
The Holographic Correspondence

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Plan

• What is it?
• Historical interlude
• How does it work?
• What is it for?
Plan

• What is it?
  • Historical interlude
  • How does it work?
  • What is it for?
The statement

Quantum Gravity in D+1 dimensions = Quantum Field Theory in D dimensions
The statement

\[
\text{Quantum Gravity in D+1 dimensions} = \text{Quantum Field Theory in D dimensions}
\]

Is this reasonable?
It is not, but…
Heuristic Hint 1: RG flow

- Renormalization Group equations in QFT are local in the energy scale $u$
  \[ u \frac{d g}{d u} = \beta(g) \]

- Idea: RG flow of a D-dim QFT as “foliation” in D+1 dims.
- RG scale $u = $ Extra dimension
Heuristic Hint 2: black holes

• Model in D+1 must have same number of d.o.f. as the QFT in D-dims
• Gravity is a good candidate: it is “holographic”
• See black hole physics
Black holes… are not so black

- Quantum effects: emit thermal radiation.
- Obey laws of thermodynamics
- Entropy scales like horizon area
  [Bekenstein, Hawking 1974]

\[
S_{BH} = \frac{k_B c^3 A_H}{\hbar 4G}
\]

- The holographic “principle” ['t Hooft, Susskind 1994]
- Quantum gravity, whatever it is, is holographic.

Degrees of freedom of QG in D+1 dim. spacetime volume = Degrees of freedom of QFT in D dim. boundary
Heuristic Hint 3: String theory

- Assumption: fundamental constituents are string-like
- Point particles are different modes of a vibrating string

<table>
<thead>
<tr>
<th>String</th>
<th>Particle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open</td>
<td>photon (gluon) +…</td>
</tr>
<tr>
<td>Closed</td>
<td>graviton +…</td>
</tr>
</tbody>
</table>
• Open/closed string duality (or: 2 ways of drawing a cylinder)

Open string loop (quantum) 
Quantum Field Theory 
$X_\mu, u$ (RG scale)

Closed string propagation (classical) 
Theory of gravity 
$X_\mu, r$ (extra dimension)
• The dual nature of Dp-branes [Polchinski, 95]

Taking low energy limit on both sides, two interacting theories [J.M. Maldacena, 97]:

• Left: 4d SU($N_c$) susy Yang-Mills. Low energy limit of open strings on Nc D3-branes
• Right: closed strings (gravity) on Anti-de-Sitter 5d background (times $S^5$)
• String theory provided the **first explicit realization** of the holographic correspondence [Maldacena 1997] a.k.a. **AdS/CFT**

\[
3+1 \text{ dim } N=4 \text{ SU}(N) \text{ Yang-Mills} = \text{ IIB string on } \text{ AdS}_5 \times S_5
\]

(Conformal Field Theory) (Quantum gravity on Anti de Sitter)

• …and a very **detailed map** between observables of the corresponding theories [Witten; Gubser, Klebanov, Polyakov, 1998]

• This has produced both **extensions** and an enormous amount of **quantitative validity checks** of the correspondence…and **concrete applications**.

• Holography is changing our way of understanding gravity and quantum field theories. It does not come out from nowhere: the connections between **strings** and gauge theories (like **QCD**) have a **long history**.
Plan

• What is it ?
• Historical interlude (oversimplified)
• How does it work ?
• What is it for ?
From Gabriele to Juan Martin (and back again)
1968

• Gabriele Veneziano was born and grew up in Florence.
• He got his M.Sc. in Physics **right in this place**, in 1965.
• In 1968 he is 26 years hold. His paper containing the famous “Veneziano Amplitude” will contribute to the **birth of string theory**, see e.g. [Cappelli, Castellani, Colomo, Di Vecchia, 2012].

• Juan Martin Maldacena, italian-argentinian nationality, was born in Buenos Aires
• In the ’50s everything looked clear: electron, proton, neutron and few other particles (muon, pion, positron)

• In the ’60s however, a pletora of other hadrons: kaons, rho mesons, Delta and Omega baryons, Lambda, Sigma, Eta, Nu, Upsilon…

• Difficult to believe they were all elementary.

• 1964. Murray Gell-Mann, Zweig: quarks as building blocks.
  - Neutron, proton and all other baryons: three quarks
  - Pion, rho and other mesons: quark-antiquark pairs.

• The new hadrons posed new problems for theoretical physics. If a strong interaction happens through an exchange of some of them, the scattering amplitude increases as the energy increases.
• 1968. **Veneziano**: if an infinite number of particles is exchanged, the scattering amplitude does not diverge with energy anymore.

• This goes with the name of **Veneziano amplitude**.

• 1970. **Nambu, Nielsen e Susskind**: Veneziano amplitude can be interpreted in terms of a theory of **strings**.

• **Regge trajectories** \(M^2 \sim J\): mesons as rotating strings with quark endpoints?

• **However** string theory gave other predictions which turned out to be in conflict with experimental data.
• A SU(3) Yang-Mills gauge theory coupled to quarks.
• QCD revealed to be the correct theory of strong interactions.

• String theory almost abandoned
• In the first ’70s only very few scientists, among which Green and Schwartz, kept working on it.

• By the way: QCD is hard.
  - Asymptotic freedom: gauge coupling is small at high energies
  - Low energy limit (hadron spectra, confinement, chiral symmetry breaking…) cannot be studied using perturbation theory.
• Even understanding QCD vacuum is really very challenging.
• Can get a lot of info studying QCD on a Euclidean Lattice [Wilson 1974]
• and using Monte Carlo numerical simulations

Moreover: are we so sure that strings do not enter into the game at all?
• In SU(3) Yang-Mills the flux tubes are confined.
• Potential energy between external quarks scales linearly with the distance.
• ‘t Hooft [1974]: consider SU(N).
• Take large N limit with $\lambda = g^2_{YM}$ N fixed
• Double line notation

\[ \begin{array}{c}
  jk \\
  \, \\
  \, \\
  ij \\
\end{array} \quad = \quad \begin{array}{c}
  i \\
  j \\
  k \\
  i \\
\end{array} \]

• At large N planar diagrams are leading, non-planar diagrams subleading
• Perturbative expansion = sum over topologies
• Planar diagrams = spheres; non planar: higher genus surfaces (torus, etc)

• Same as perturbative expansion of string amplitudes with $g_s \sim 1/N$
• 1974. Scherk and Schwarz realize that string theory contains a massless spin 2 particle which corresponds to the graviton.

• String theory = quantum gravity?

• Not so many people interested, since the theory had various inconsistencies (anomalies)

• 1984. Green and Schwarz discover that actually these inconsistencies are not there.

• Since then, the interest in string theory grew up enormously
• 1995. Witten suggests that the 5 different consistent string theories are just different manifestations of a mother theory, M-theory

• 1996. Strominger, Vafa: black hole entropy from D-branes.

• 1997. Maldacena: holographic conjecture
• String theories (and thus quantum gravity) can be equivalent to quantum field theories (like QCD) with no gravity, in at least one dimensions less.

• This raises the hope to find a string theory model which is equivalent to QCD, coming back to the origins in a sense.

• (It is fair to say that we do not have found a string dual to QCD, yet)
Plan

• What is it?
• *Historical interlude*
• How does it work?
• What is it for?
“It works in a very subtle way, as a strong/weak coupling duality.”


“Certain regimes where QFT is strongly interacting, mapped into classical (i.e. weakly interacting) gravity!”

(and the other way around)

- For the master example [Maldacena 1997]:
  \[ \mathcal{N} = 4 \, SU(N_c) \, SYM \text{ in } D = 4 \text{ dual to gravity on } AdS_5 \times S^5. \]
  Classical gravity regime: \( N_c \ll 1, \quad \lambda = g_{YM}^2 N_c \ll 1. \)

- Large number of d.o.f. (large N)
- Strong coupling

- Non-perturbative QFT problems can be solved by classical gravity!
- Quantum gravity from a dual perturbative QFT!
How to compute?
QFT vacuum $\Rightarrow$ Gravity background

$Z_{QFT} = Z_{QG(String)} \approx e^{-S_{\text{gravity (on-shell)}}}$

[Gubser, Klebanov, Polyakov; Witten, 1998]
QFT vacuum $\rightarrow$ Gravity background

**QFT at finite temperature**

$$Z = Tr e^{-\frac{H}{T}}$$

$\log Z \approx - S[\text{gravity on shell}]$

[Witten, 98]
QFT vacuum \rightarrow Gravity background

QFT at finite temperature and density

\[ Z = Tr e^{-\frac{H - \mu N}{T}} \]

Charged Black hole

QFT charge density = electric flux on the boundary

\[ \log Z \approx - S[\text{gravity on shell}] \]
CFT d

\[ x_\mu \rightarrow \lambda x_\mu \]
\[ E \rightarrow \lambda^{-1} E \]

[\[ E \approx r \]
[\[ ds^2 = dx_\mu dx^\mu + dr^2 \]}

Francesco Bigazzi
The Holographic Correspondence
CFT $d$

\begin{align*}
 x_\mu &\rightarrow \lambda x_\mu \\
 E &\rightarrow \lambda^{-1} E
\end{align*}

$E \approx r$

AdS $d+1$

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{R^2}{r^2} dr^2$$

(R=AdS radius)
\[
\begin{align*}
CFT \ d & \hspace{1cm} \text{AdS } d+1 \\
& \\
E \to \lambda^{-1} E & \hspace{1cm} ds^2 = \frac{r^2}{R^2} d\mathbf{x}_\mu d\mathbf{x}^\mu + \frac{R^2}{r^2} dr^2 \\
E \approx r & \hspace{1cm} (R = \text{AdS radius}) \\
CFT \text{ at finite } T & \hspace{1cm} \text{AdS black hole} \\
Z = \text{Tr} e^{-\frac{H}{T}} & \hspace{1cm} ds^2 = \frac{r^2}{R^2} \left[-b[r] dt^2 + dx_i dx_i\right] + \frac{R^2}{r^2} \frac{dr^2}{b[r]} \\
& \\
T_{CFT} = T_{BH} = \frac{r_h d}{4\pi R^2} & \hspace{1cm} b[r] = 1 - \frac{r_h}{r^d} \\
& \\
S_{CFT} = S_{BH} = \frac{A_h}{4G_N} \sim V_{d-1} T^{d-1} & \hspace{1cm} \text{Charged RN-AdS black hole} \\
& \\
Z = \text{Tr} e^{-\frac{H-\mu N}{T}} & \hspace{1cm} A_t \sim \mu - \frac{\rho}{r^d} \\
& \\
&
\end{align*}
\]
Correlators

• In a D dim. QFT we compute the generating functional
  \[ Z_{QFT}[\phi_0] = \int D\Psi \exp \left( i[S_{QFT} + \int d^D x \phi_0(x) \mathcal{O}[\Psi](x)] \right) \]

• N-point correlators of \( \mathcal{O}[x] \): from n-th derivarives of Log Z w.r.t. external source \( \phi_0(x) \).

• Holography: treat external source \( \phi_0(x) \) as boundary value of a gravity field \( \phi(x,r) \) which is “dual” to the operator \( \mathcal{O}[x] \).
Correlators

- Then: [Gubser, Klebanov, Polyakov; Witten; 1998]

\[ Z_{QFT}[\phi_0] = Z_{QG/String} \approx e^{iS_{gravity}[\phi_0]} |\phi(x,r)\rightarrow\phi_0(x)| \]

- So that for example:

\[ \langle \mathcal{O}(x)\mathcal{O}(y) \rangle \sim \frac{\delta^2 S_{grav}[\phi_0]}{\delta\phi_0(x)\delta\phi_0(y)} |_{\phi_0=0} \]

- Can compute correlators, just solving equation of motion for \(\phi(x,r)\)!
- Retarded correlators at finite temperature: incoming b. c. at horizon

**Operator** \(O(x)\) \(\rightarrow\) **Gravity field** \(\phi(x,r)\)

- Example 1 (stress tensor): \(T^{\mu\nu}(x) \rightarrow g_{\mu\nu}(x, r)\)
- Example 2 (conserved current): \(J^{\mu}(x) \rightarrow A_{\mu}(x, r)\)
- Example 3 (scalar operator): \(Tr F^2(x) \rightarrow \Phi(x, r)\)

- Global symmetries in QFT are mapped into local symmetries in the gravity side
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Strongly correlated QFTs arise in many places:

- **QCD**: confinement, mass gap, quark-gluon plasma phase, neutron star core…
- **Quantum critical regions** in condensed matter: high Tc superconductors, strange metals …

• Often **non-equilibrium**, **finite density** challenging: need novel tools.
• **Holography** is emerging as a promising one.

• Often **analytic** control on the models. Novel intuitions.
• Can deal with both static and **real-time** dynamical properties.
• Still limited to **toy models**.
• Provide **benchmarks** and info on **universal** behavior at strong coupling.
Example 1: Holography, black holes and the Quark-Gluon Plasma
The Quark-Gluon-Plasma


A novel state of matter where QCD is deconfined.
The Quark-Gluon-Plasma

• Heavy ion collisions at RHIC and LHC indicate that QGP:
  - behaves like liquid
  - Very small shear viscosity over entropy density ratio
  - Hence: strongly coupled
  - Nearly scale invariant
  - Very opaque (large jet quenching)

• Challenging, dynamical system. Try with holography

• Strongly coupled thermal QFT \( \Rightarrow \) Black Hole in higher dim.

• QFT Thermodynamics \( \Rightarrow \) Black Hole thermodynamics.

• Hydrodynamics \( \Rightarrow \) Fluctuations around black hole background
Shear viscosity from holography

\[ \eta = \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G^{R}_{xy,xy}(\omega, 0) \]

\[ G^{R}_{xy,xy}(\omega, 0) = \int dt \, dx \, e^{i\omega t} \, \theta(t) \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle \]

- Compute using holography: T_{xy}(x) dual to g_{xy}(x,r)

QFT correlator = classical scattering of gravitons from black hole

\[ \frac{\eta}{s} = \frac{1}{4\pi} \frac{\hbar}{K_{B}} \]

[T. Damour, Ph.D. Thesis 1979; Policastro, Son, Starinets, 2001; Kovtun, Son, Starinets 2004]

- **Universal**: for any isotropic fluid with classical gravity dual
- **Surprisingly close** to the measured value for QGP
Second order hydrodynamics from holography

[Romatschke 2009; F.B., Cotrone, Tarrio; F.B., Cotrone, 2010]

Model: conformality broken by marginally relevant operator

$$\Delta \equiv (1 - 3c_s^2)$$

$$\frac{\eta}{s} = \frac{1}{4\pi}$$
$$\frac{T \lambda_1}{s} = \frac{1}{8\pi^2} \left(1 + \frac{3}{4}\Delta\right)$$
$$\frac{T \lambda_2}{s} = -\frac{1}{4\pi^2} \left(\log 2 + \frac{3\pi^2}{32}\Delta\right)$$
$$\frac{T \lambda_3}{s} = 0$$
$$\frac{T \lambda_4}{s} = 0$$
$$\frac{\zeta}{\eta} = \frac{2}{3} \Delta$$
$$\frac{T \xi_2}{s} = \frac{2 - \log 2}{36\pi^2} \Delta$$
$$\frac{T \xi_3}{s} = \frac{T \xi_4}{s} = 0$$
$$\frac{T \xi_5}{s} = \frac{1}{12\pi^2} \Delta$$
$$\frac{T \xi_6}{s} = \frac{1}{4\pi^2} \Delta$$

Note: QCD fireball nearly conformal and strongly coupled for $1.5T_c \lesssim T \lesssim 4T_c$ (RHIC, LHC).
Jet quenching parameter

- Transport coefficient characterizing probe parton energy loss
- Evaluated holographically in $N=4$ SYM [Liu, Rajagopal, Wiedemann 06]
- Adding $N_f$ dynamical flavors [Bigazzi, Cotrone, Mas, Paredes, Ramallo, Tarrio 2009]

$$\hat{q} = \frac{\pi^{3/2} \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \sqrt{\lambda} T^3 \left[ 1 + \frac{2 + \pi}{64 \pi^2} \frac{N_f}{N_c} + \ldots \right]$$

Quarks enhance jet quenching

Extrapolating to QGP: $N_c=N_f=3$, $\lambda=6\pi$, $T=300$ MeV, get

$$q \approx 4\div5 \text{ GeV}^2/\text{fm}$$

right in the ballpark of data
Example 2:
Holography and condensed matter
• **Strongly correlated electrons** (Condensed Matter) (often layered, 2+1)

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<table>
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<tr>
<th>“High-T_c” Cuprates</th>
<th>Heavy Fermions</th>
</tr>
</thead>
<tbody>
<tr>
<td>YBa$_2$Cu$<em>3$O$</em>{7-x}$ (YBCO)</td>
<td>CeCu$_{6-x}$Au$_x$</td>
</tr>
<tr>
<td>La$_{2-x}$Sr$_x$CuO$_4$ (LSCO)</td>
<td>CeCoIn$_5$</td>
</tr>
<tr>
<td>Bi$_2$Sr$_2$CaCu$<em>2$O$</em>{8+x}$ (BSCCO)</td>
<td>YbRh$_2$Si$_2$</td>
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```

- Quantum phase transitions (T=0). **Scale invariant** QFT. Very large correlations
2. Strongly correlated electrons (Condensed Matter) (often layered, 2+1)

- Quantum phase transitions (T=0). Scale invariant QFT. Very large correlations
- Quantum critical region. Affected by quantum critical point at T=0 [Sachdev]
- Exhibit phases which escape standard paradigms based on quasi-particle description

- E.g. strange metallic phase with linear (in T) resistivity
- No satisfactory theoretical understanding

![Diagram showing quantum critical points in heavy fermion metals and phase diagrams of cuprates and heavy fermions.](Image)
- 2. Strongly correlated electrons (Condensed Matter)

"High-Tc" Cuprates

- YBa$_2$Cu$_3$O$_{7-x}$ (YBCO)
- La$_{2-x}$Sr$_x$CuO$_4$ (LSCO)
- Bi$_2$Sr$_2$CaCu$_2$O$_{8+x}$ (BSCCO)

Heavy Fermions

- CeCu$_{6-x}$Au$_x$
- CeCoIn$_5$
- YbRh$_2$Si$_2$

- Example of challenging observable at strong coupling: (optical) conductivity

- Ohm’s law: $J = \sigma E$. $\sigma$: retarded correlator of U(1) current $J$; ($\rho = 1/\text{Re}[\sigma(0)]$)

$$\sigma(\omega) = \text{Lim}_{k \to 0} \frac{G^R_{JJ}(\omega, k)}{i\omega}$$

$$G^R_{JJ}(\omega, k) = -i \int d^{d-1}x \, dt \, e^{i\omega t - ik \cdot x} \theta(t) \langle [J(t, x), J(0, 0)] \rangle$$
Optical conductivity in $d=2+1$ from holography

From fluctuations around dual charged black hole [Herzog, Kovtun, Sachdev, Son, 07]

- $J_x = \sigma E_x = -i\omega \sigma A_x$

- Cfr with graphene (at low energy a relativistic theory in $2+1$) [Li et al 2008]
Example 3:
Holography, black holes and quantum information
Holography and Entanglement Entropy

- Consider region A in a space A+B
- If ignorant on B, define reduced density matrix \( \rho_A = \text{tr}_B \rho \)
- Entanglement entropy \( S_A = -\text{tr}_A \rho_A \log \rho_A \)
- Holographically [Ryu, Takayanagi 2006]

\[
S_A = \text{extremum}_{\partial M = \partial A} \frac{\text{area}(M)}{4G_N}
\]

- For AdS\(_3\) this gives,
  \[
  S_A = \frac{c}{3} \log \frac{l}{a}
  \]
- Same as in 2d CFT [Wilczek; Cardy, Calabrese]
Concluding comments

• Holography: theory of strings (QG) = lower dim. QFT without gravity.
• In a sense this could provide a background independent definition of QG.

• An enormous amount of quantitative checks.

• Many applications to strongly coupled QFT, from hep to cond-mat

• In recent years many efforts for going in the other direction:
  - Gravity as an emergent quantum many-body phenomenon?
  - What role do quantum information concepts such as entanglement and circuit complexity play in this connection?

• A lot of fun!
Thank you for your time