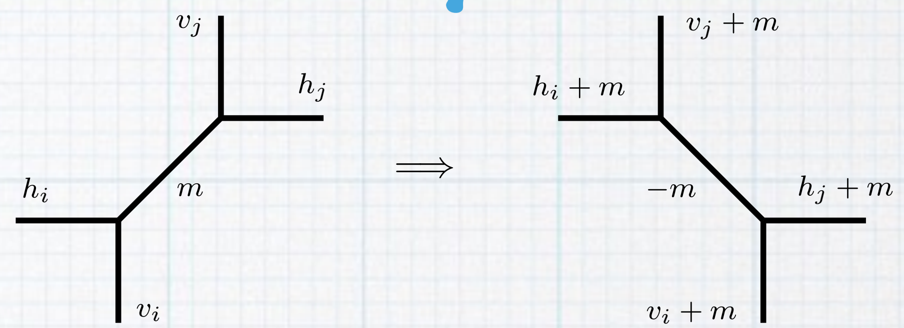

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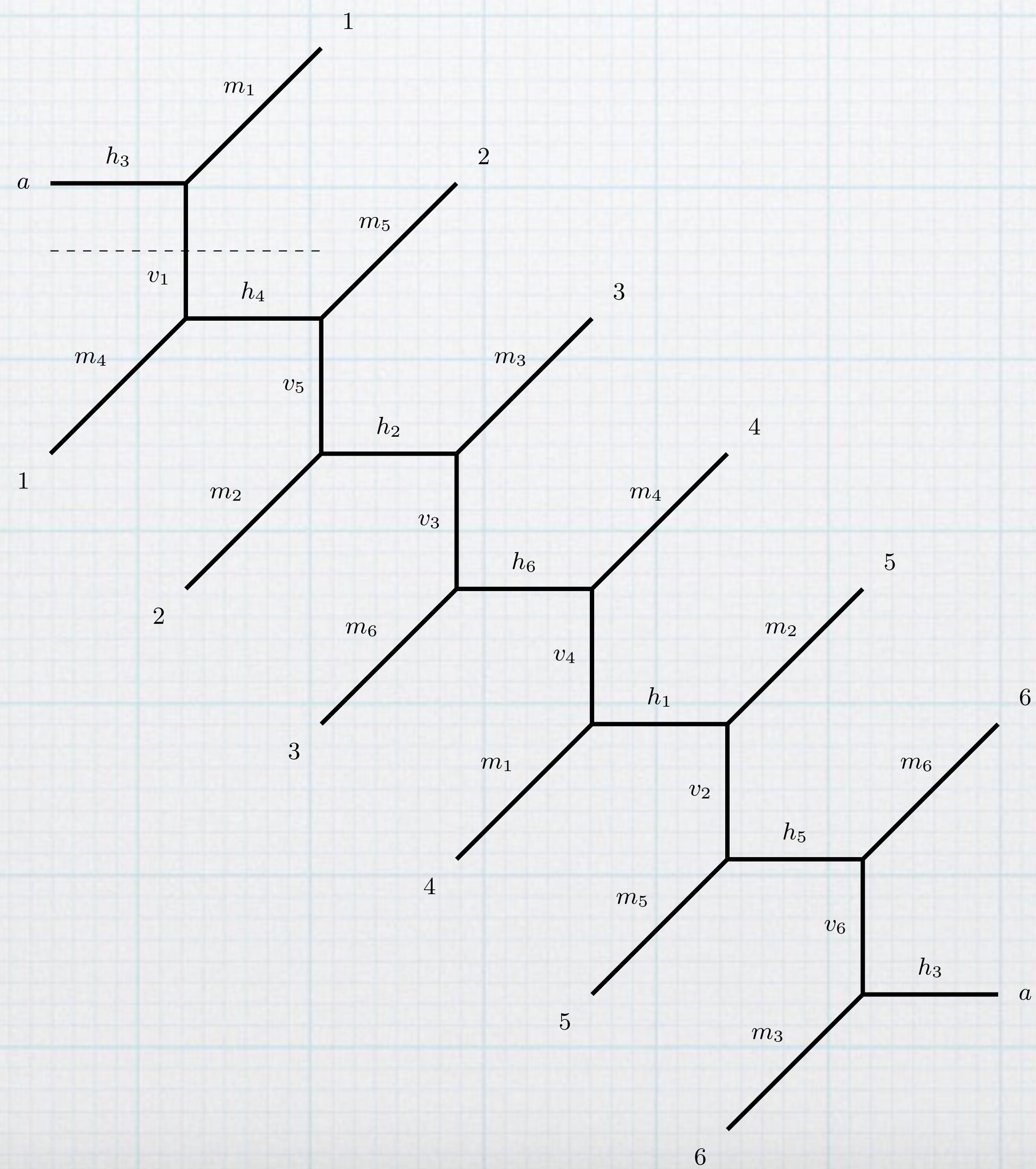

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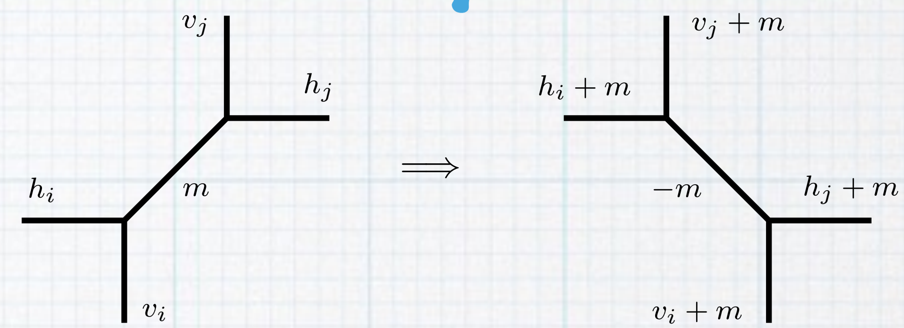

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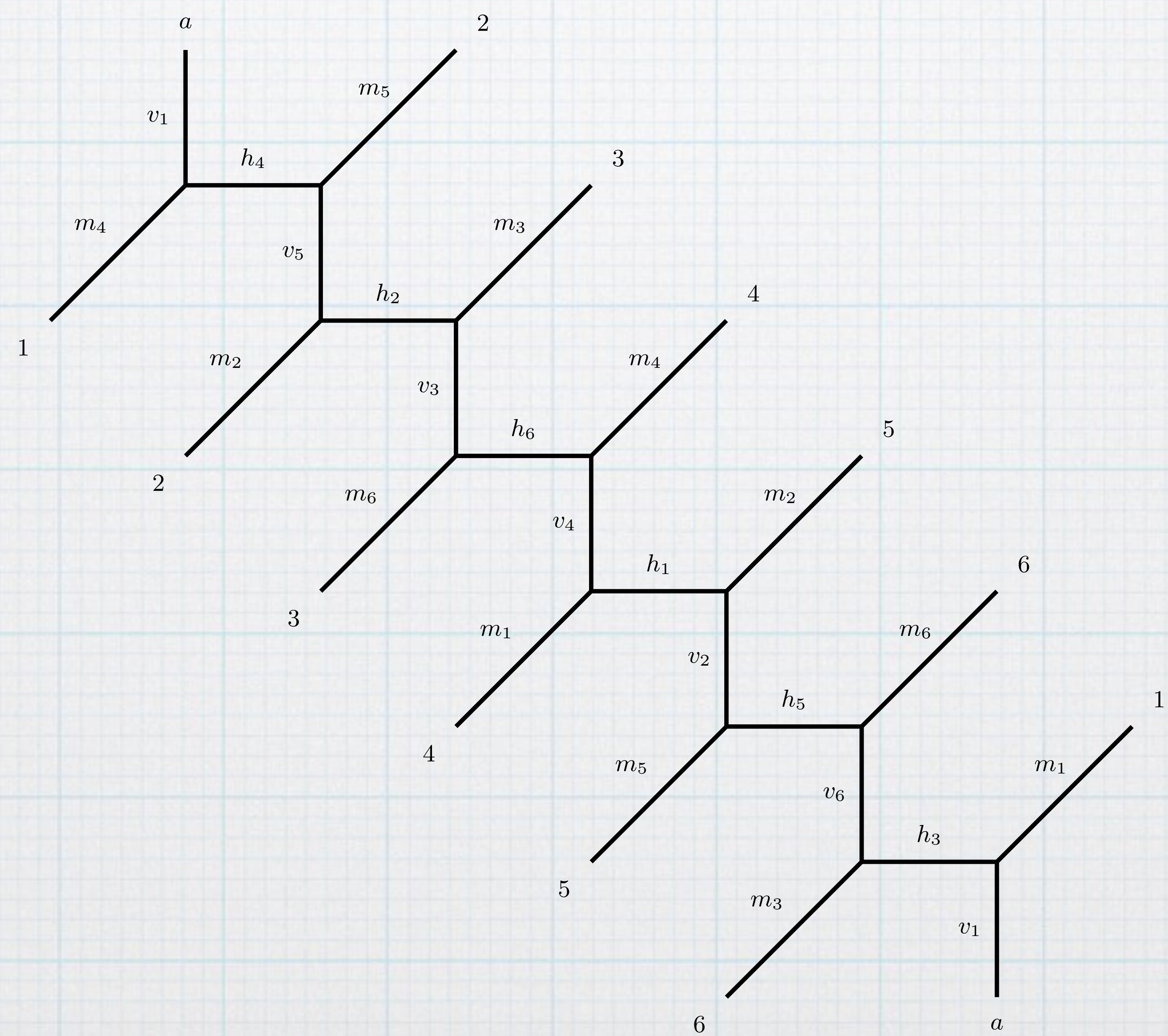

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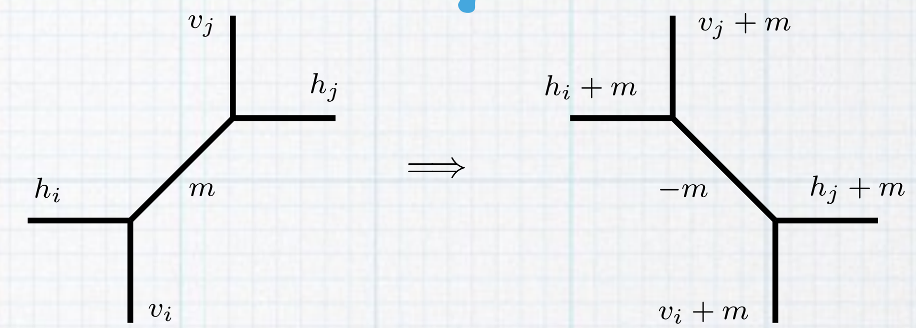

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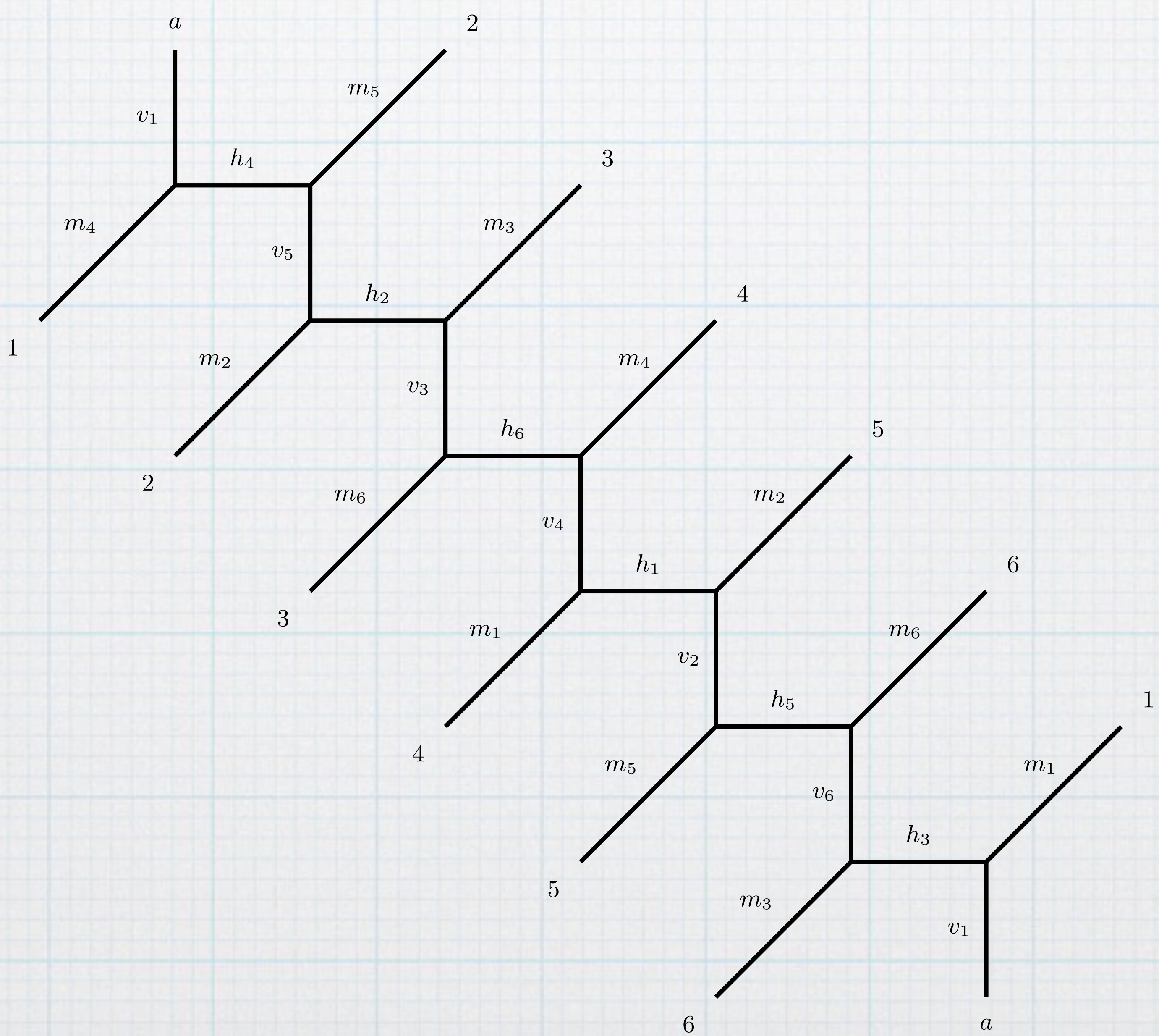

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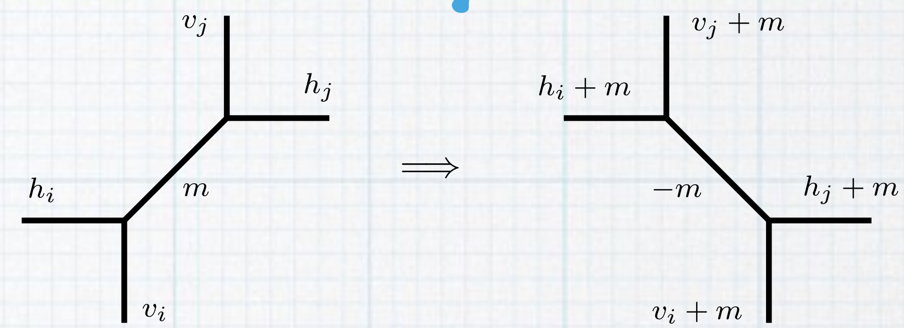

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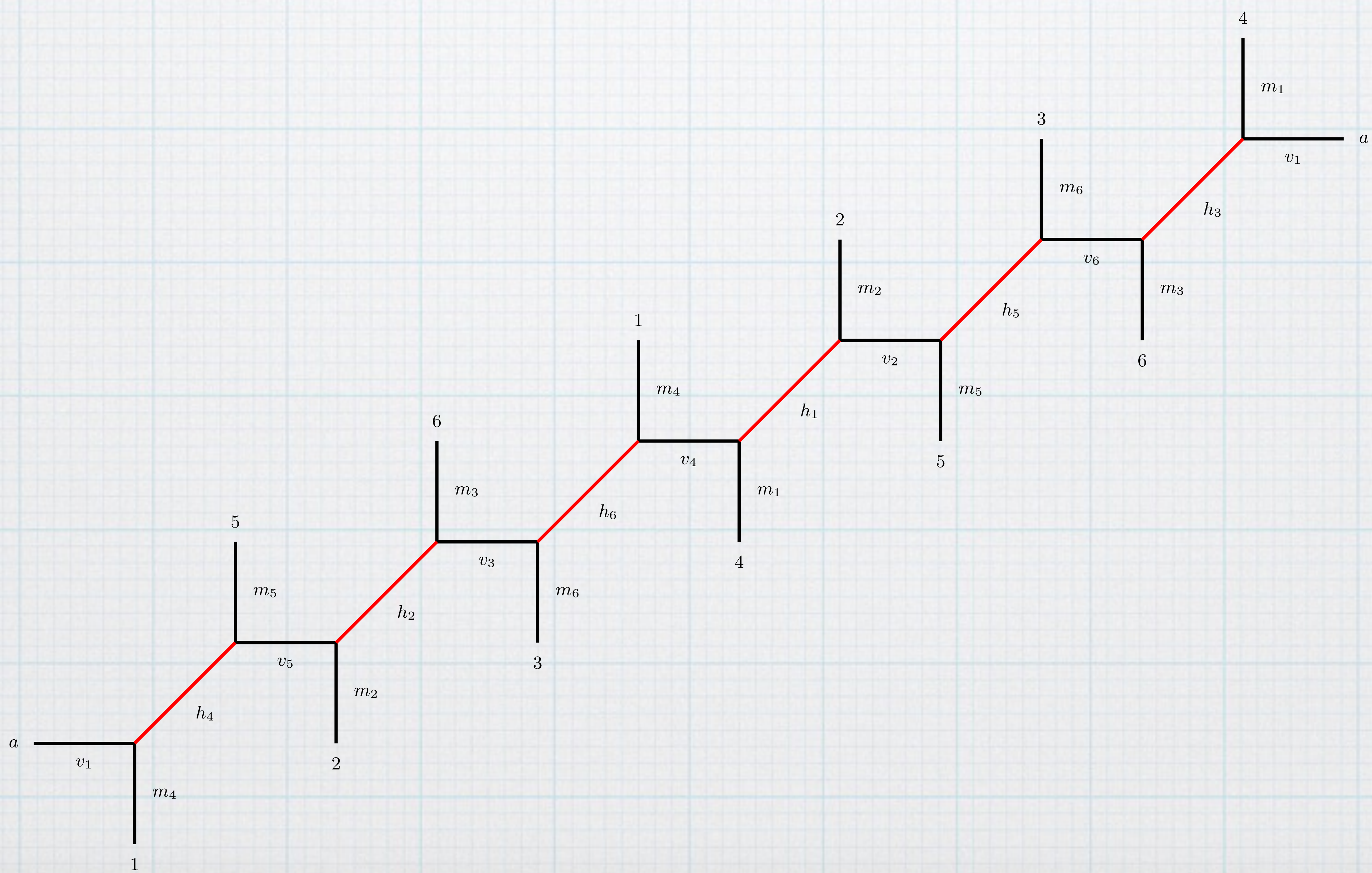

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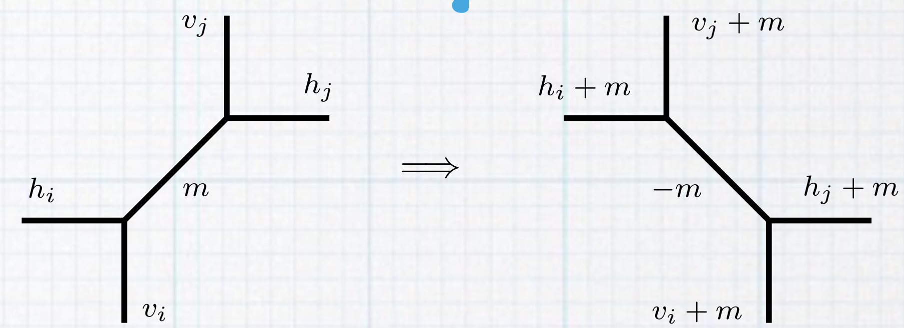

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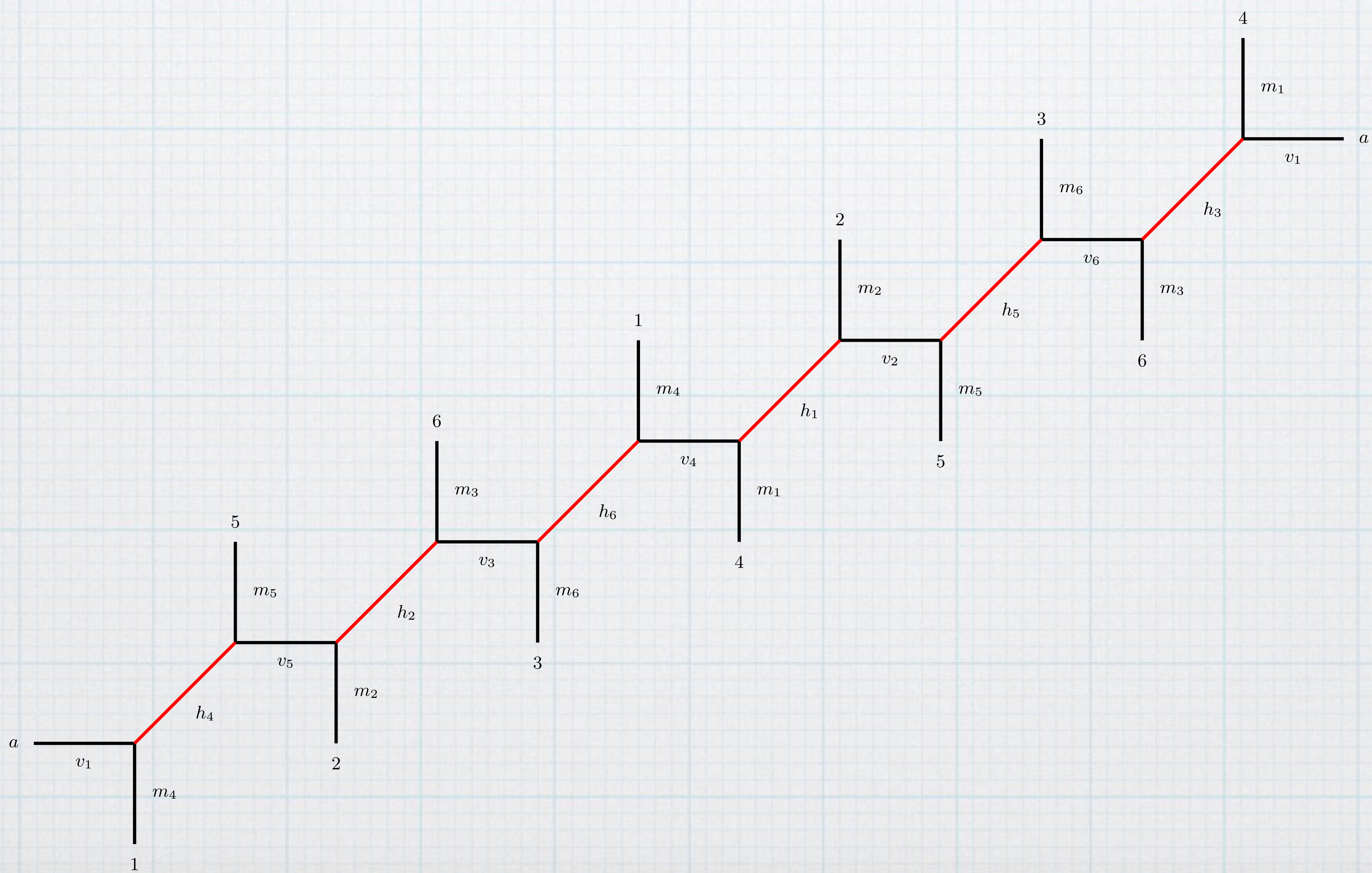

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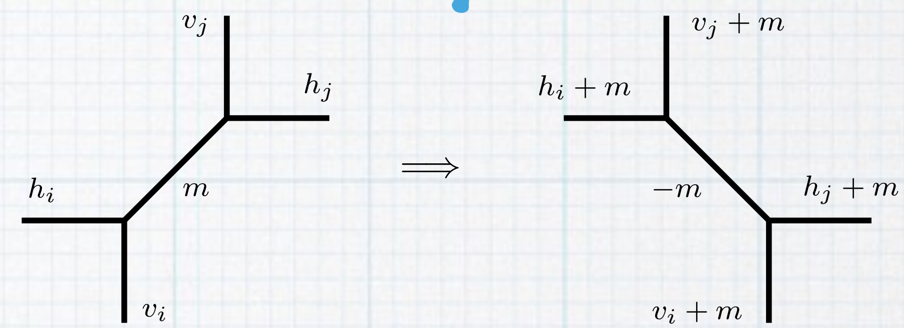

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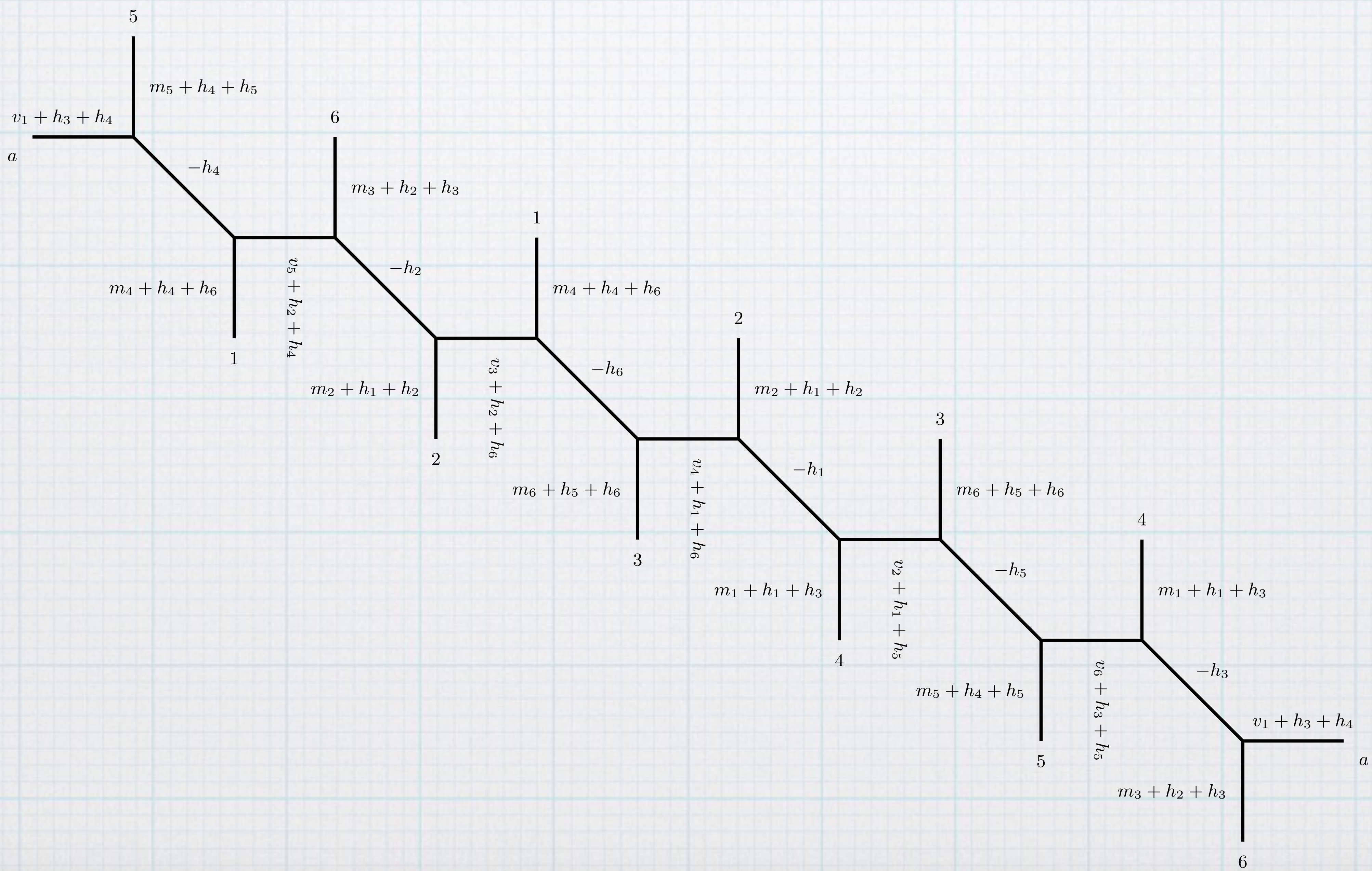

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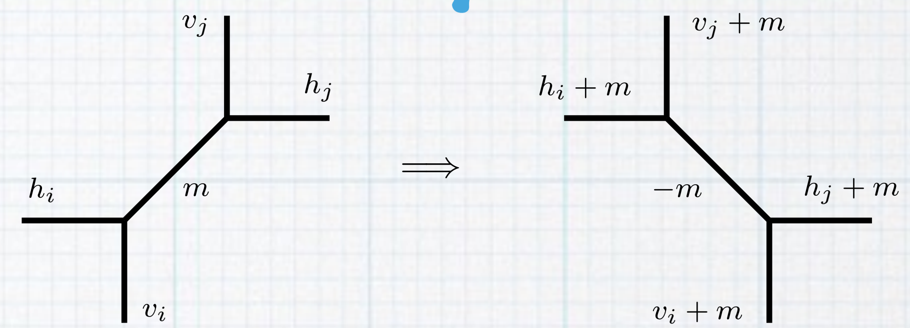

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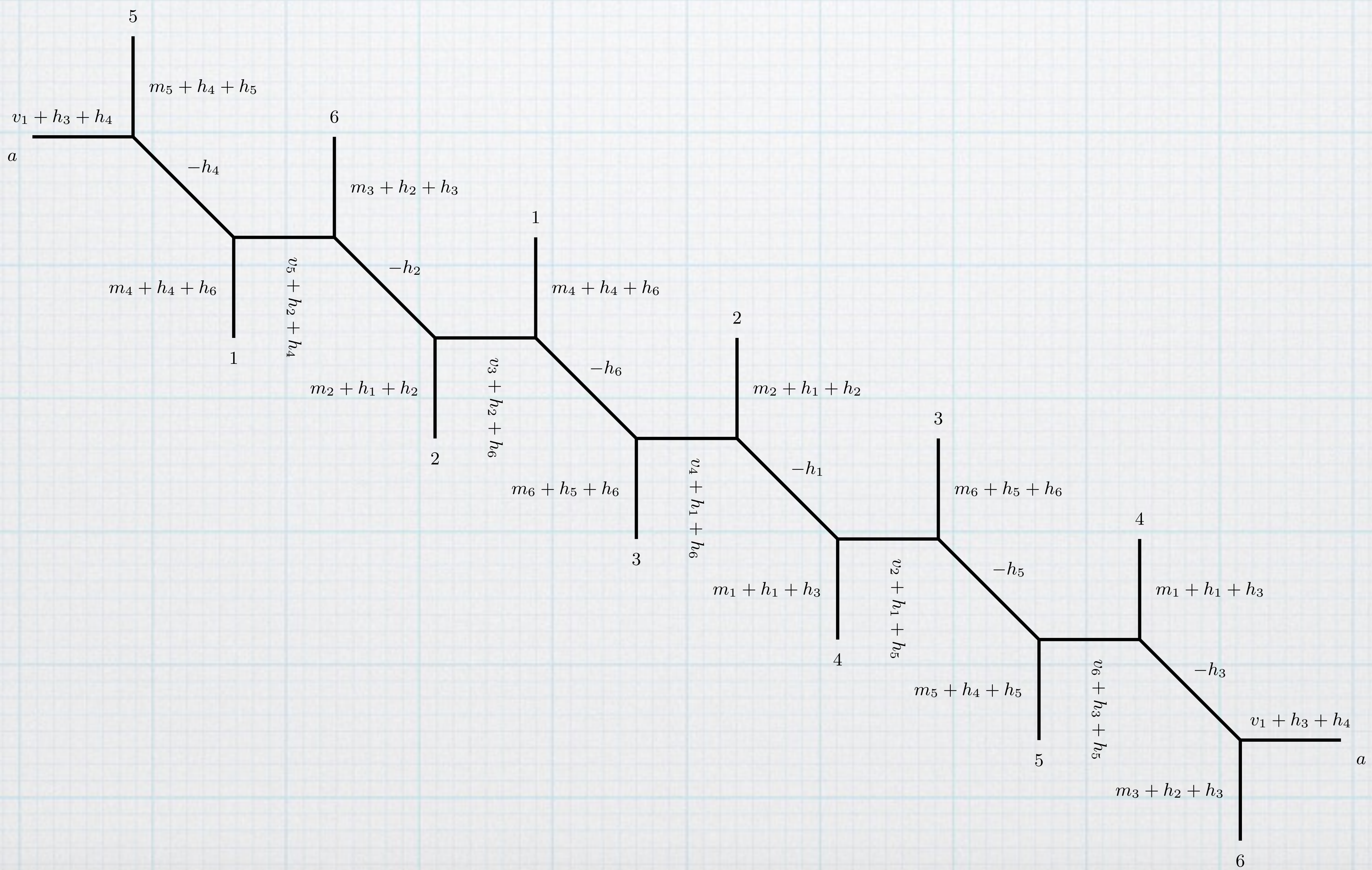

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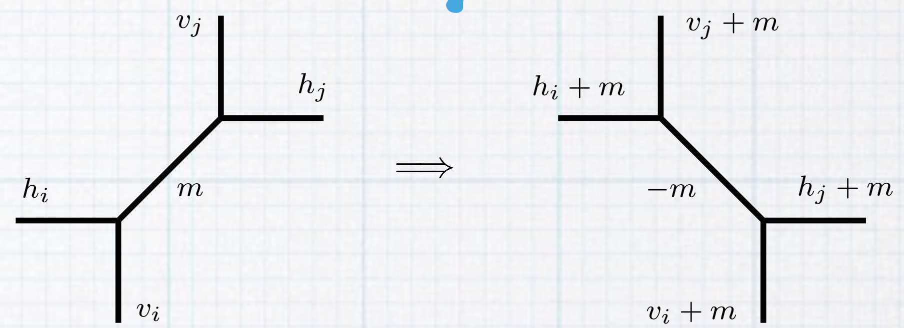

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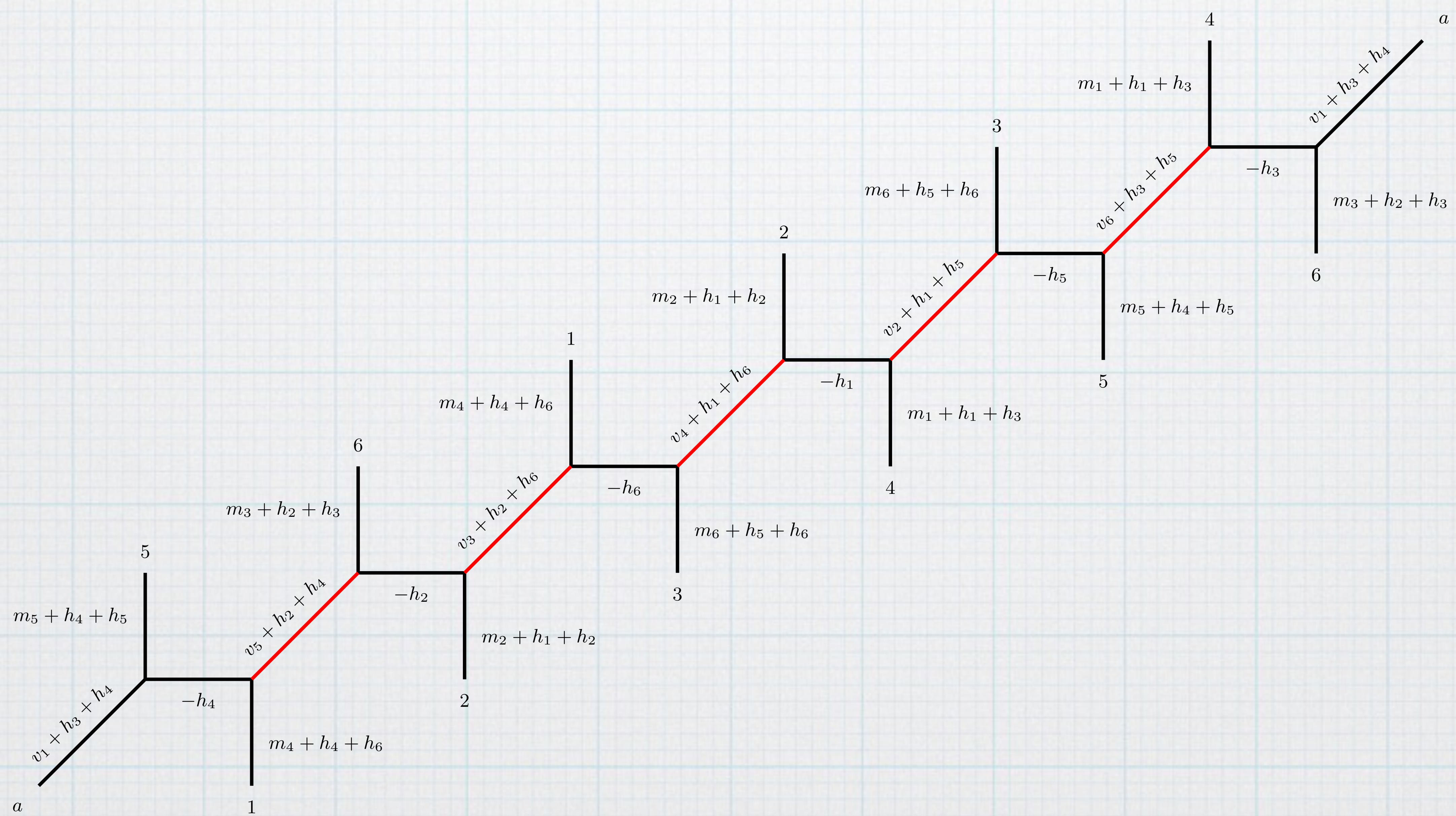

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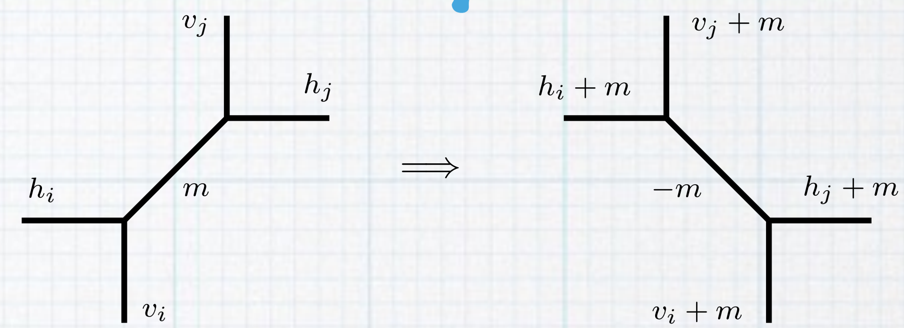

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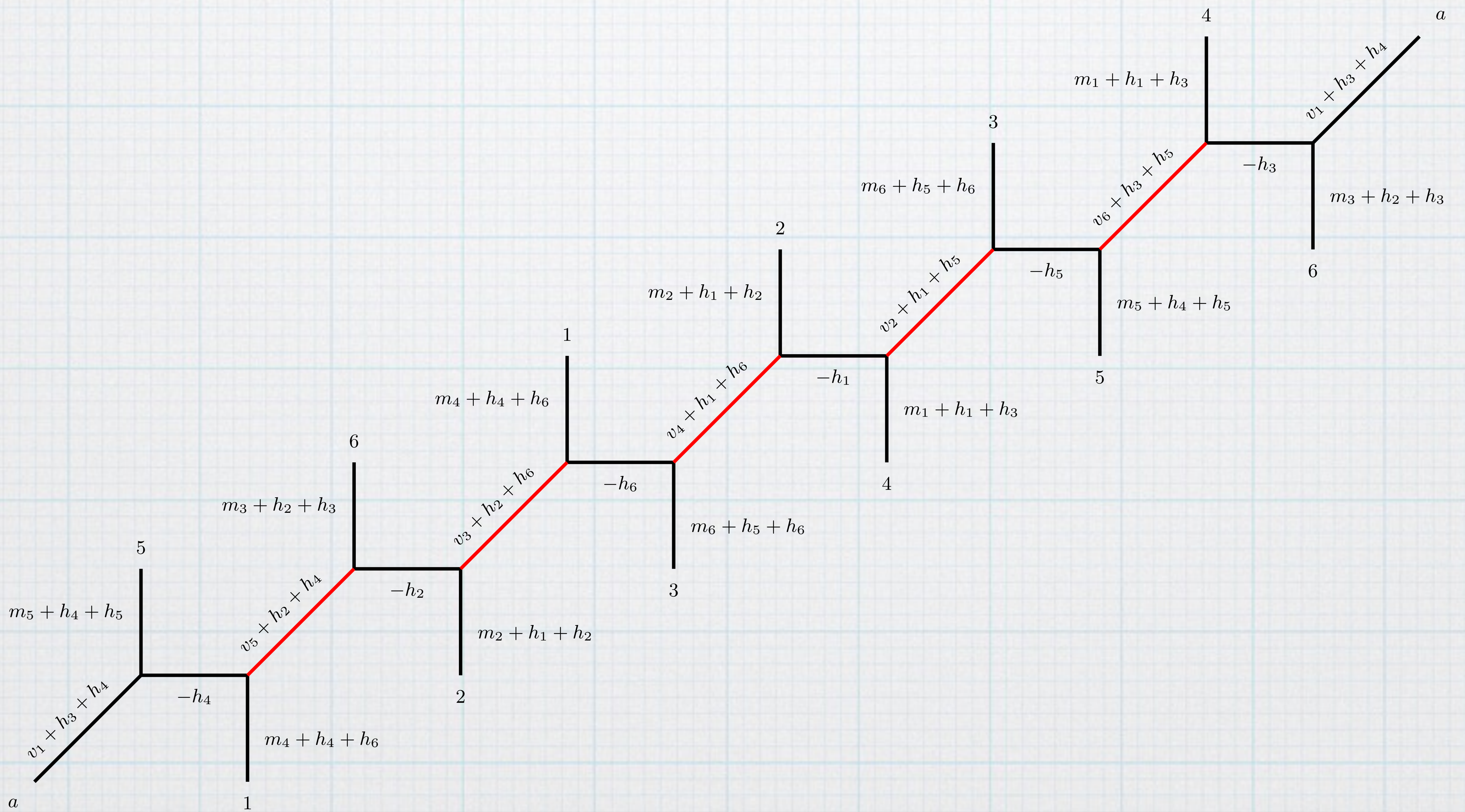

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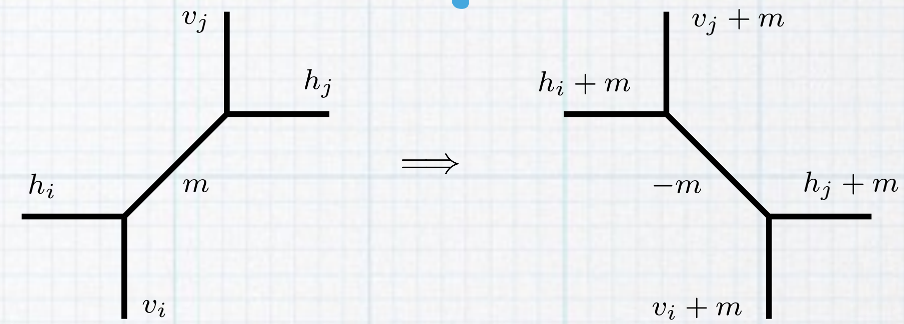

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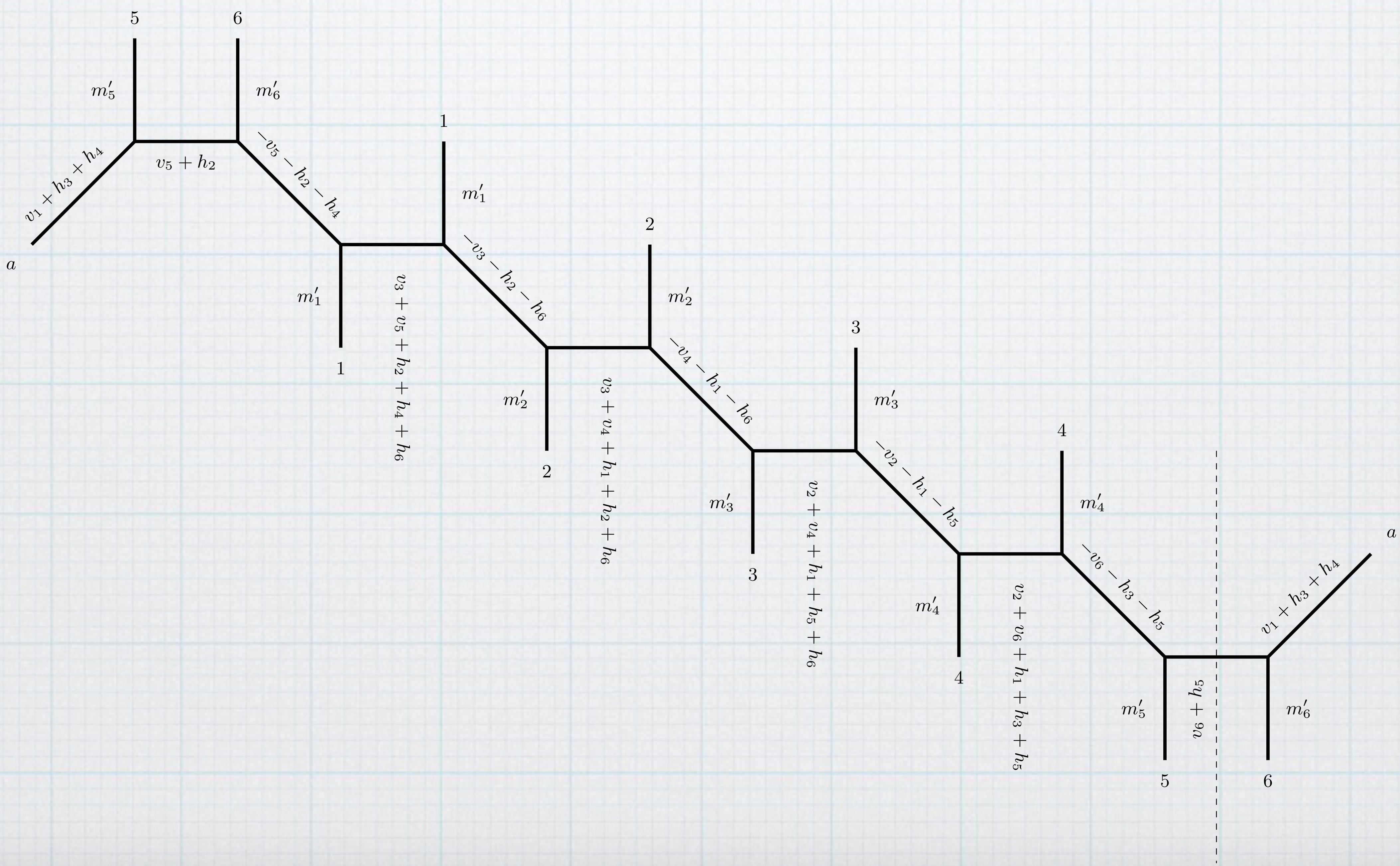

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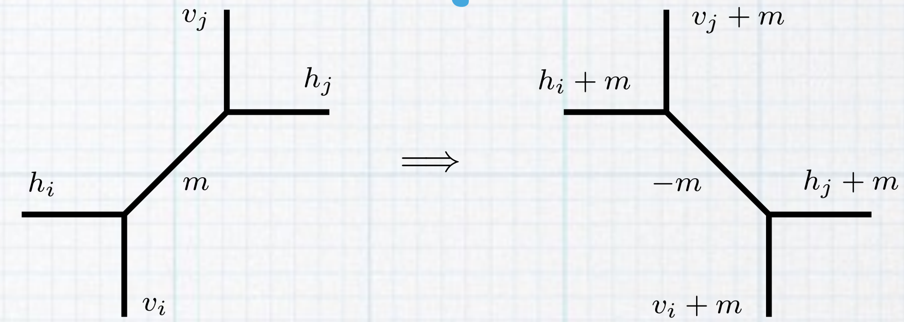

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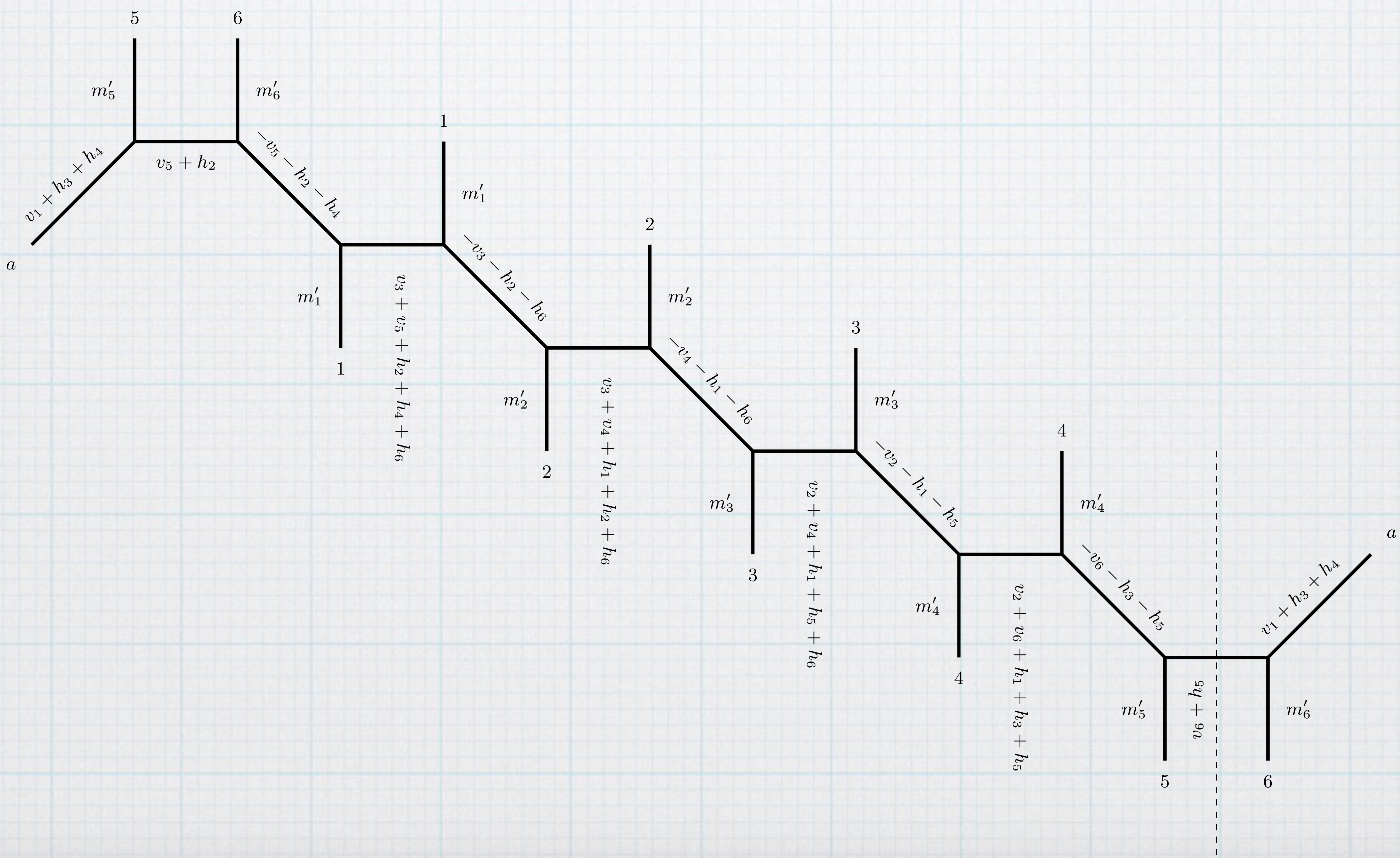

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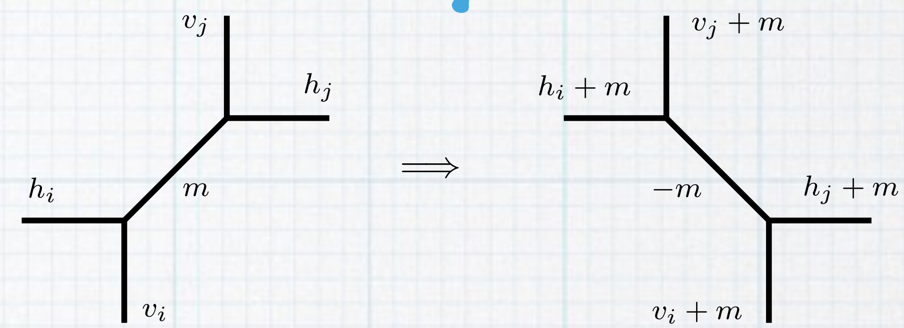

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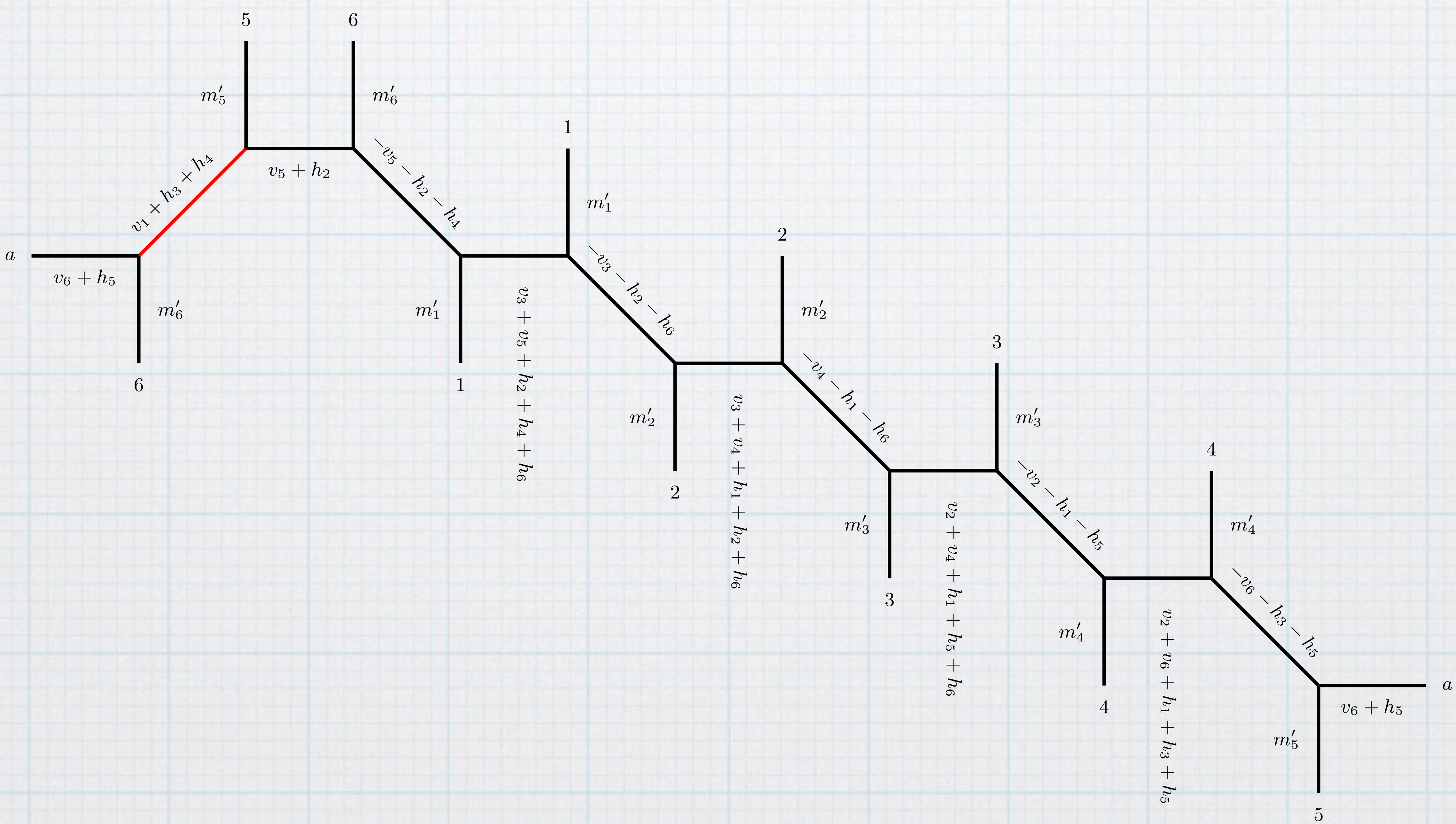

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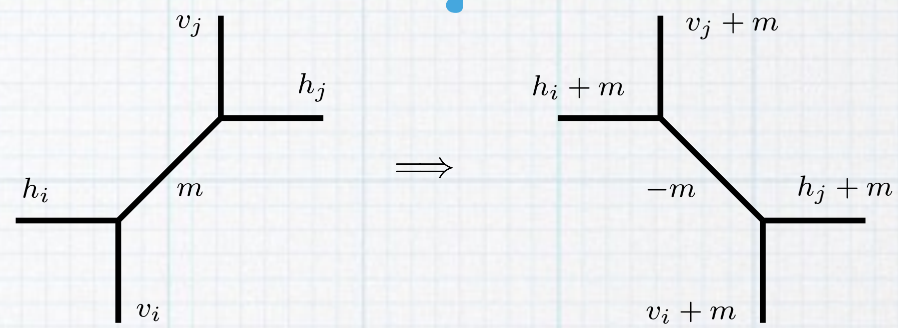

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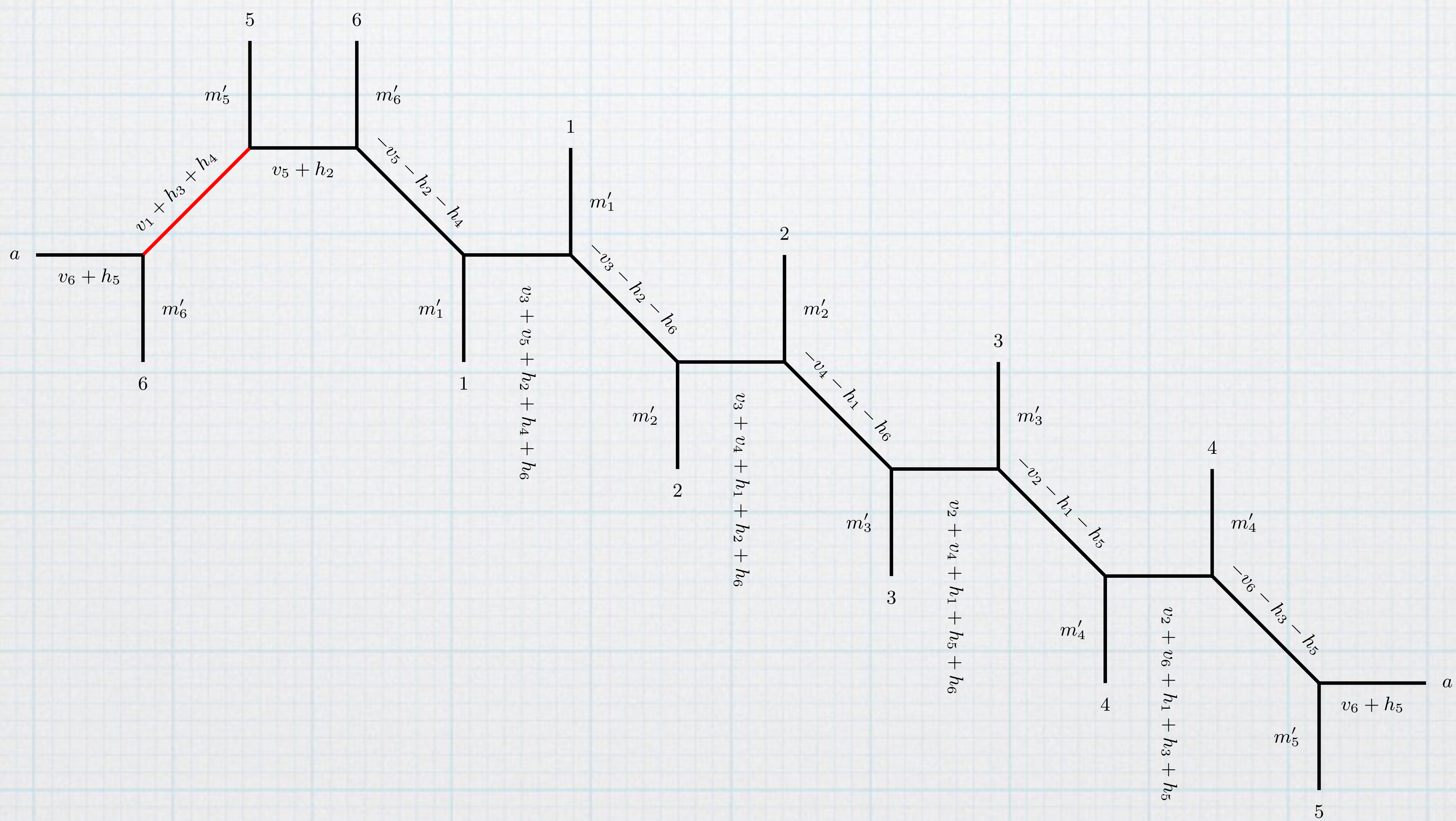

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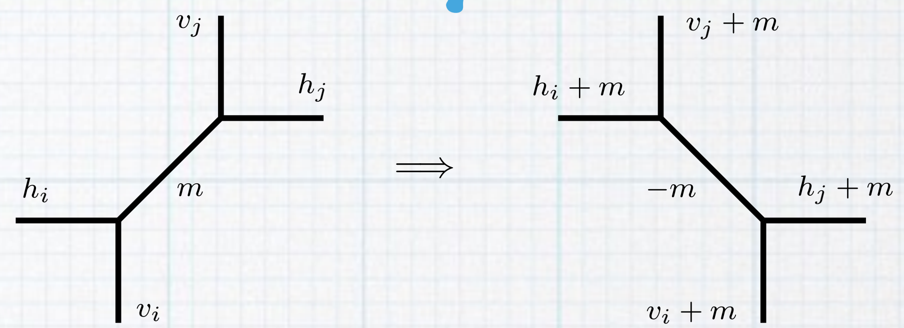

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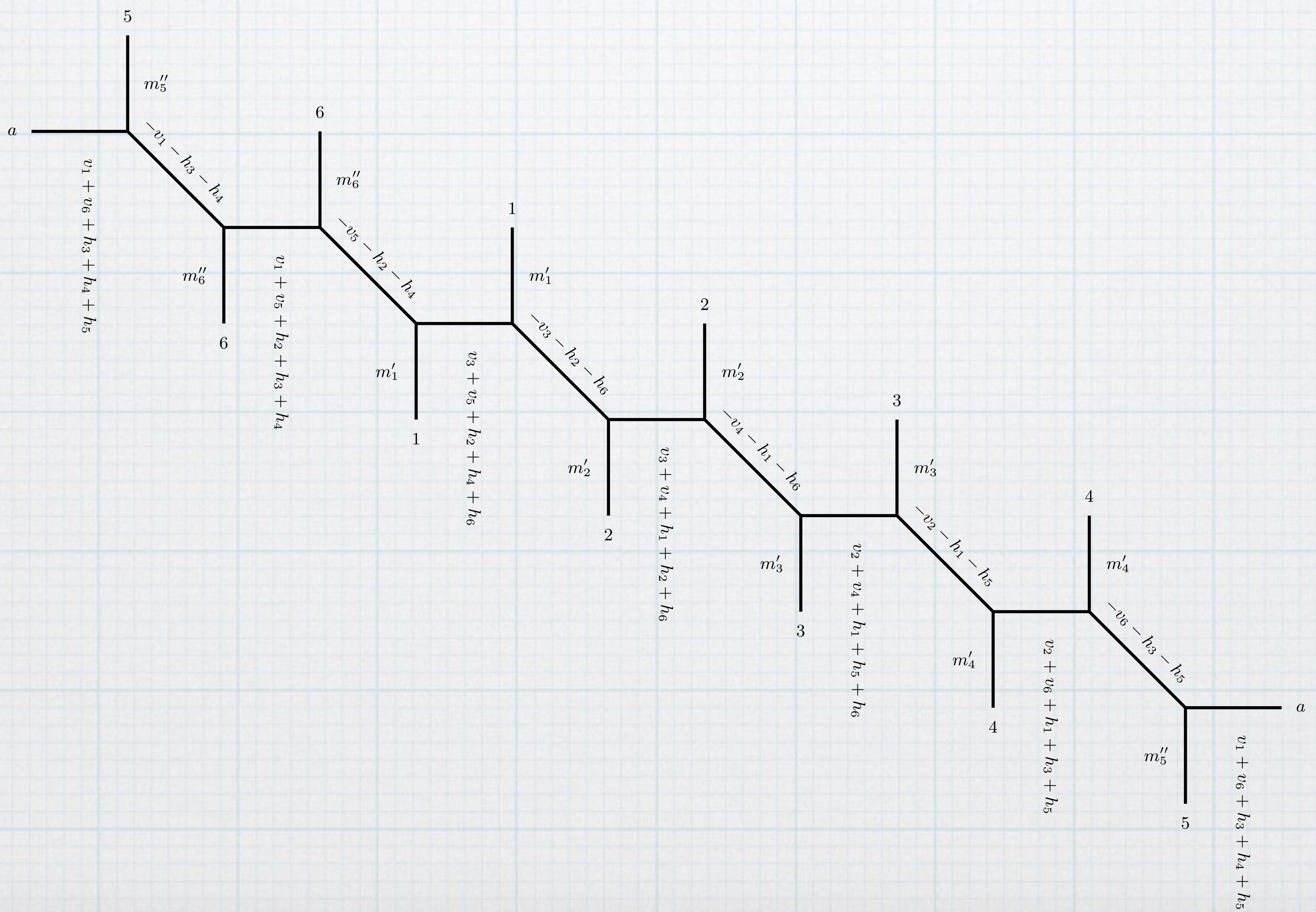

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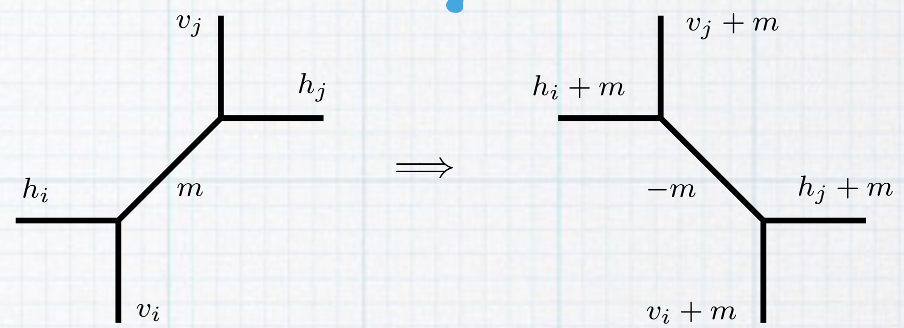

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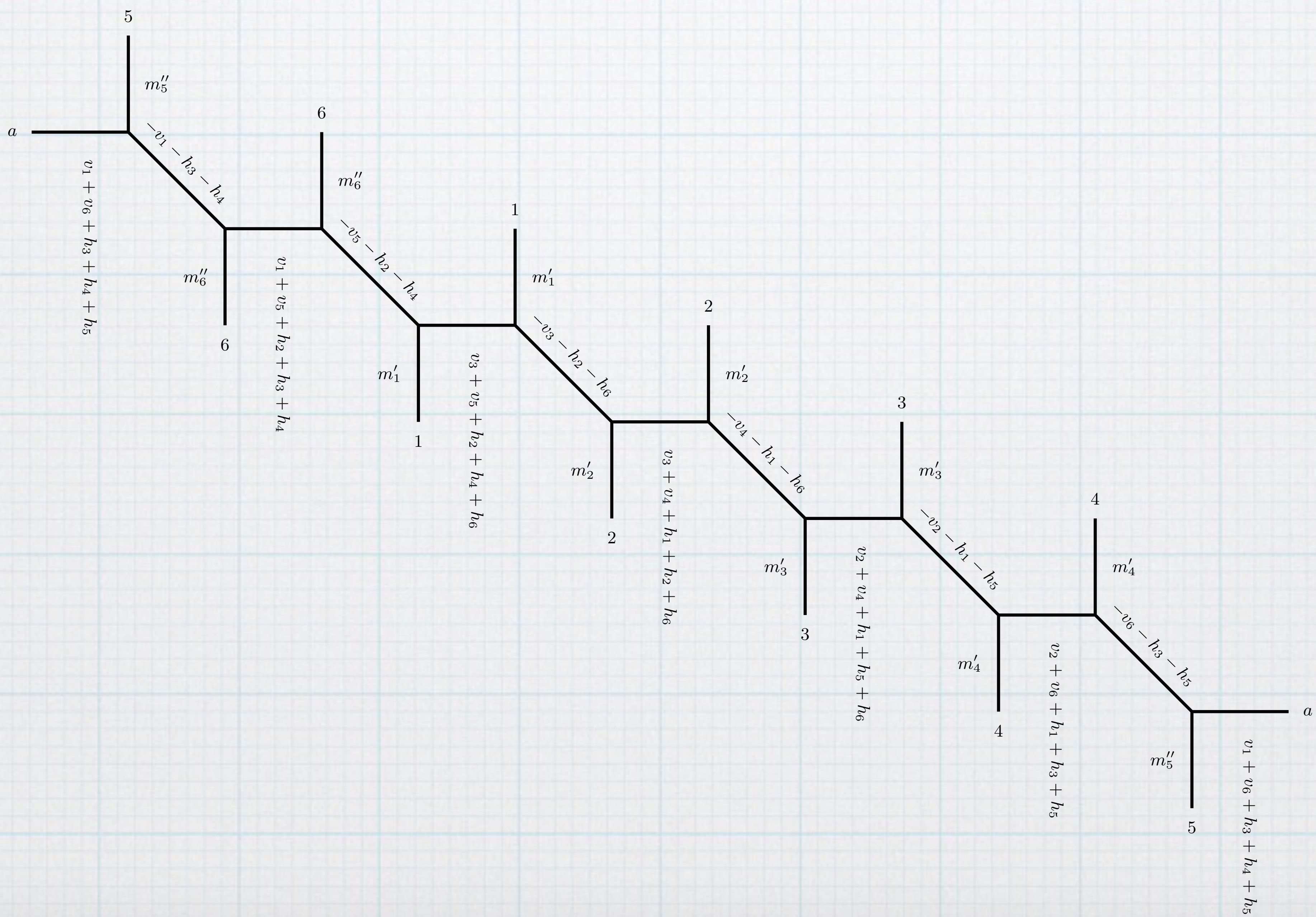

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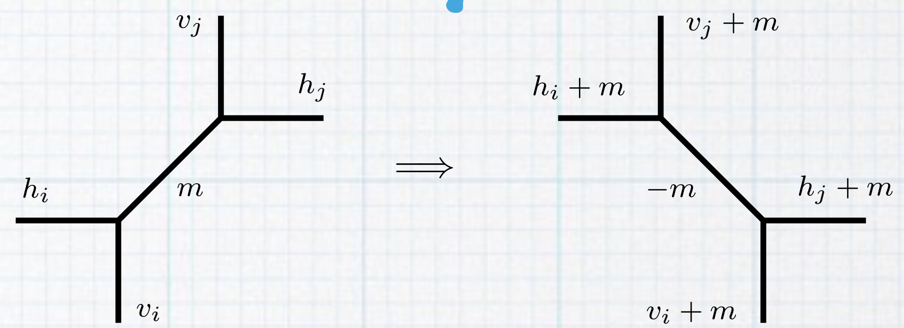

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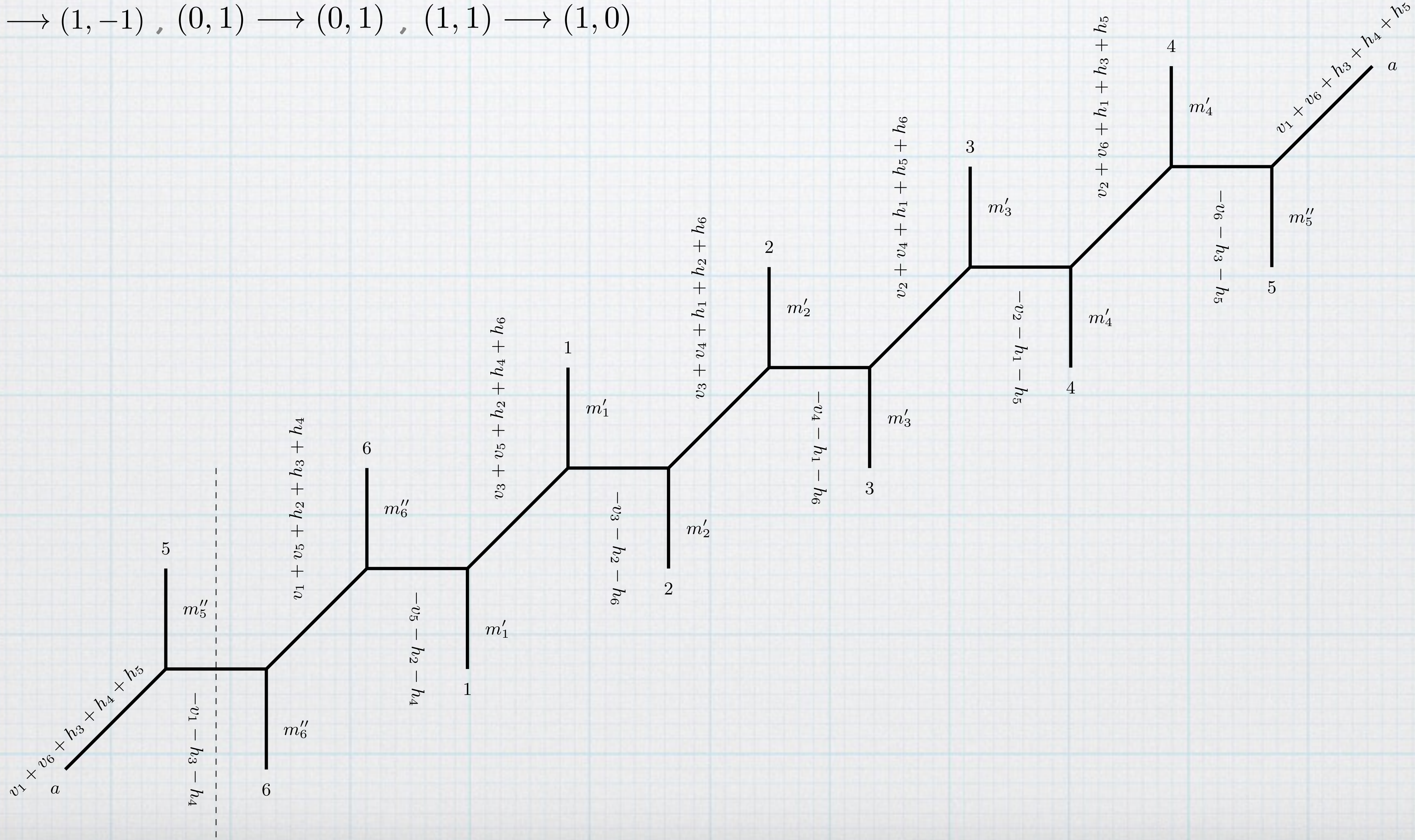

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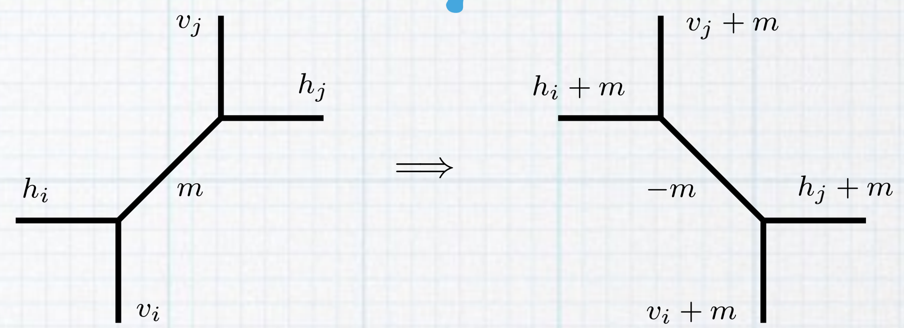

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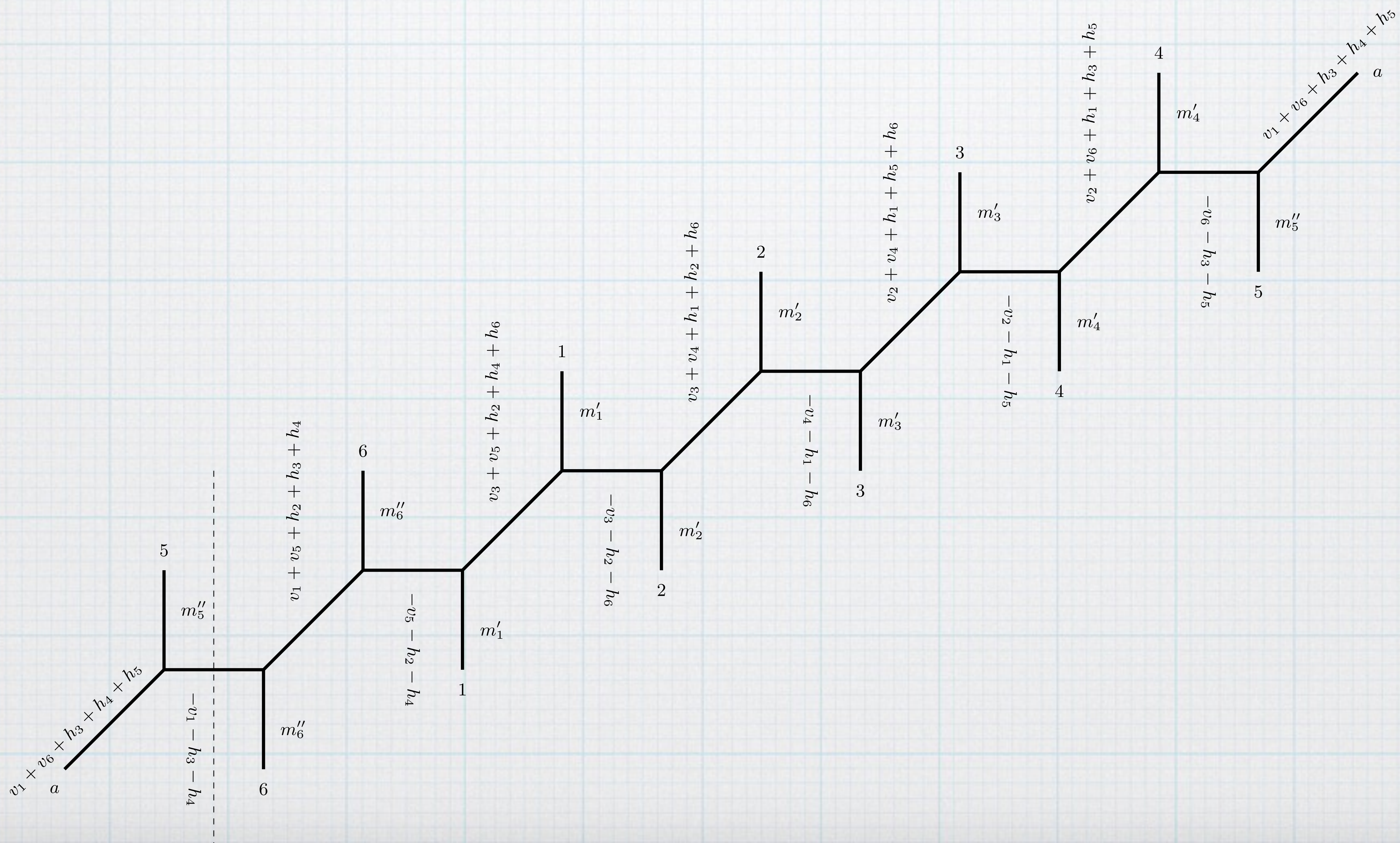

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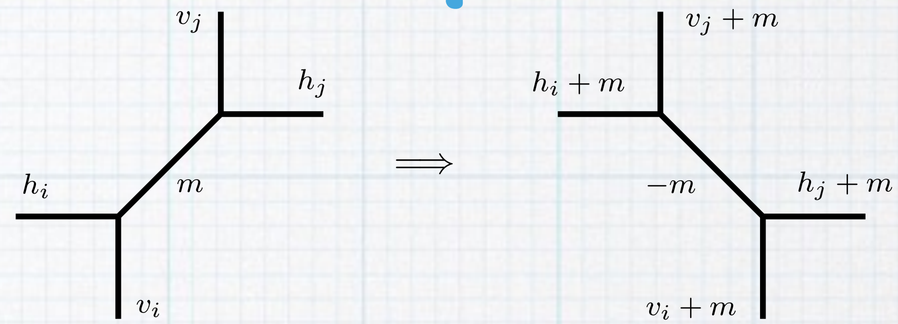

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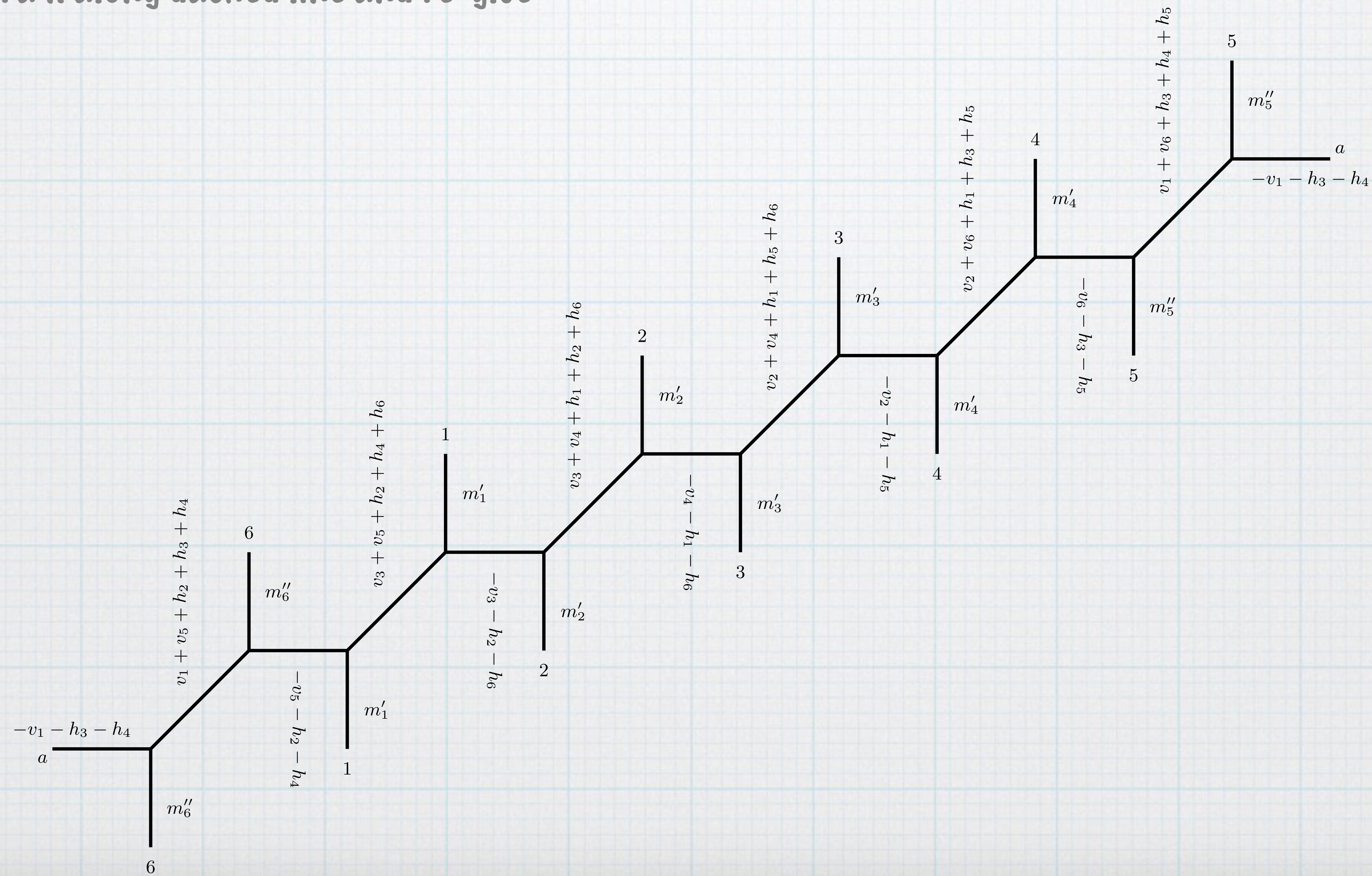
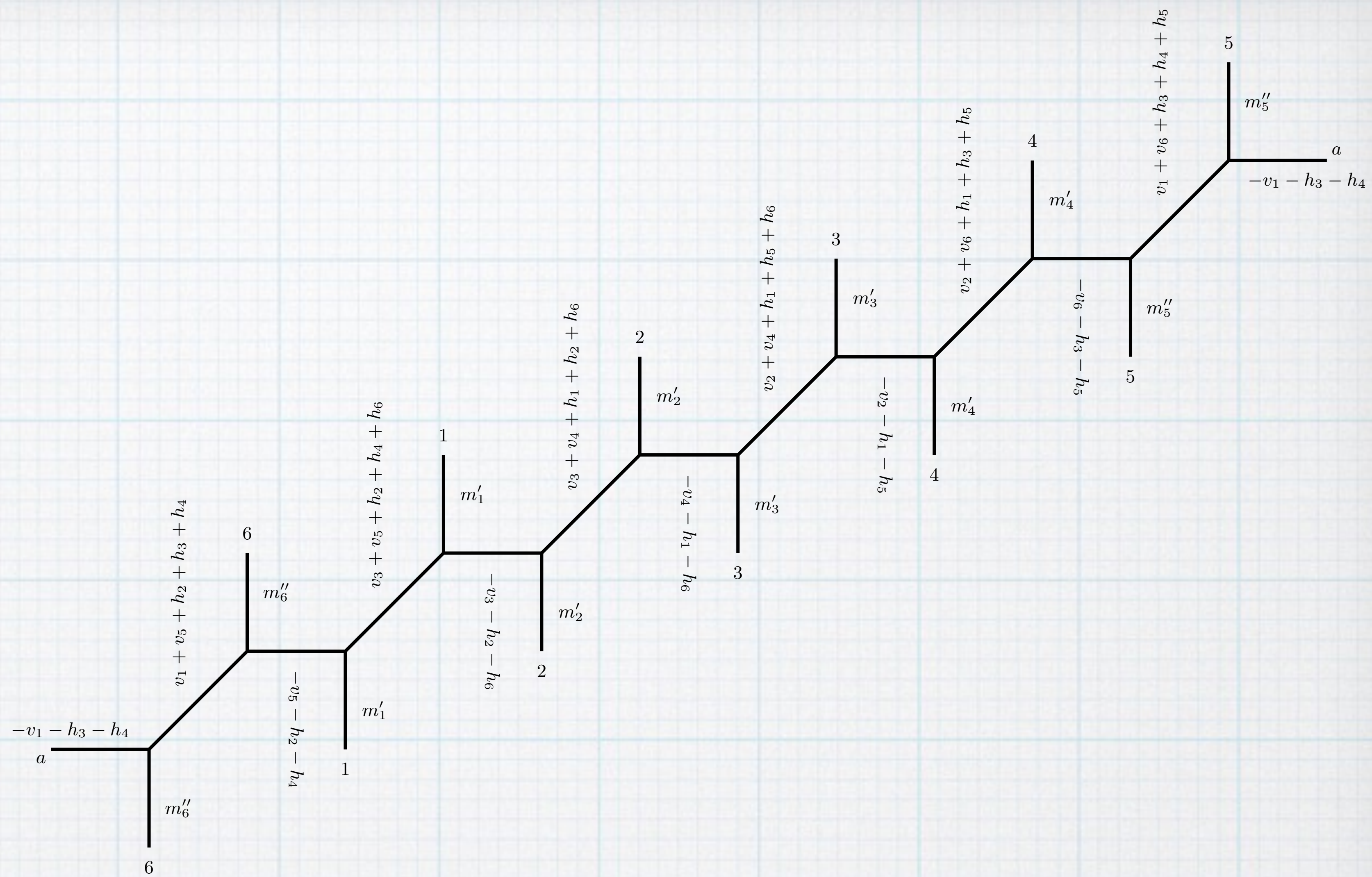
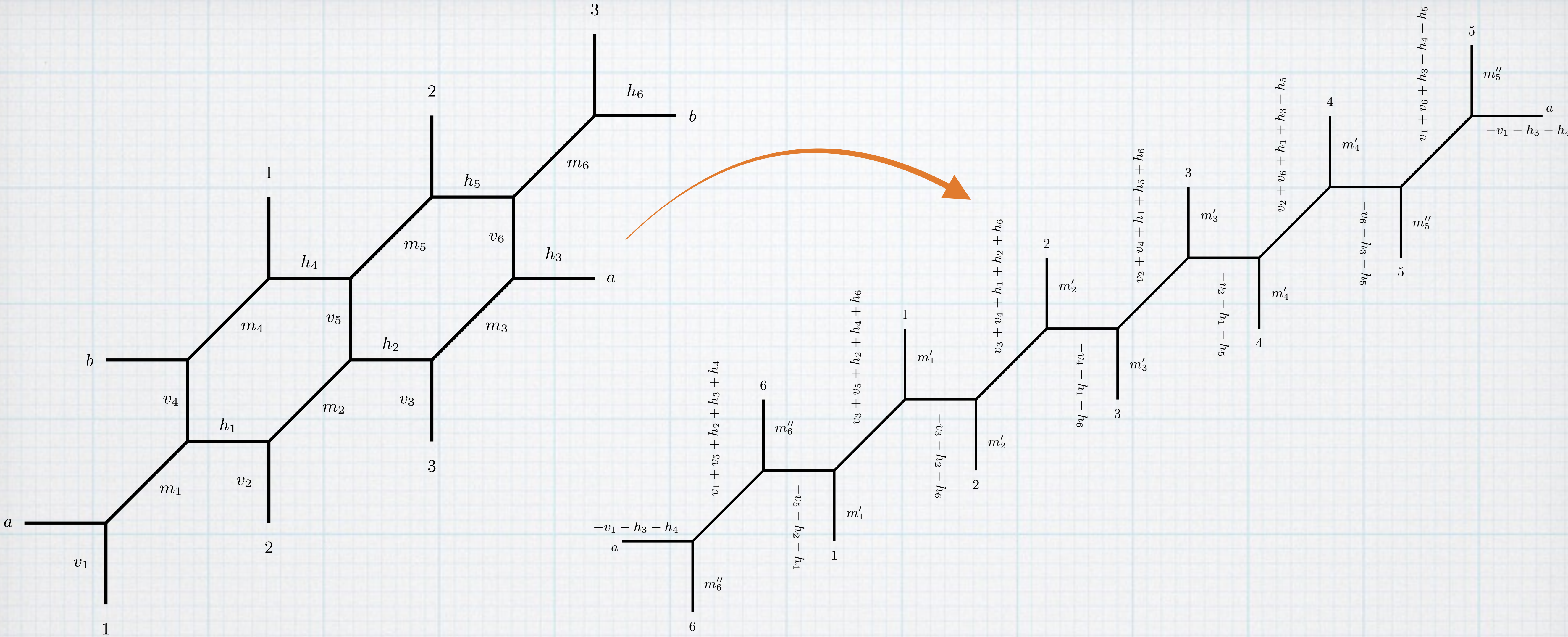




a
$-v_1-h_3-h_4$
m_6''
6
$v_1+v_5+h_2+h_3+h_4$
$-v_5-h_2-h_4$
m_1'
1
$v_3+v_5+h_2+h_4+h_6$
$-v_3-h_2-h_6$
m_2'
2
$v_3+v_4+h_1+h_2+h_6$
$-v_4-h_1-h_6$
m_3'
3
$v_2+v_4+h_1+h_5+h_6$
$-v_2-h_1-h_5$
m_4'
4
$v_2+v_6+h_1+h_3+h_5$
$-v_6-h_3-h_5$
m_5''
5
$v_1+v_6+h_3+h_4+h_5$



h_6
b
h_5
m_6
h_4
m_5
v_6
h_3
a
m_4
v_5
h_2
m_3
b
v_4
v_3
h_1
m_2
m_1
v_2
a
v_1
$-v_1 - h_3 - h_4$
a
m_6''
$v_1 + v_5 + h_2 + h_3 + h_4$
$-v_5 - h_2 - h_4$
m_6''
m_1'
$v_3 + v_5 + h_2 + h_4 + h_6$
m_1'
$-v_3 - h_2 - h_6$
m_2'
$v_3 + v_4 + h_1 + h_2 + h_6$
m_2'
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m_5''
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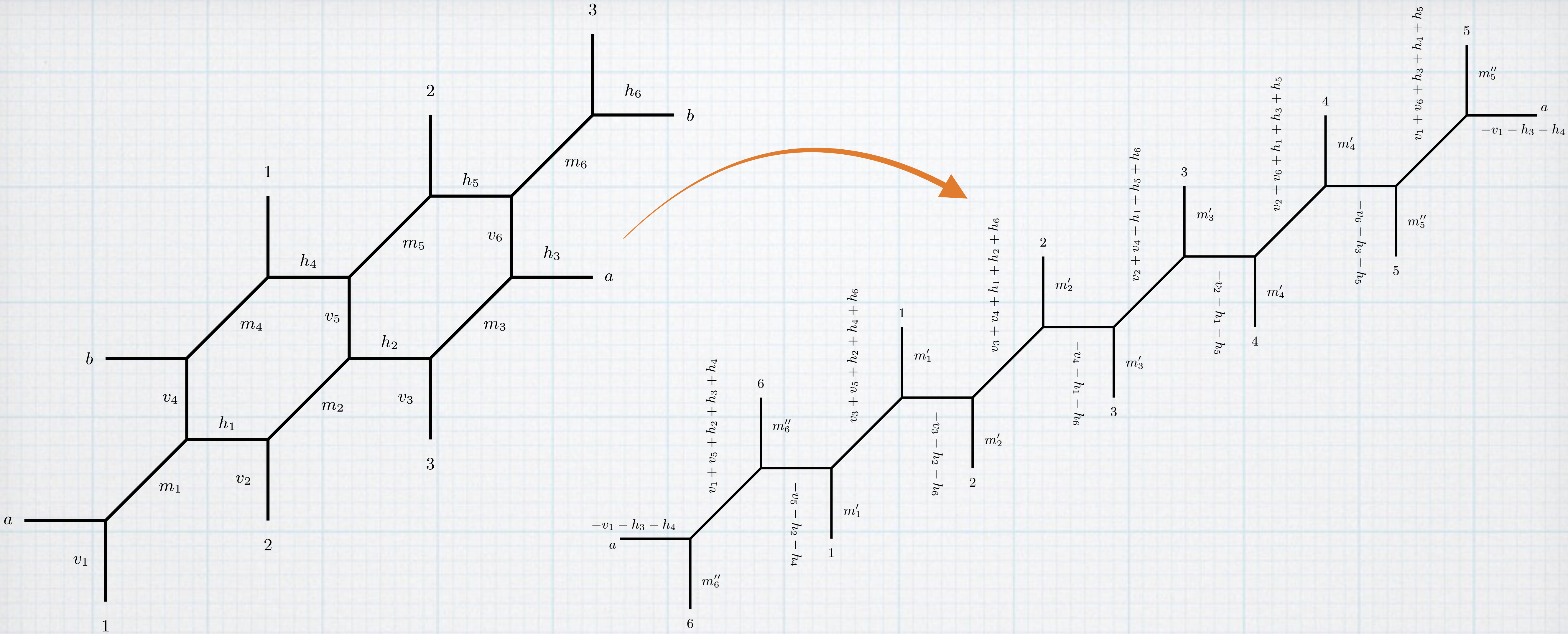
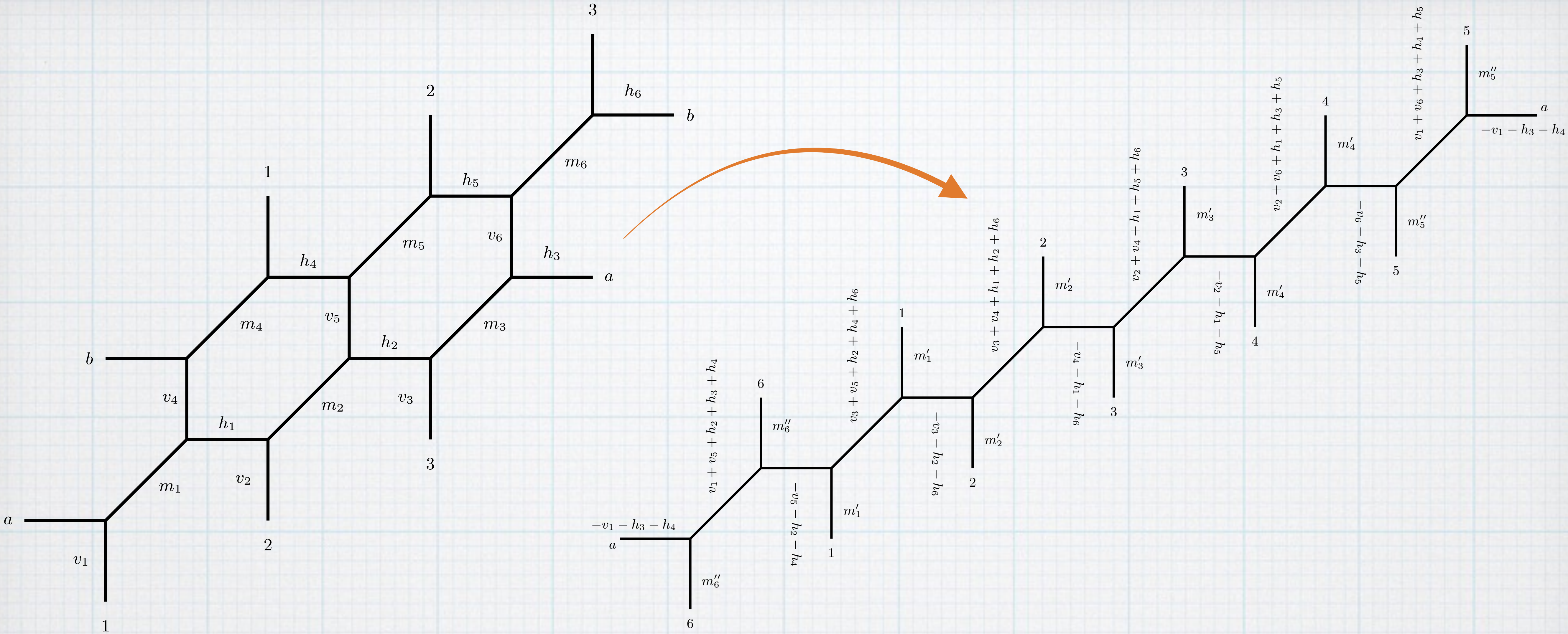
m_5''
a
$-v_1 - h_3 - h_4$



Duality leaves partiton function invariant

$$\mathcal{Z}_{3,2}(\{h\},\{v\},\{m\},\epsilon_{1,2}) = \mathcal{Z}_{6,1}(\{h'\},\{v'\},\{m'\},\epsilon_{1,2})$$

[Bastian, SH, Iqbal, Rey 2017]



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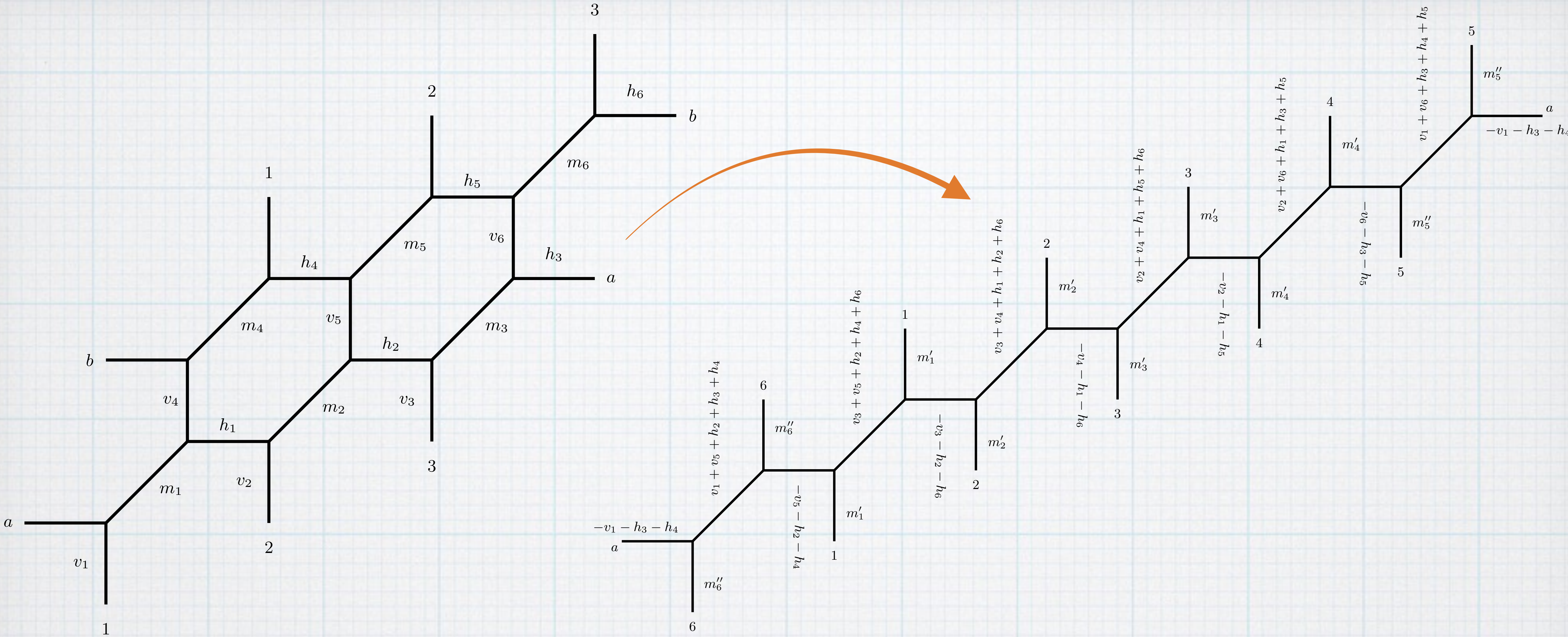
Kähler parameters implied by duality transformation



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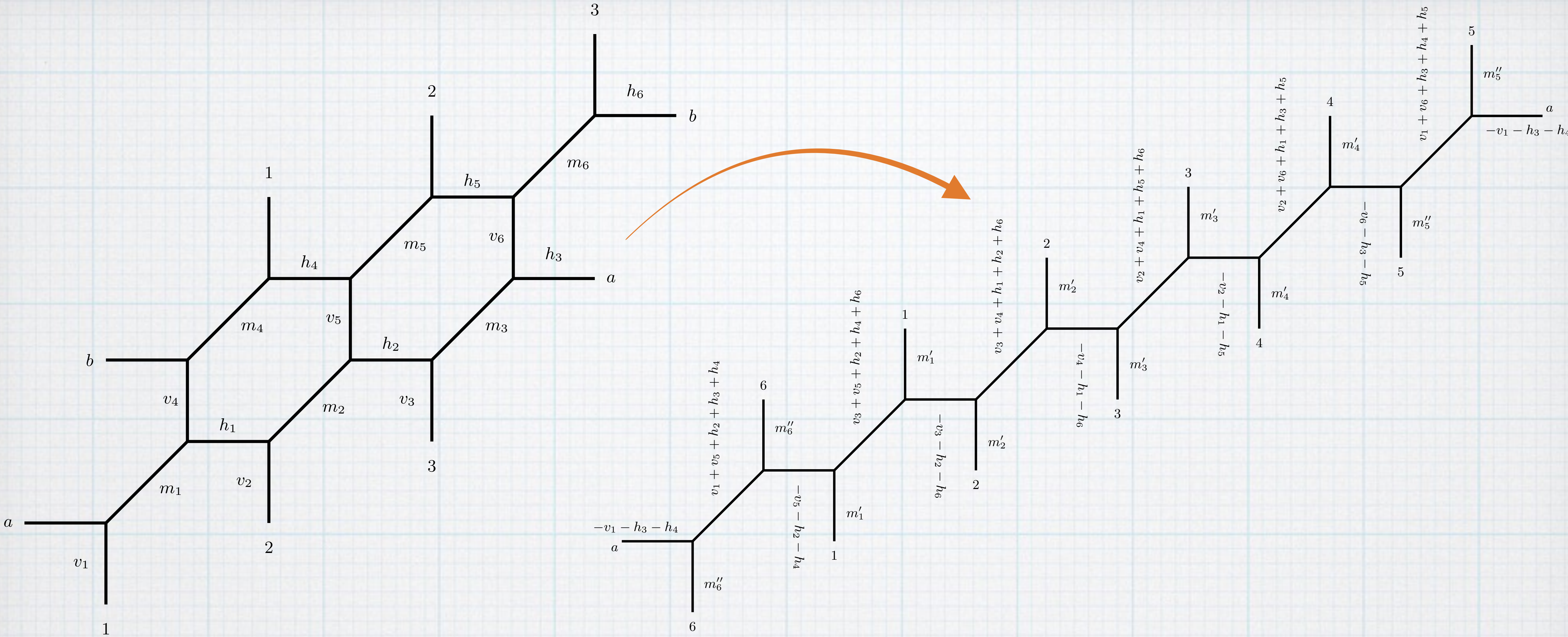
Vertical expansion of $\mathcal{Z}_{6,1}$ gives rise to a gauge theory with gauge group $U(6)$ and part. fct. $\mathcal{Z}^{(6,1)}_{\text{vert}}$



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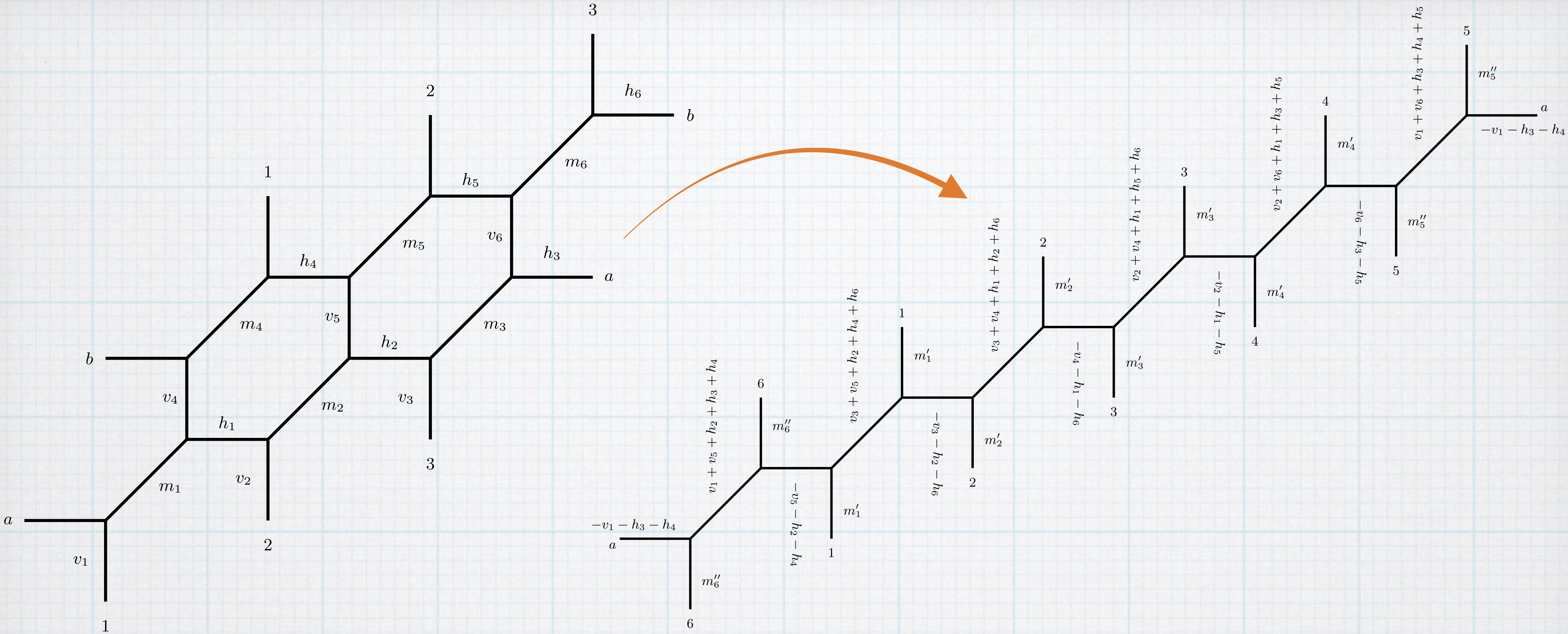

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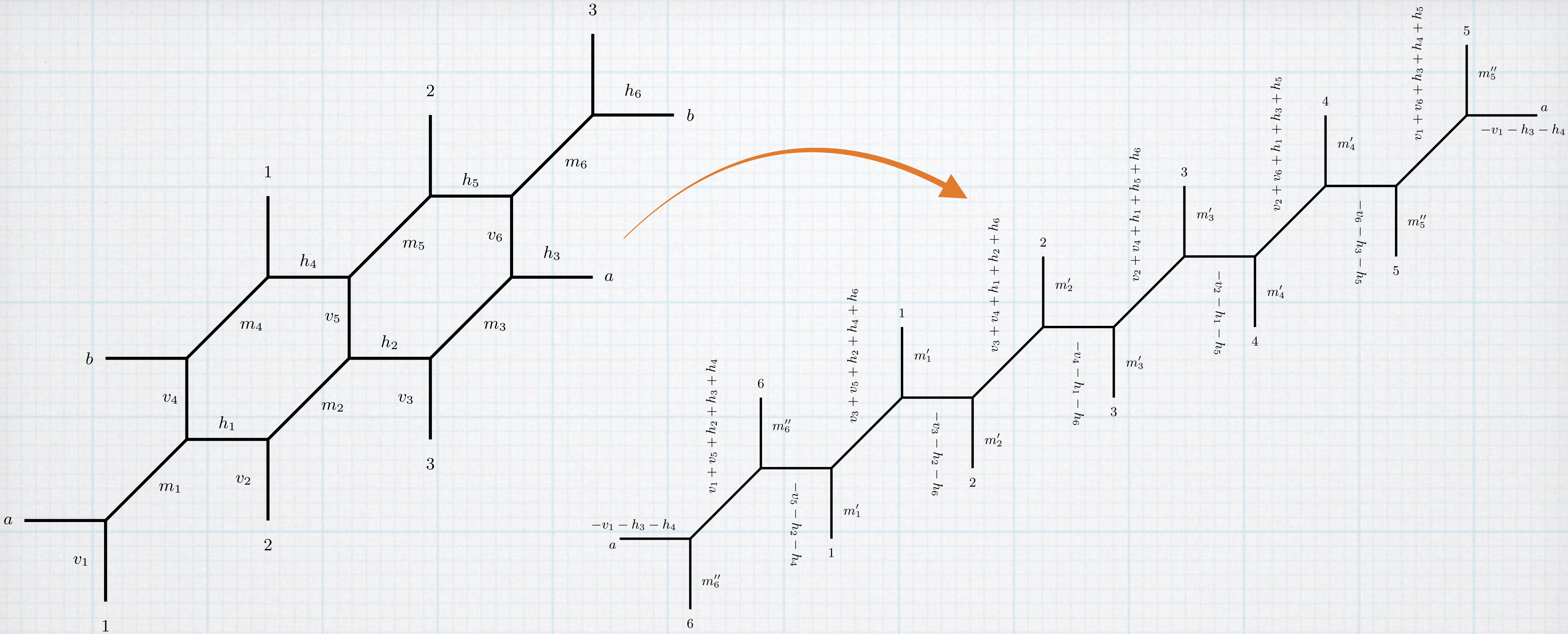
related to coupling constant of $\mathcal{Z}^{(3,2)}_{\text{diag}}$



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Symmetry transformations do not flop any curve whose area is proportional to V

$\Longrightarrow$ partition functions $\mathcal{Z}^{(3,2)}_{\text{diag}}$ and $\mathcal{Z}^{(6,1)}_{\text{vert}}$ have same asymptotic expansion

Generalisation to (N,M)

Duality conjectured to hold for generic (N, M)

$$X_{N,M} \sim X_{NM/k,k} \qquad \text{where} \qquad k = \gcd(N, M)$$

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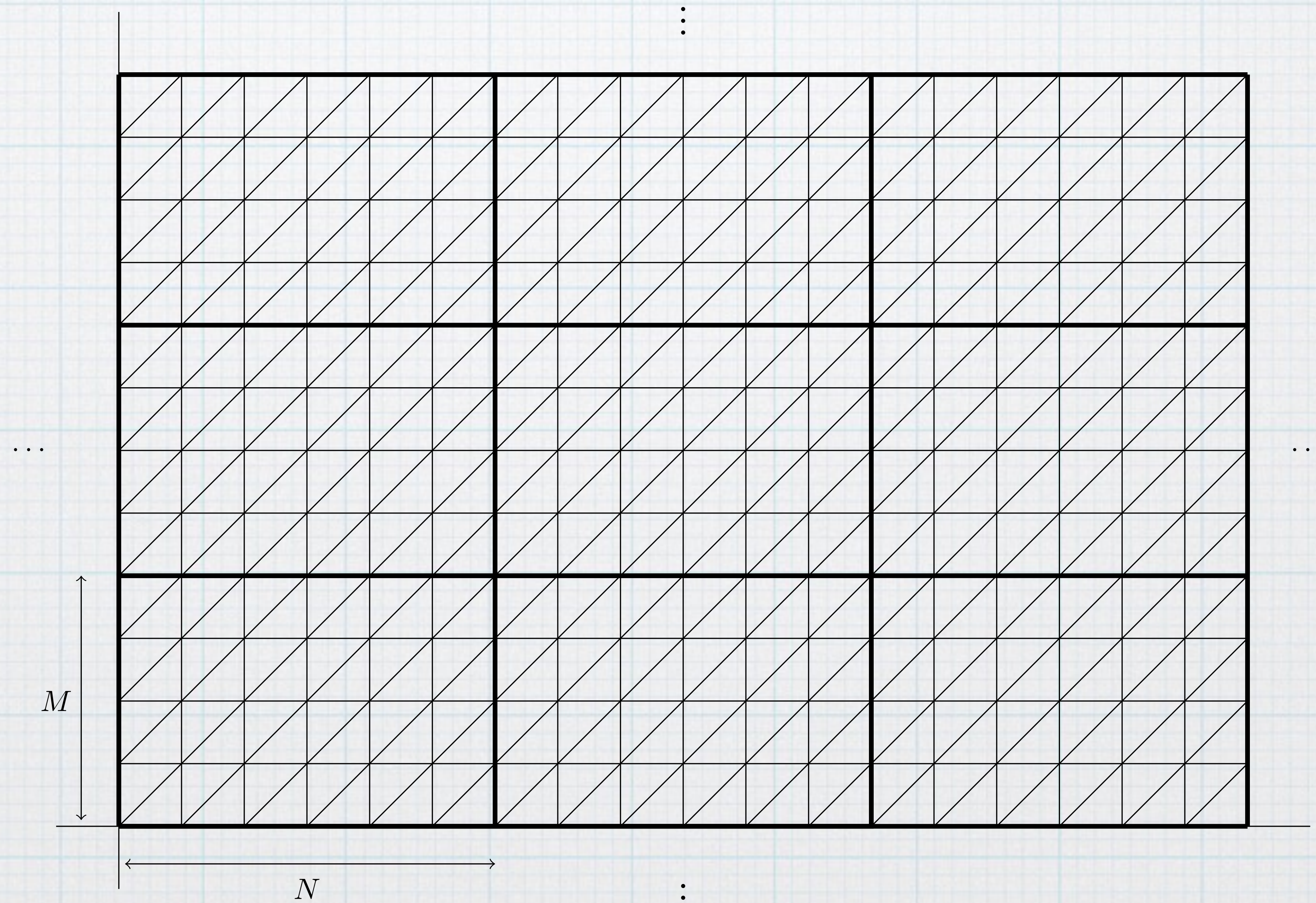
Newton polygon (dual of the topic web diagram)



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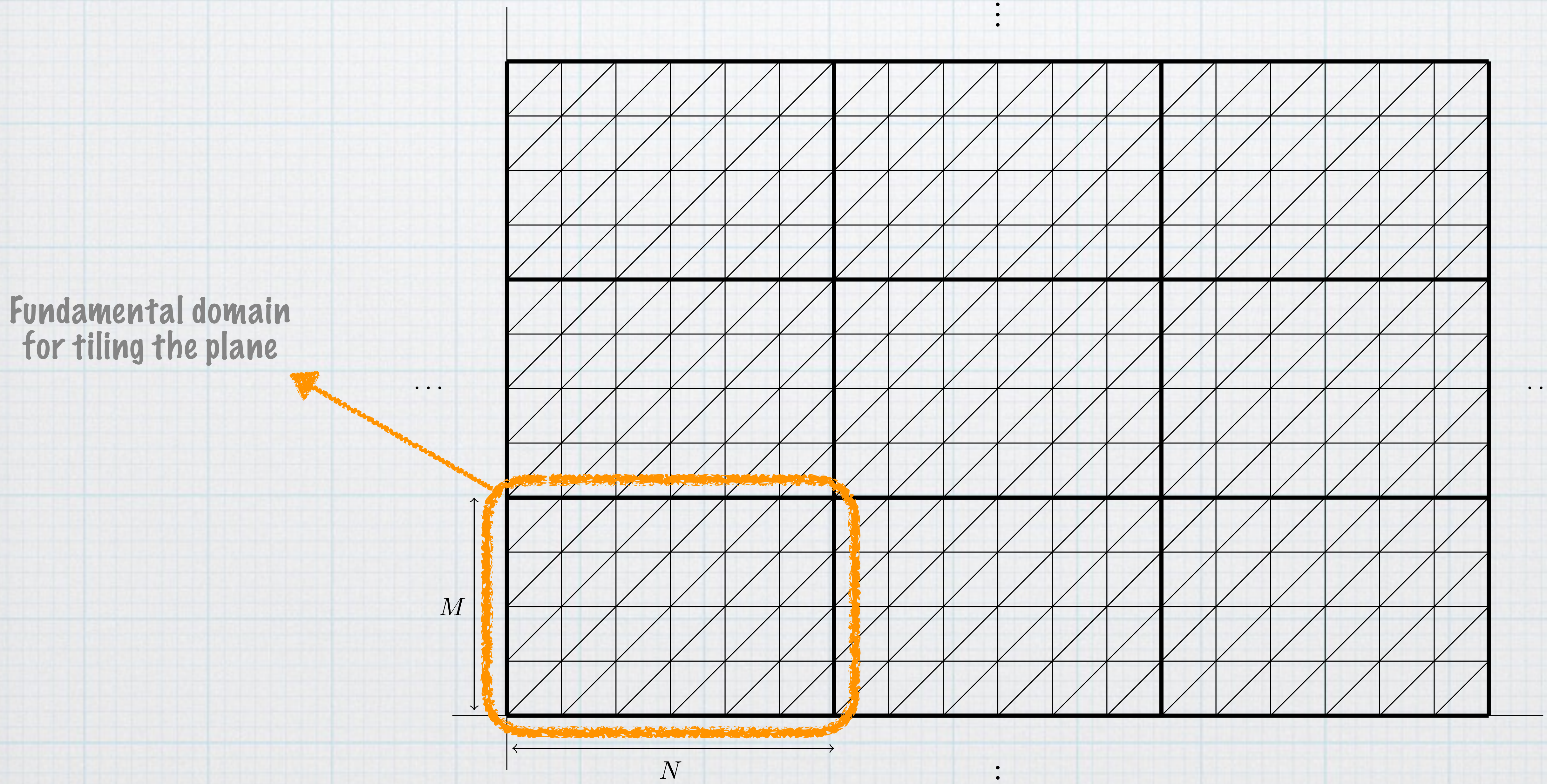

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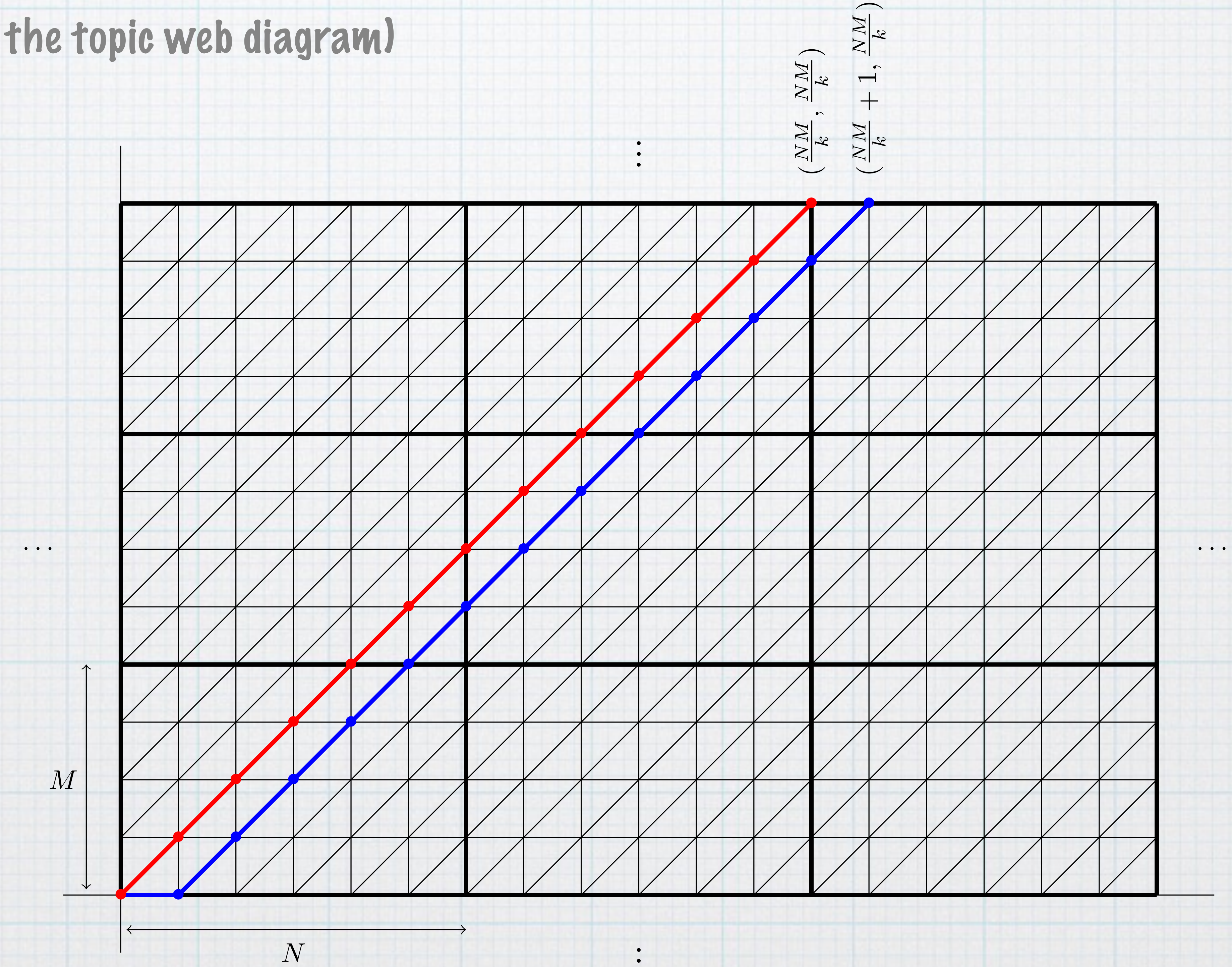

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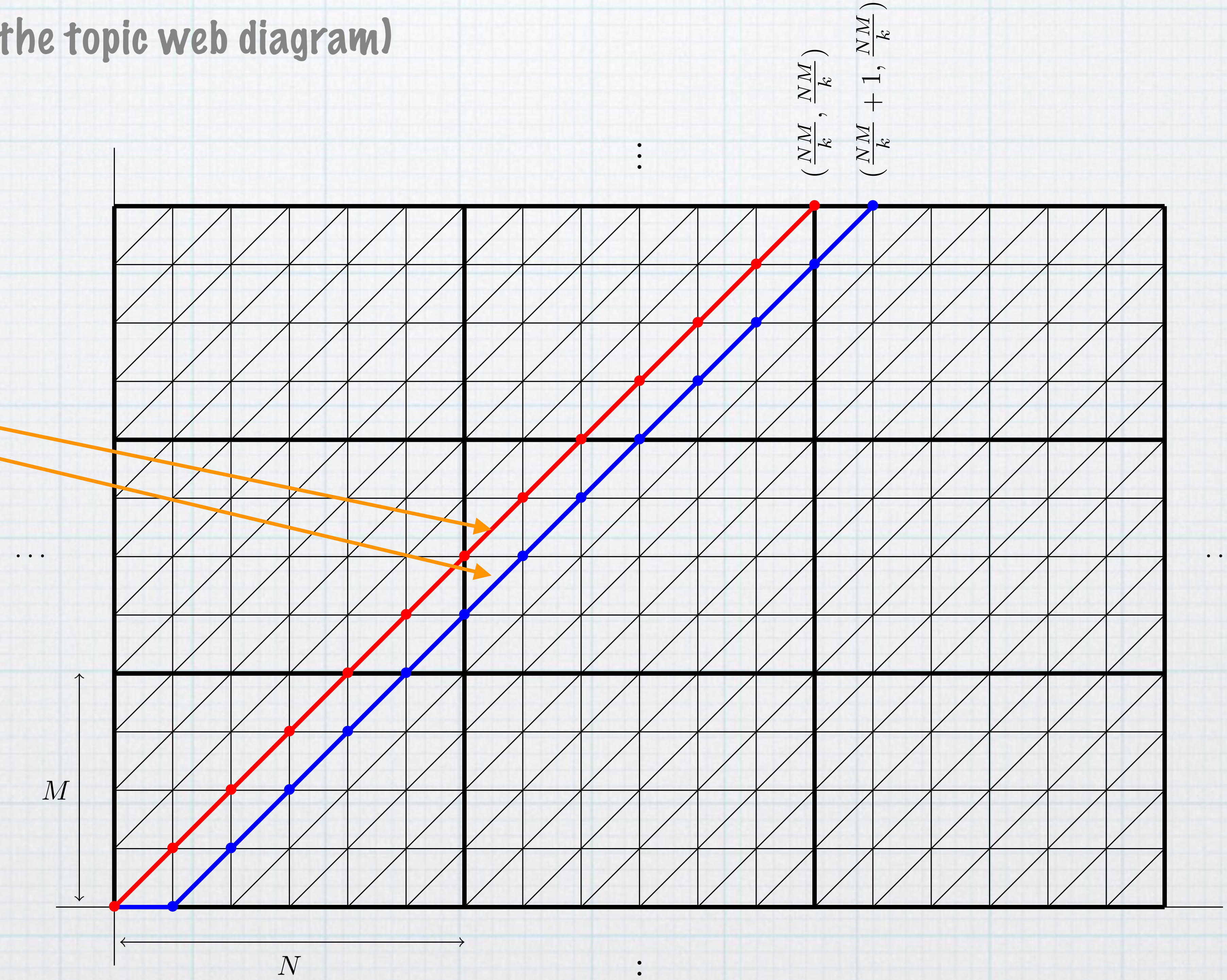
equivalent tiling of the plane by these two lines, which visit every inequivalent point exactly once



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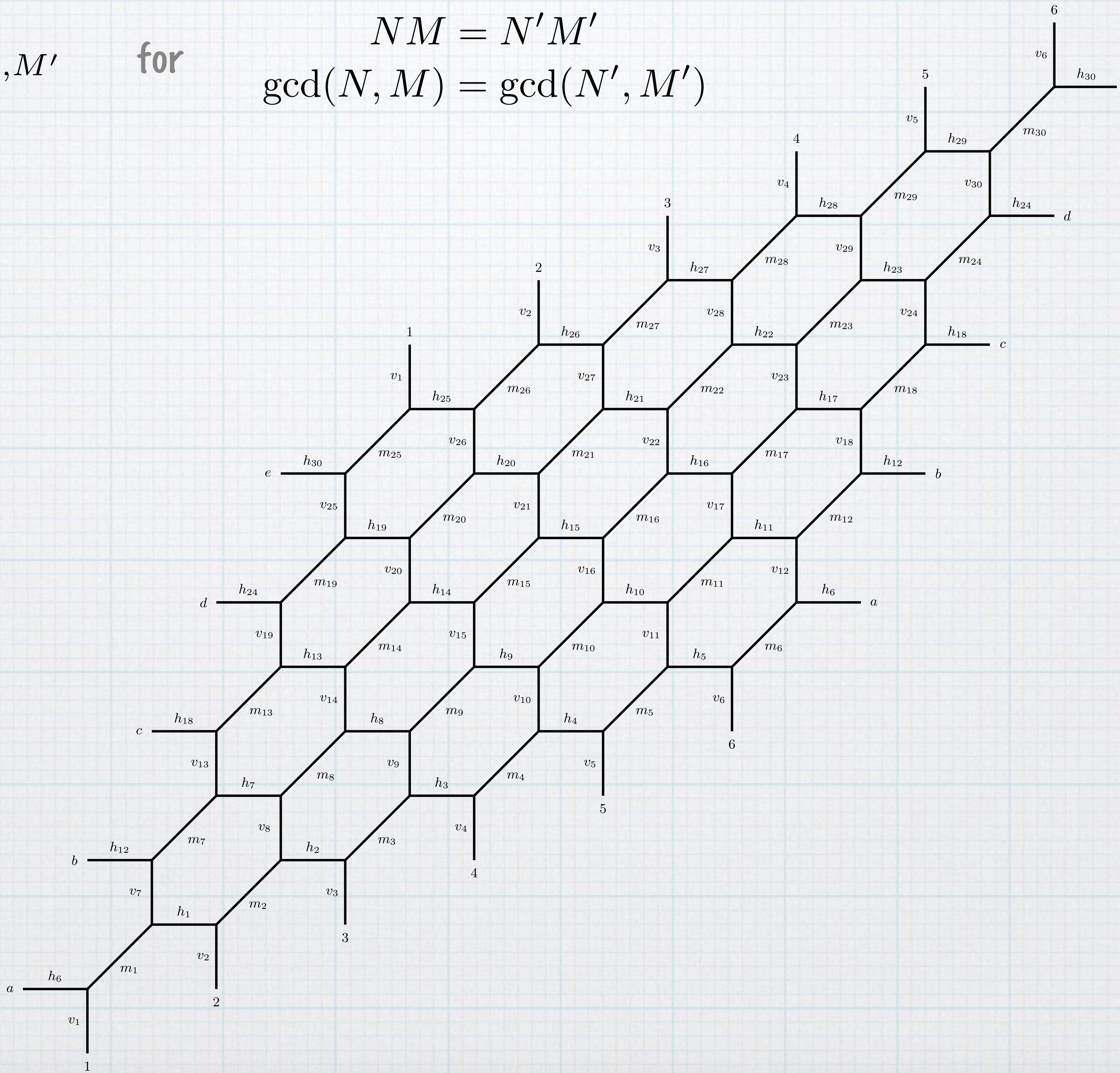

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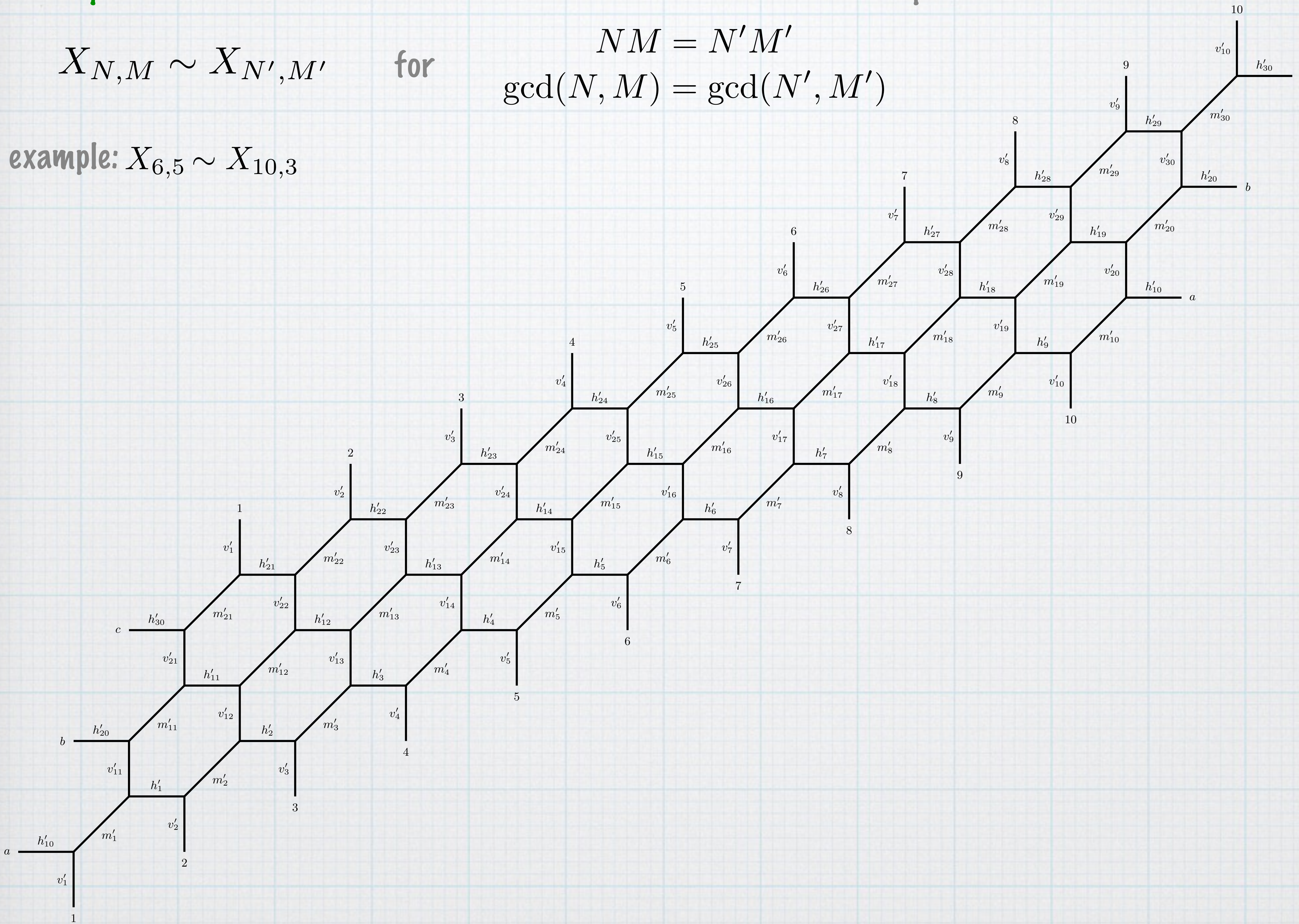

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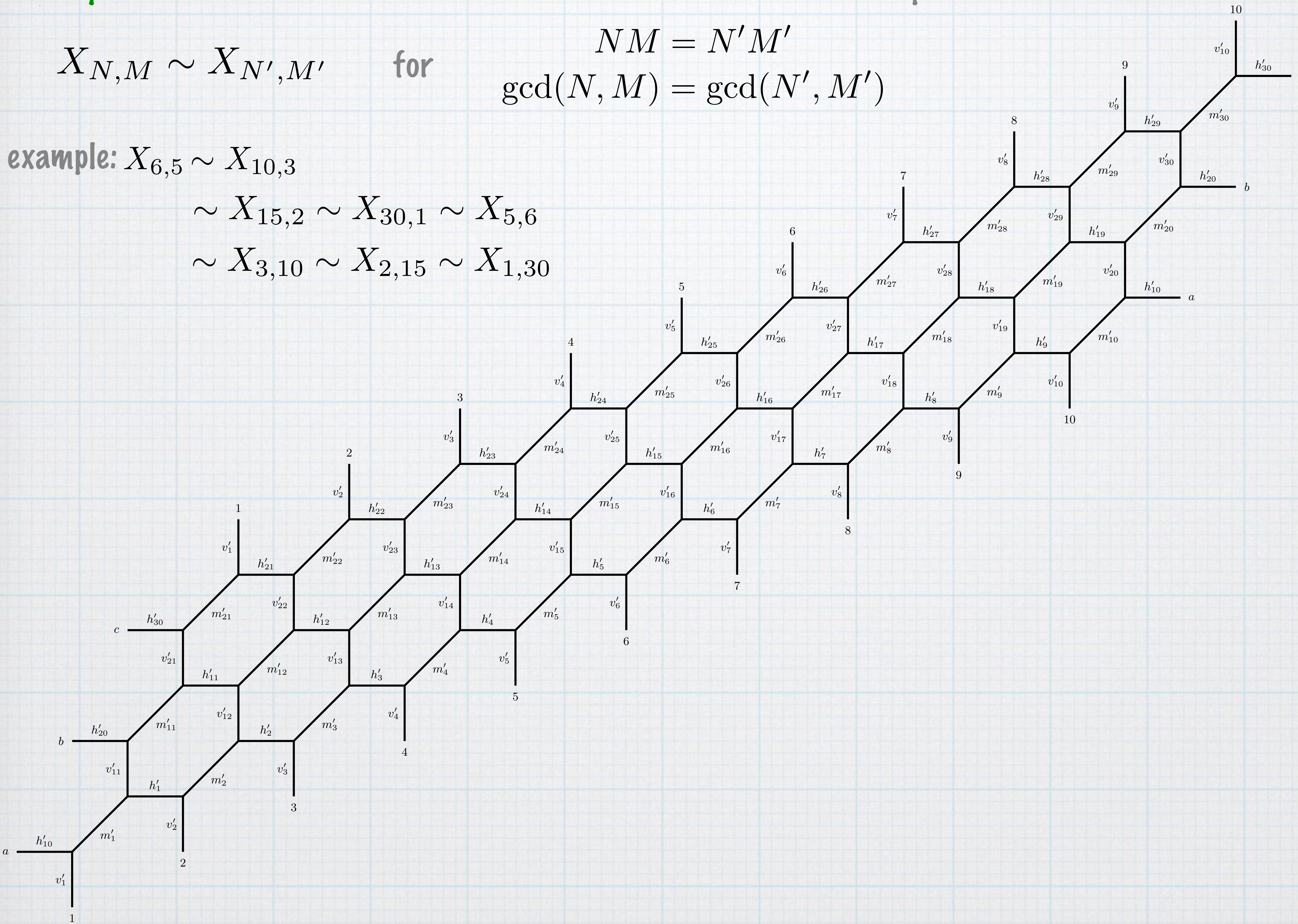
$$\sim X_{3,10} \sim X_{2,15} \sim X_{1,30}$$



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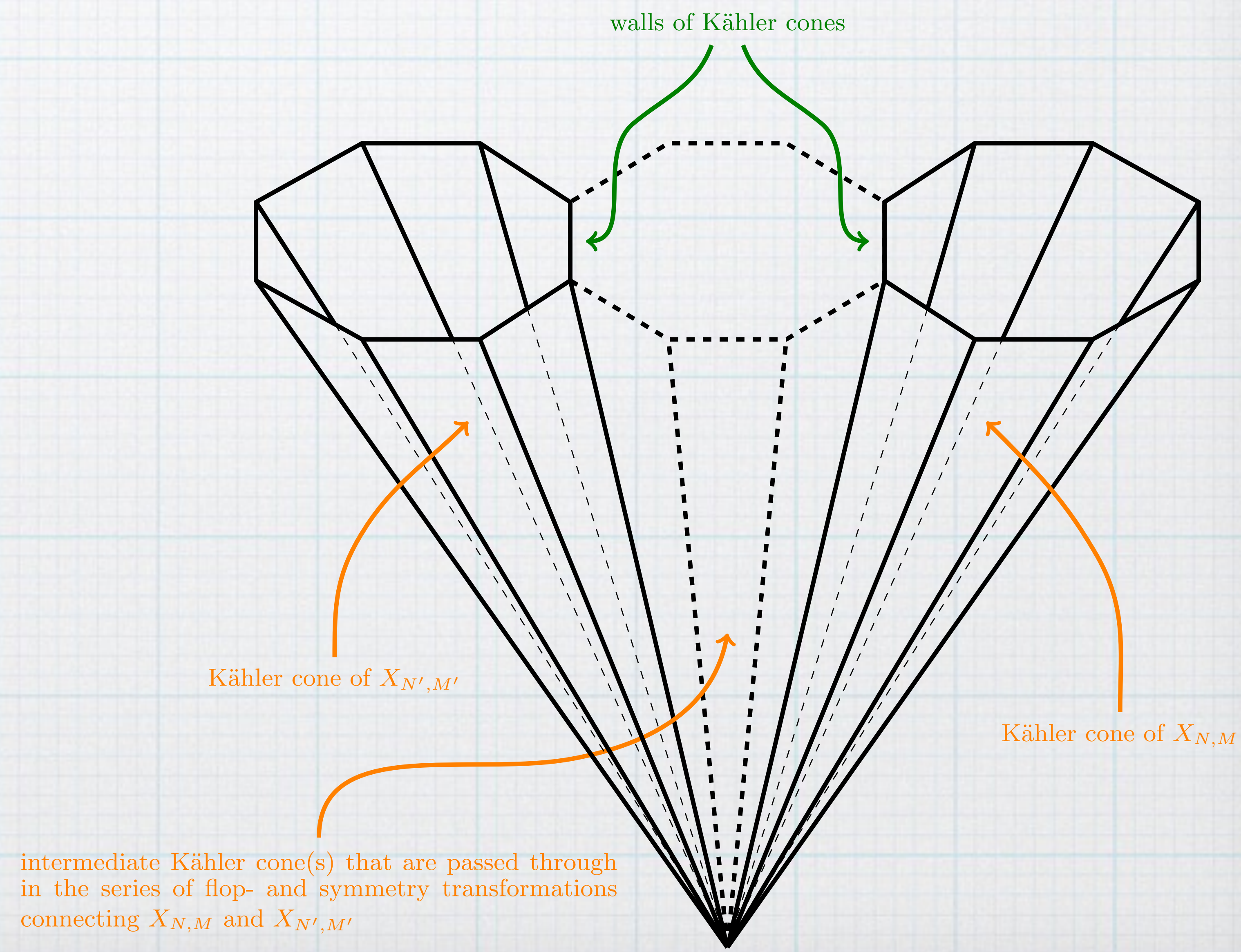

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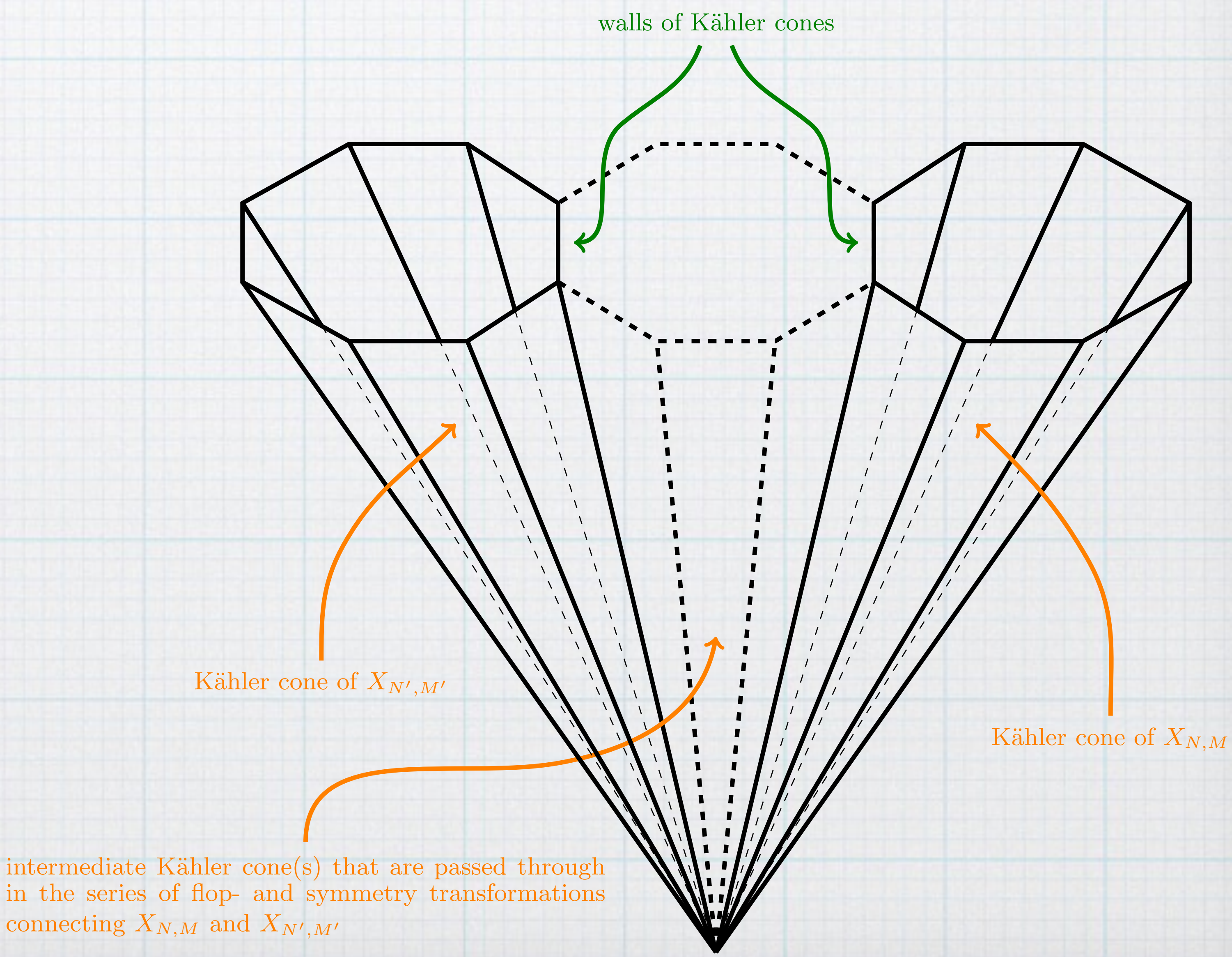

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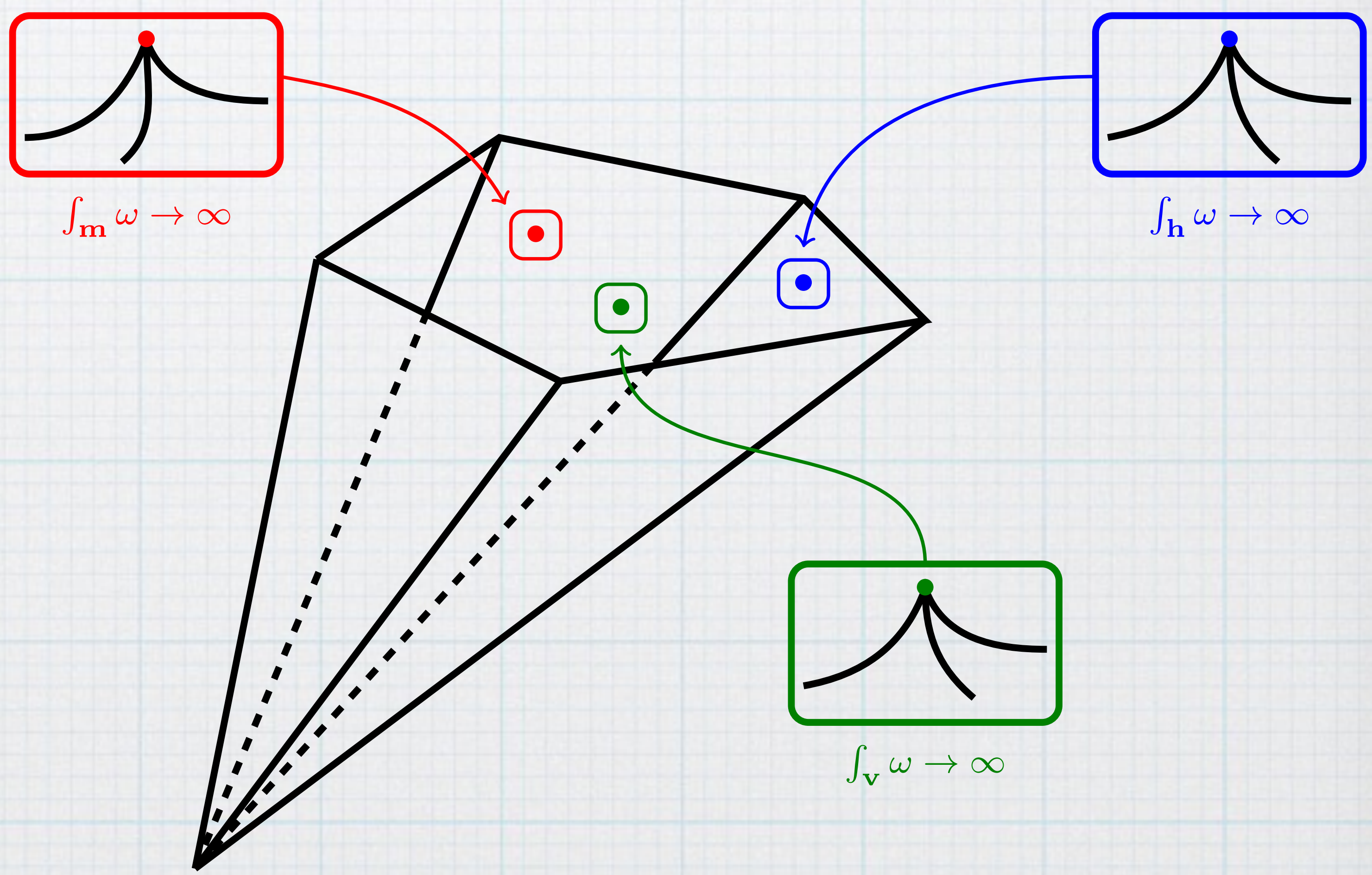

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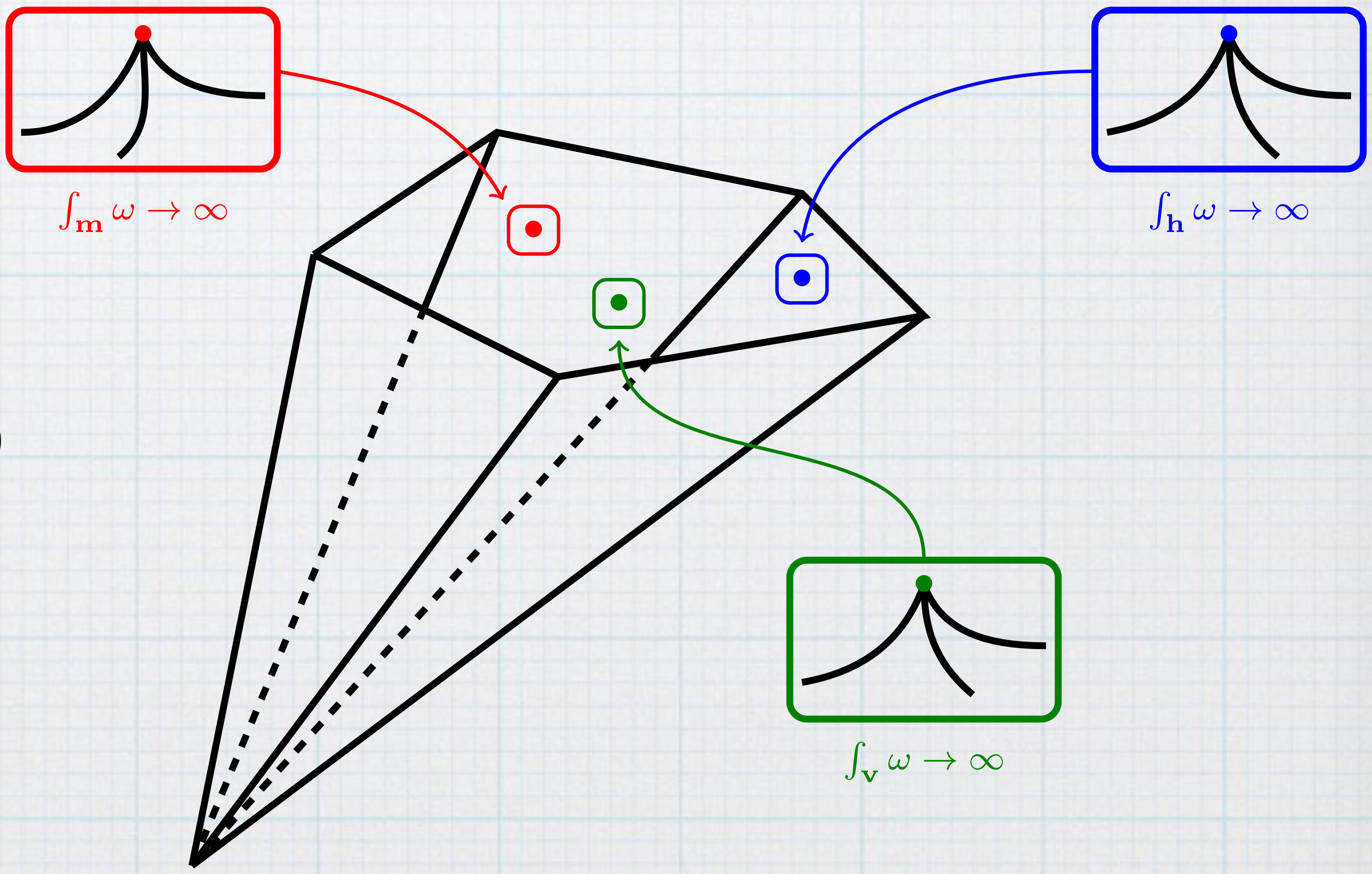

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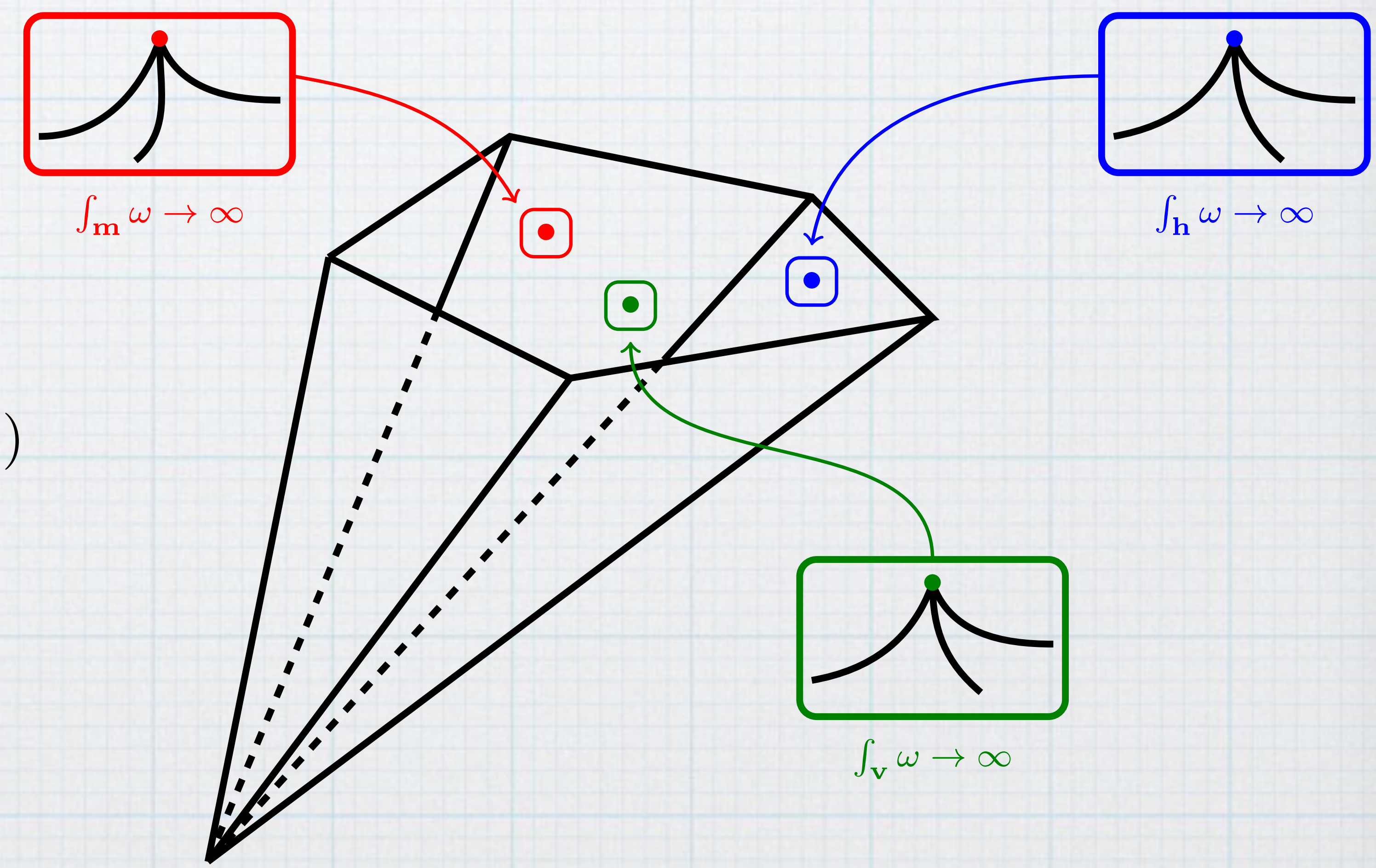

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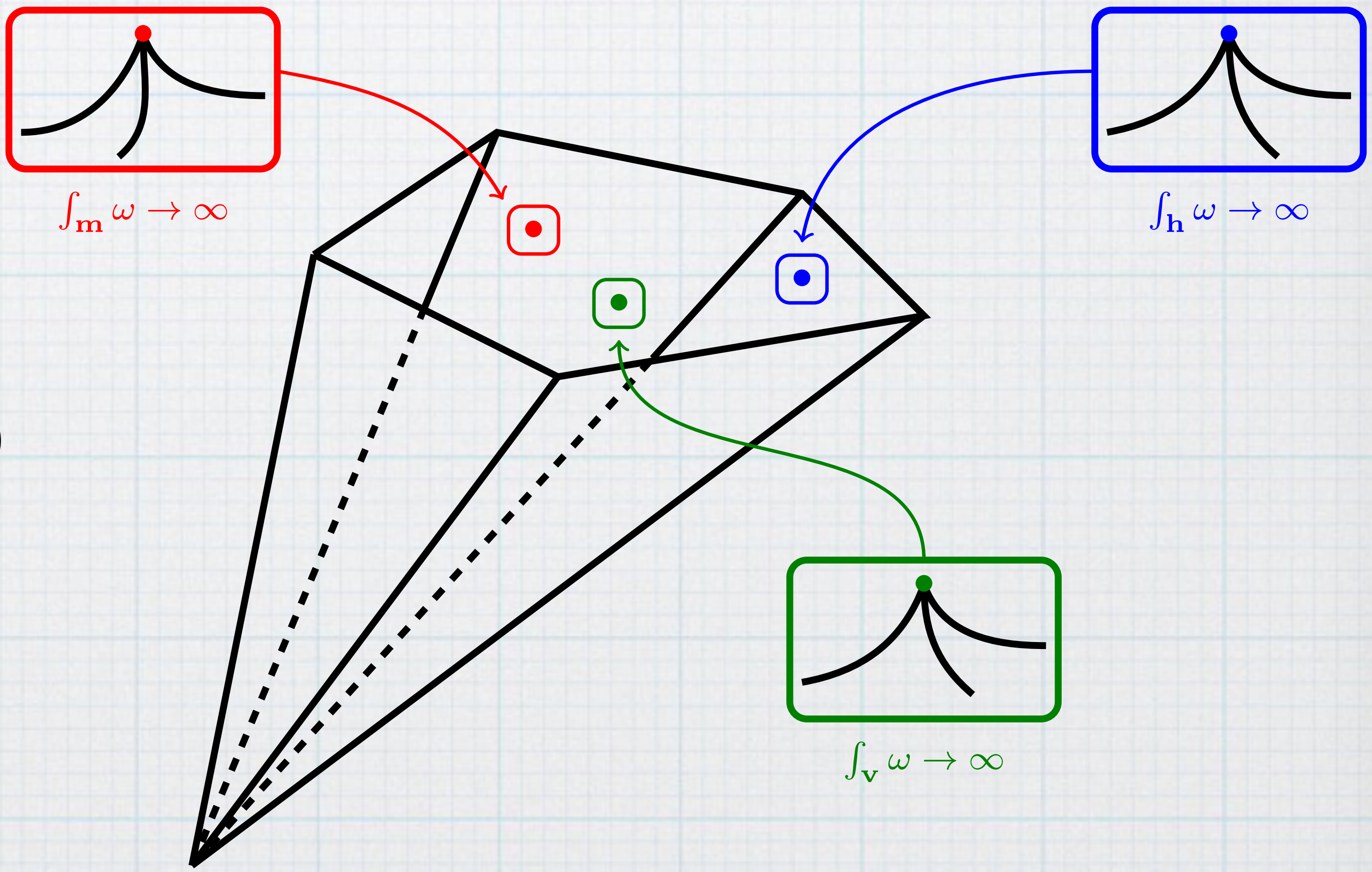

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Web of dualities among different theories can be turned into symmetries for individual theories

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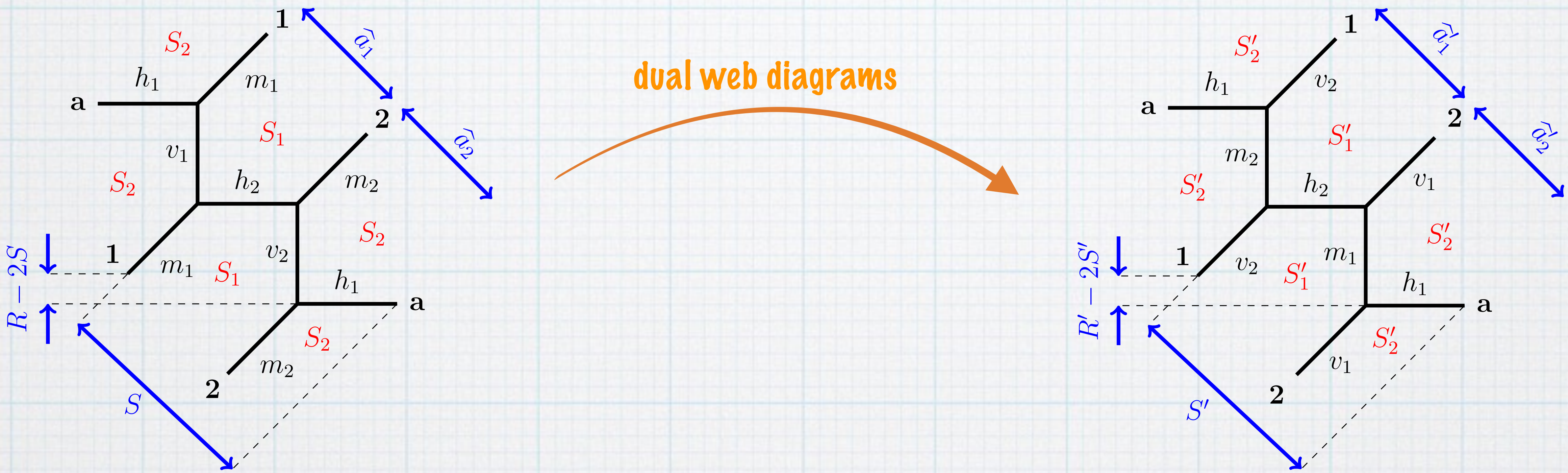

$$\widehat{a}_1 = v_1 + h_2\,, \qquad \widehat{a}_2 = v_2 + h_1\,,$$
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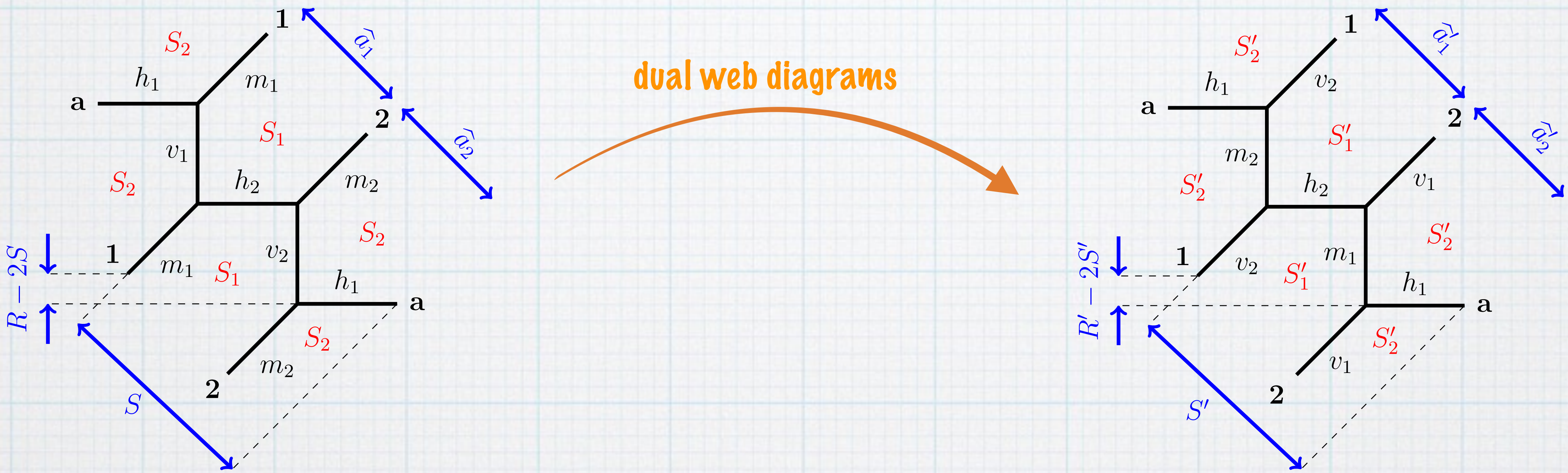

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Implies the following symmetry of the partition function:

$$\begin{pmatrix} \widehat{a}_1 \\ \widehat{a}_2 \\ S \\ R \end{pmatrix} = G_1 \cdot \begin{pmatrix} \widehat{a}'_1 \\ \widehat{a}'_2 \\ S' \\ R' \end{pmatrix}$$

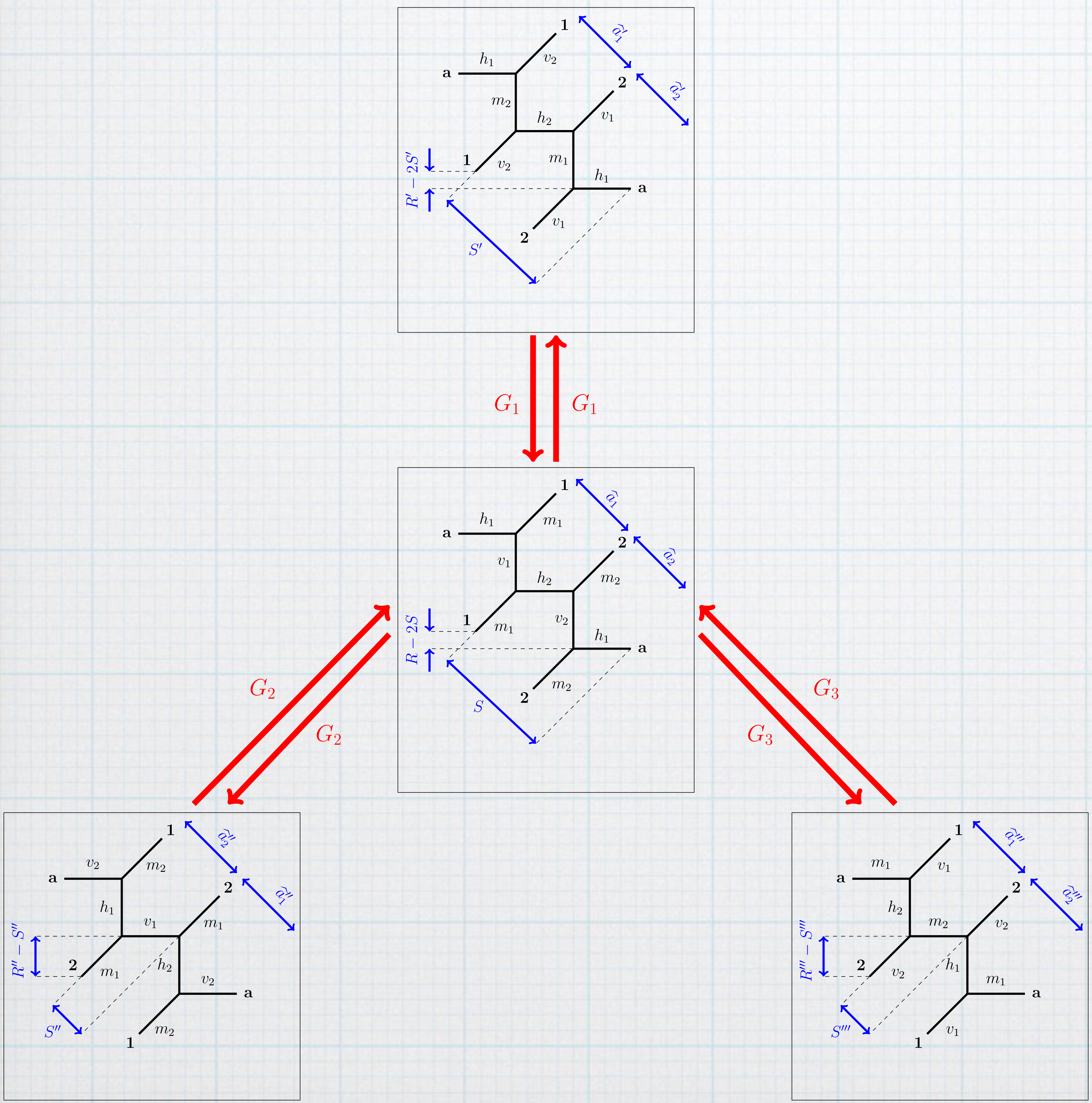
where

$$G_1 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$\det G_1 = 1$$
$$G_1 \cdot G_1 = 1\!\!1_{4\times 4}$$

Generalising to include other duality transformations:



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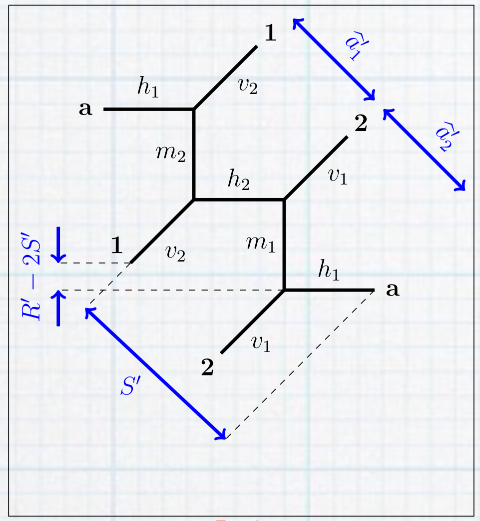

G_1 G_1

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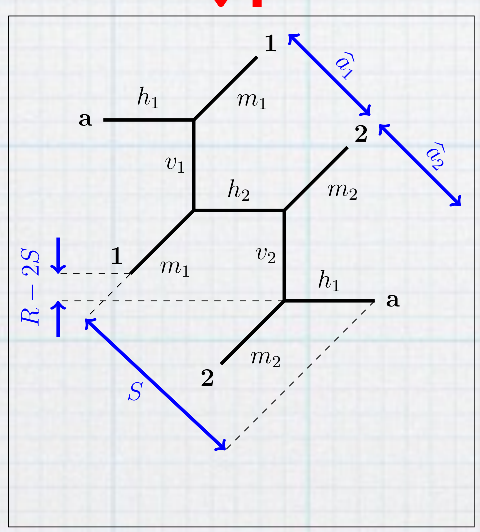

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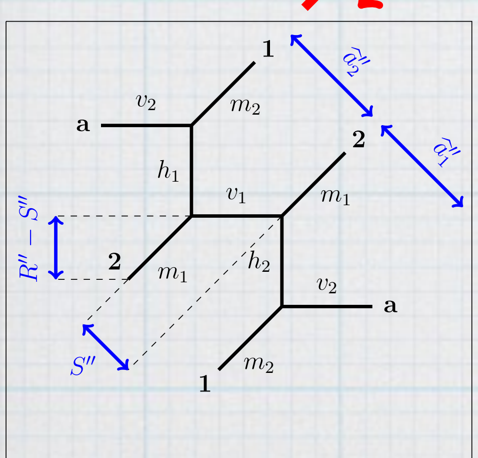
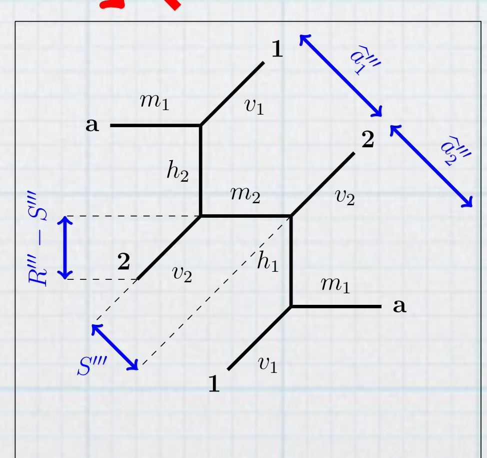
G_2 G_2 G_3 G_3





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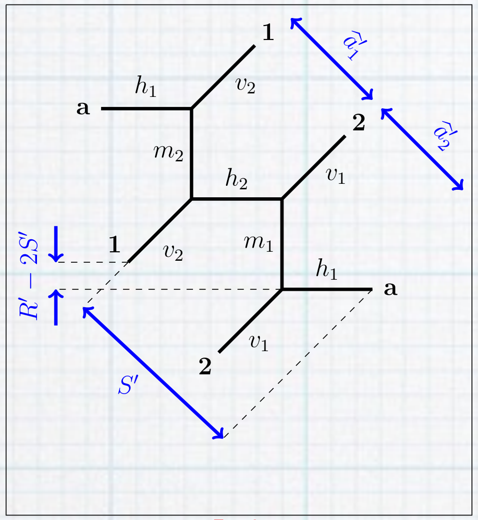

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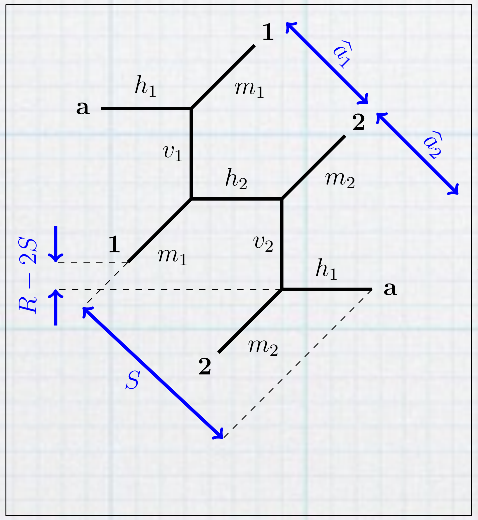

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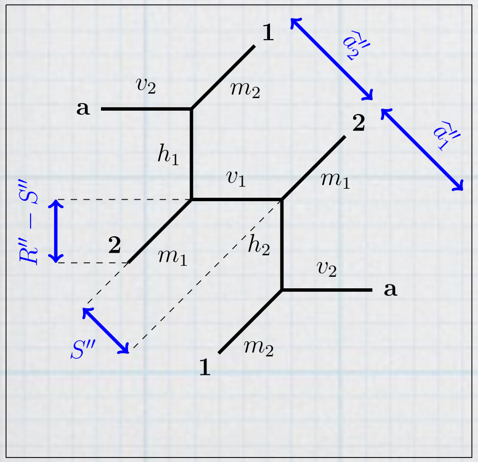

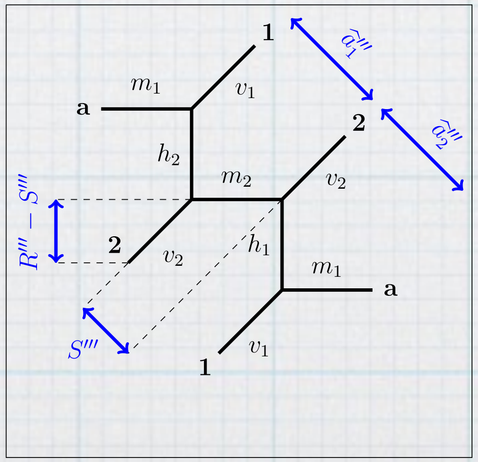

	$\mathbb{1}_{4\times4}$	G_1	G_2	G_3
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G_1	G_1	$\mathbb{1}_{4\times4}$	G_3	G_2
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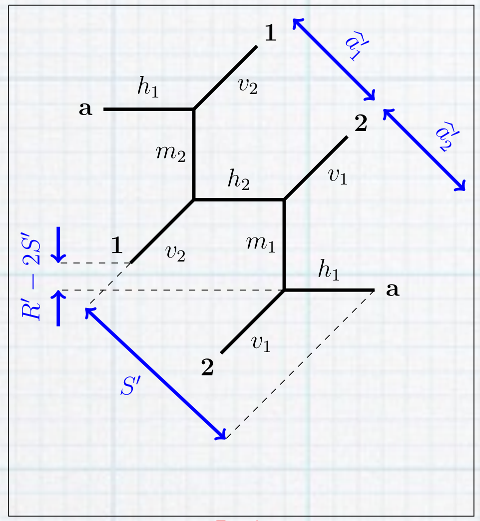

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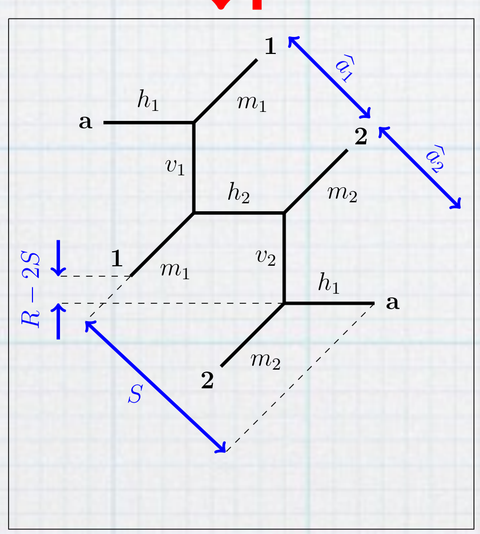

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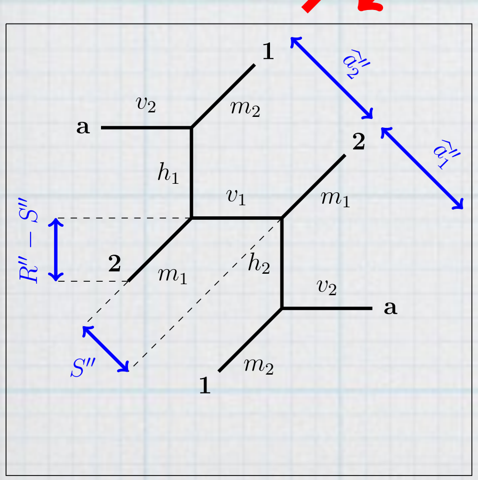

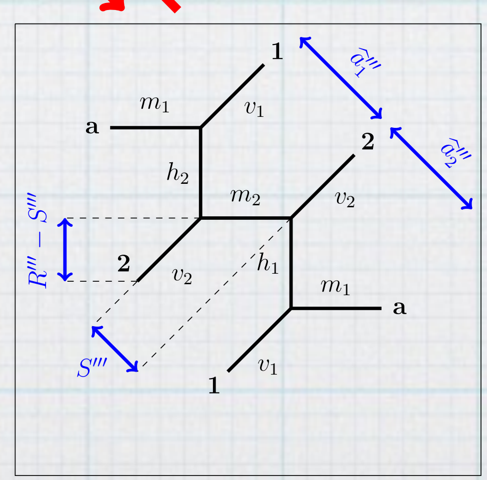

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G_1	G_1	$\mathbb{1}_{4\times4}$	G_3	G_2
G_2	G_2	G_3	$\mathbb{1}_{4\times4}$	G_1
G_3	G_3	G_2	G_1	$\mathbb{1}_{4\times4}$

Group Structure:

$$\{\mathbb{1}_{4\times4}, G_1, G_2, G_3\} \cong \mathrm{Dih}_2$$

Generalisation to (N,1): Symmetry group

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Explicitly

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with the $(N+2) \times (N+2)$ matrices

$$\mathcal{G}_2(N) = \left(\begin{array}{ccccc} & & & 0 & 0 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & 0 & 0 \\ 1 & \cdots & 1 & -1 & 0 \\ N & \cdots & N & -2N & 1 \end{array} \right) \quad \text{and} \quad \mathcal{G}_2'(N) = \left(\begin{array}{ccccc} & & & -2 & 1 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & -2 & 1 \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{array} \right)$$

Conclusions and Further Directions

Studied dualities in a class of Little String Orbifolds:

* efficiently described by dual F-theory compactification on a class of toric CY3folds $X_{N,M}$
* partition function $\mathcal{Z}_{N,M}$ compute as topological string partition function on $X_{N,M}$
* Kähler cone of $X_{N,M}$ contains three weak coupling regions in which web diagram decomposes into parallel strips
* weak coupling regions give rise to different (but equivalent) expansions of $\mathcal{Z}_{N,M}$ that can be interpreted as instanton partition functions, realising a **triality** of 5dim quiver gauge th.:

$$G_{\text{hor}} = [U(M)]^N \quad \Longleftrightarrow \quad G_{\text{vert}} = [U(N)]^M \quad \Longleftrightarrow \quad G_{\text{diag}} = [U(\tfrac{MN}{k})]^k \quad \text{for} \quad k = \text{gcd}(N,M)$$

* implies (dihedral) symmetries of the partition function

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* partition function $\mathcal{Z}_{N,M}$ compute as topological string partition function on $X_{N,M}$
* Kähler cone of $X_{N,M}$ contains three weak coupling regions in which web diagram decomposes into parallel strips
* weak coupling regions give rise to different (but equivalent) expansions of $\mathcal{Z}_{N,M}$ that can be interpreted as instanton partition functions, realising a **triality** of 5dim quiver gauge th.:

$$G_{\text{hor}} = [U(M)]^N \quad \Longleftrightarrow \quad G_{\text{vert}} = [U(N)]^M \quad \Longleftrightarrow \quad G_{\text{diag}} = [U(\tfrac{MN}{k})]^k \quad \text{for} \quad k = \text{gcd}(N,M)$$

* implies (dihedral) symmetries of the partition function

Future directions:

* study implications of triality on W-algebras associated with AGT dual theories
* Generalisation to other LSTs than A-series
* study extended web of dualities by considering further weak coupling regions in the extended moduli space of $X_{N,M}$

further dualities: $[U(M)]^N \Longleftrightarrow [U(M')]^{N'}$

for $NM = N'M'$

$\text{gcd}(N,M) = \text{gcd}(N',M')$