$\mathcal{N}=2$, conformal gauge theories at large R-charge: the $SU(N)$ case

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Introduction and Motivation

Double Scaling Limits

- 't Hooft realized that $SU(N)$ gauge theory simplifies in the limit $g \to 0$, $N \to \infty$, with $g^2 N$ a constant.
- This is the prototypical example of a double scaling limit.
- Another class of examples comes from considering a QFT with some coupling g and studying the operators with large charge n under a global symmetry. [Hellerman et al. -2015; Arias-Tamargo et al - 2019]
- $\mathcal{N} = 2$ superconformal theories with gauge group $SU(N)$ are an attractive setup. We will study correlation functions of Coulomb branch operators with large $U(1)$ R-charge.
- The goal is to exhibit the simplicity that emerges in the double scaling limit.
- As we will see this limit enables us to probe some massive BPS states in the theory.

Two Point Functions in Conformal Field Theories

- For isolated conformal field theories, two point functions of primary operators are trivial: they are fixed by conformal symmetry up to normalization.
- For conformal field theories that allow exactly marginal deformations, the normalization is not global: the two point functions have non a non-trivial dependence on exactly marginally couplings.
- The complexified gauge coupling τ is always exactly marginal for a superconformal $\mathcal{N} = 2$, $SU(N)$ theory.
- For a superconformal primary $\mathcal{O}$, the two point function is:

$$\left\langle \mathcal{O}(x), \bar{\mathcal{O}}(y) \right\rangle = \frac{G_{\mathcal{O}\bar{\mathcal{O}}}(\tau, \bar{\tau})}{(x-y)^{2\Delta(\mathcal{O})}}$$

Coulomb Branch Operators and Localization

- The Coulomb branch operators of an $\mathcal{N}=2$, $SU(N)$ theory are generated by $\operatorname{tr}\phi^k$ with $1 < k < N$.
- Their VEVs parameterize the Coulomb branch of vacua.
- Using supersymmetric localization the partition function of any superconformal $\mathcal{N}=2$ theory on 4-sphere can be reduced to finite dimensional integral over the Coulomb branch. [Pestun - 2007].
- For an $SU(N)$ gauge theory, this is a one matrix model i.e an integral over a matrix M that depends only on traces of M.

$$Z_{S^4} = \int [\mathrm{d}a] \exp\left(-4\pi \operatorname{Im}\tau \operatorname{tr} a^2\right) Z_{\text{1-loop}}(\operatorname{tr} a^2, \operatorname{tr} a^3, \cdots)$$

With $[\mathrm{d}a] = \prod_{\mu=1}^{N} \mathrm{d}a_\mu \prod_{\nu<\mu}(a_\mu - a_\nu)^2 \delta\Big(\sum_\mu a_\mu\Big)$.

Correlation functions from Localization [Gerchkovitz et al. 2017]

- We will consider the simplest infinite sequence of Coulomb branch operators with increasing R-charge $\mathcal{O}_n = (\text{tr}\phi^2)^n$.
- On S^4, the correlation function can be evaluated using localization.

$$\left\langle \mathcal{O}_n(N)\bar{\mathcal{O}}_m(S)\right\rangle_{S^4} = \partial_\tau^n \partial_{\bar{\tau}}^m Z_{S^4}$$

- This is not diagonal! Metric on sphere and flat space are conformally equivalent but due to conformal anomaly the map between flat space operators and those on S^4 is not trivial.
- To get flat space operator : $\mathcal{O}_n$: we need to perform Gram-Schmidt orthogonalization on $1, \mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_n$. [Bourget et al - 2018]

A Double Scaling Limit ?

- Let's consider the double scaling limit:

$$F(\kappa) = \lim_{n\to\infty} \frac{\left\langle \mathcal{O}_n(x), \bar{\mathcal{O}}_n(y) \right\rangle^{\mathcal{N}=2}}{\left\langle \mathcal{O}_n(x), \bar{\mathcal{O}}_n(y) \right\rangle^{\mathcal{N}=4}}$$

  With κ the finite coupling $\frac{2\pi n}{\operatorname{Im}\tau}$
- Does this limit even exist? Maybe it is trivial?
- Localization seems to provide a path to answer this question but it is complicated by conformal anomaly.
- Progress can be made by exploiting the integrable structures in $\mathcal{N} = 2$ theories. For SQCD see:[Bourget et al - 2018, Beccaria - 2018]
- Grassi, Komargodski and Tizzano realized that for $SU(2)$ the Gram-Schmidt process is hiding another "dual" matrix model.
- This observation in fact generalizes to higher rank case.

Large n Correlators and Positive Matrices

Correlators from Determinants

- Define the the $n \times n$ matrix $M^{(n)}$ by $M^{(n)}_{kl} = \partial^k_\tau \partial^l_{\bar\tau} Z_{S^4}$.
- Then the flat space correlator can be written as a ratio of determinants.

$$G_{2n} = \frac{\det M^{(n+1)}}{\det M^{(n)}}$$

- Using the localization result $M^{(n)}$ is a matrix with each element a finite dimensional integral. We can exchange $\det M^{(n)}$ for an integral over determinants.

$$\det \mathcal{M}^{(n)} = \frac{1}{n!} \int \prod_{i=0}^{n-1} [\mathrm{d}a_i]\, e^{-4\pi \,\mathrm{Im}\,\tau \,\mathrm{tr}\, a_i^2} Z_{\text{1-loop}}(a_i) \prod_{j<i} (\mathrm{tr}\, a_i^2 - \mathrm{tr}\, a_j^2)^2$$

- We have an integral over a matrix W whose eigenvalues are $\mathrm{tr}\, a_i^2$!

The Dual Matrix Model

- The result is that we are dealing with a matrix integral

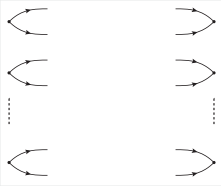
$$\det \mathcal{M}^{(n)} = \frac{1}{n!} \int [\mathrm{d}\, W] \exp(-V(W))$$

- Eigenvalues of W are $\operatorname{tr} a_i^2$: W is a positive matrix.
- The large n-limit of potential V can be determined from the interacting action of the $\mathcal{N} = 2$ theory.
- It turns out that if rank of gauge group is greater than 1, V contains higher traces of all orders!
- The higher trace operators are suppressed just right to contribute at the same order as single trace operators.
- So the large n limit exists but it is not planar.

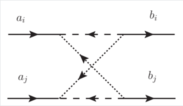
Planarity and Diagrams

- This non-planarity has a very interesting analog in the super-diagram analysis.
- In the 't Hooft limit only the planar diagrams contribute to leading order in N.
- In contrast the large n limit is dominated by diagrams that maximize genus at a given order in gauge coupling.
- Concretely the relevant diagrams are all possible completions of the skeleton.
- The 1-loop correction is planar but the 2-loop correction has genus 1 due to an insertion of the box diagram.

Basic Skeleton



The box diagram



Perturbative results

- In summary, we have an efficient algorithm for perturbative calculations able to quickly produce long series expansion to very high order. For example for $\mathcal{N} = 2$ Superconformal QCD we obtain:

$$\log F(\kappa) = -\frac{9\,\zeta(3)}{2}\,\kappa^2 + \frac{25(2N^2-1)\,\zeta(5)}{N(N^2+3)}\,\kappa^3 - \frac{1225(8N^6+4N^4-3N^2+3)\,\zeta(7)}{16N^2(N^2+1)(N^2+3)(N^2+5)}\,\kappa^4 + \cdots$$

- The algorithm is completely generic and doesn't require any assumptions beyond a simple gauge group and the input of partition function on S^4 as an integral over Coulomb branch.
- But non-planarity makes it hard to resum the perturbative results in a way amenable to probing the large κ regime, in contrast to $SU(2)$ where it is possible [Beccaria 2019, Grassi et al. 2019].

One Point Functions in the Presence of Wilson Loop

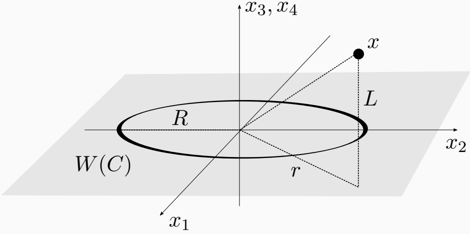


Figure credits: M. Billo, F. Galvagno, P. Gregori and A. Lerda

Wilson Loops

- For a more striking simplification we turn to one point functions of chiral operators in the presence of Wilson loops.
- These can also be computed using localization,

$$\langle :\mathcal{O}_n: \mathcal{W}\rangle \propto \int [\mathrm{d}a] :\mathcal{O}_n: \operatorname{tr}\exp(2\pi a)\exp\left(-4\pi \operatorname{Im}\tau \operatorname{tr} a^2\right) Z_{\text{1-loop}}(a)$$

- It turns out that the large n limit is the same as that of two point functions, $\langle :\mathcal{O}_n: \mathcal{W}\rangle \to \left\langle :(\operatorname{tr} a^2)^n: \operatorname{tr} a^{2n}\right\rangle$.
- The large n limit of this two point function is captured by an "$SU(2)$" like matrix model!

$$Z_{\text{eff}} = \int \mathrm{d}r\, r^{N^2-2} \exp\left(-4\pi \operatorname{Im}\tau\, r^2\right) Z_{\text{1-loop}}(r a_0).$$

- $a_0 = \left(\frac{1}{\sqrt{N(N-1)}}, \frac{1}{\sqrt{N(N-1)}}, \cdots, \frac{1}{\sqrt{N(N-1)}}, -\sqrt{\frac{N-1}{N}}\right)$ is the point on S^{N-1} that maximizes $\operatorname{tr} a^{2n}$.

A Simple Final Result

- As a result the large n-limit is planar. This allows us to conjecture all order resummations that reveals a strikingly simple structure.

$$\lim_{n\to\infty} \log \frac{\langle : \mathcal{O}_n : W \rangle^{\mathcal{N}=2}}{\langle : \mathcal{O}_n : W \rangle^{\mathcal{N}=4}} = \int_0^\infty \frac{\mathrm{d}t\, e^t}{t(e^t-1)^2} \mathcal{J}(t)$$

- The "$SU(2)$ like" Z_{eff} is an integral over the line ra_0 in Coulomb branch. On this line, the VEVs of ϕ break $SU(N) \to U(N-1)$.
- The supermultiplets split as representations of this $U(N-1)$. The VEVs of ϕ also give mass to some of resulting fields.
- Each such massive representation r of $U(N-1)$ contributes a term to to $\mathcal{J}(t)$ which is $\pm 2 \dim \mathrm{r}\, [J_0(\sqrt{2} m_{\mathrm{r}} t) - 1]$.
- m_r is the mass of r at the point κa_0 of the moduli space.

An Example: $\mathcal{N} = 2$ SQCD

- $2N$ hypermultiplets in the fundamental of $U(N)$.
- Each fundamental hypermultiplet splits into a fundamental and a singlet of $U(N-1)$. At κa_0,
  - $U(N-1)$ fundamental has mass $\sqrt{\frac{\kappa}{N(N-1)}}$.
  - $U(N-1)$ singlet has mass $\sqrt{\frac{\kappa(N-1)}{N}}$.
- The vector multiplet splits as
  - Adjoint of $U(N-1)$ which is massless as expected from unbroken $U(N-1)$ gauge symmetry.
  - 2 massive W-bosons in the fundamental of $U(N-1)$ with mass $\sqrt{\frac{\kappa N}{N-1}}$.

The large n limit we are after is

$$4 \int_0^\infty \frac{dt\, e^t}{t\,(e^t-1)^2} \left[N\, \widetilde{J}_0\!\left(t\,\sqrt{\frac{2\,(N-1)\,\kappa}{N}}\right) + N\,(N-1)\,\widetilde{J}_0\!\left(t\,\sqrt{\frac{2\,\kappa}{N\,(N-1)}}\right) - (N-1)\,\widetilde{J}_0\!\left(t\,\sqrt{\frac{2\,N\,\kappa}{N-1}}\right)\right]$$

This contains both perturbative and exponentially suppressed non-perturbative terms.

Conclusions and Outlook

- Leveraging localization and random matrix theory we can learn about the large R-charge limit of $\mathcal{N} = 2$, $SU(N)$ theories.
- This requires choosing the observables (or rather the sequence of observables) carefully.
- Is there a similar story for generic sequence of operators with increasing R-charge?
- The large n limit of one point functions of chiral operators in the presence of Wilson loop remains planar for $SU(N)$ gauge group.
- It also admits a simple and interesting interpretation in terms of the mass spectrum at the relevant point in moduli space.
- Maybe it contains a hint of an EFT description generalizing the $SU(2)$ case?