

# Feynman integrals reduction and Intersection theory

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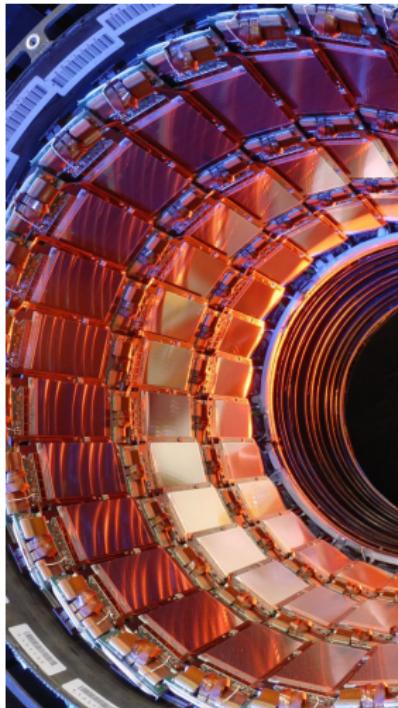
Based on:

- Pierpaolo Mastrolia and Sebastian Mizera  
*Feynman integrals and Intersection theory*  
JHEP 1902 (2019) 139
- H.Frellesvig, F.Gasparotto, S.Laporta,  
M.Mandal, P. Mastrolia, L.M., S.Mizera  
*Decomposition of Feynman Integrals on the  
Maximal Cut by Intersection Numbers*  
JHEP 1905 (2019) 153
- H.Frellesvig, F.Gasparotto, M.Mandal, P.  
Mastrolia, L.M., S.Mizera  
*Vector Space of Feynman Integrals and  
Multivariate Intersection Numbers*  
Phys.Rev.Lett. 123 (2019) no.20, 201602

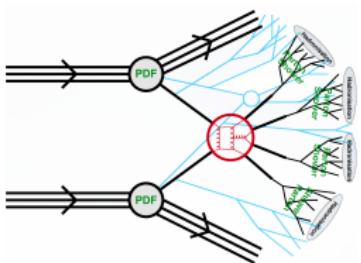


# Motivation

Higgs discovery  
Success of Standard Model  
↓  
SM as effective theory  
(Dark Matter, Dark Energy...)  
↓  
New physics hide in subtle effects  
↓  
Precision calculation needed  
↓  
**Scattering amplitudes** are  
powerful means to access it



# Motivation

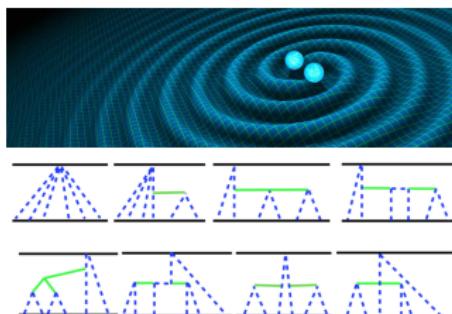


Scattering amplitudes are at the core of cross sections measured in colliders

very effective tool to know **gravitational waveforms** with high precision in weak field approximation

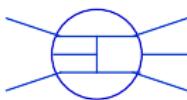
[Foffa, Sturani, Mastrolia, Sturm (2016)]

[Foffa, Sturani, Mastrolia, Sturm, Torres Bobadilla (2019)]...



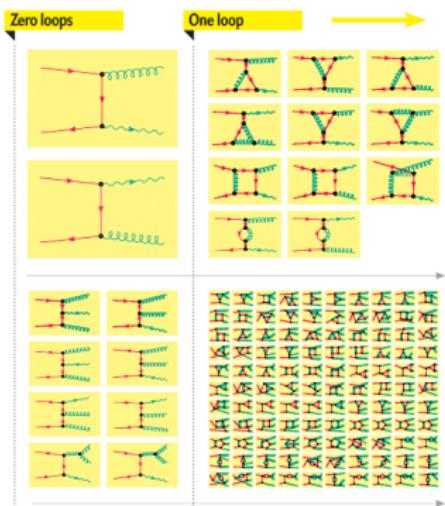
# Motivation

Scattering amplitudes are built out of many Feynman integrals:

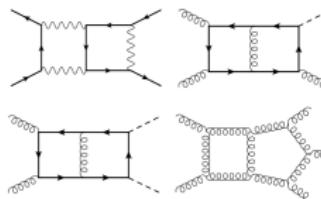


$$I_{a_1, \dots, a_N} = \int \prod_{i=1}^L d^d k_i \frac{1}{D_1^{a_1} \dots D_N^{a_N}}$$

Higher precision  $\Rightarrow$  Higher loop  
Complexity of the calculations increases quickly



State of the art calculations at 2 loop,  
such as



requires  $\mathcal{O}(10000)$  integrals. Needs to evaluate them all? No!

[Bern, Dixon, Kosower (2012)]

# Integration By Parts Identities

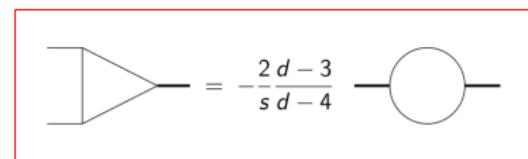
Linear relations among integrals: **Integration By Parts Identities - IBPs**

[Chetyrkin, Tkachov (1981)]

[Laporta (2001)]...

$$\int \prod_{i=1}^L d^d k_i \frac{\partial}{\partial k_j^\mu} \left( \frac{v^\mu}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0 \Rightarrow c_1 I_{a_1+1, \dots, a_N} + \dots + c_N I_{a_1, \dots, a_N+1} = 0$$

**Integrals related by a total derivative**



**Linear System  $\Rightarrow$  Gauss Elimination  $\Rightarrow$  Master Integrals  $\{J_i\}$  - MIs**

Decomposition of an Integral in terms of **MIs**

$$I_{a_1, \dots, a_N} = \sum_{i=1}^{\nu} c_i J_i$$

# Integration By Parts Identities

[Kotikov '91, Remiddi '97, Gehrmann & Remiddi '00, Argeri & Mastrolia '07, Henn '13]  
[Tarasov '96, Lee '07]

**MIs** as solutions of

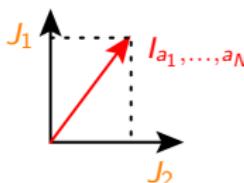
- Differential equations
  - Dimensional recurrence relations
- ⇒ Built by means of **IBPs**

$$\partial_s J_k = \sum_{i=1}^{\nu} c_i J_i \quad \Rightarrow \quad \partial_s \mathbf{J} = \mathbb{A} \mathbf{J}$$
$$\mathbb{A} = \begin{pmatrix} \textcolor{red}{\blacksquare} & & & \\ * & \textcolor{orange}{\blacksquare} & & \\ * & & \textcolor{orange}{\blacksquare} & \\ * & & * & \textcolor{blue}{\blacksquare} \end{pmatrix}$$

**IBPs Drawbacks:**

- # equations grows dramatically
- manipulation large expressions

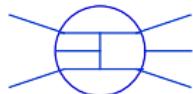
Possible **bottleneck** of multiloop calculations  
Can we directly project integrals on MI?



# Intersection theory

In Baikov representation

[Aomoto, Kita, Matsumoto, Mizera, ...]  
 [Mastrolia, Mizera (2018)]



$$I_{a_1, \dots, a_N} = \int_{\mathcal{C}} u(z) \varphi(z)$$

$$u(z) = \prod_i P_i(z)^{\gamma_i} \Rightarrow u(z) \text{ multivalued function s.t. } P_i(\partial \mathcal{C}) = 0$$

$$\varphi(z) = \hat{\varphi}(z) dz \Rightarrow \varphi(z) \text{ single valued form}$$

Total derivative translates to

$$\begin{aligned} \int_{\mathcal{C}} d(u\varphi) &= \int_{\mathcal{C}} du \varphi + u d\varphi = \int_{\mathcal{C}} u \left( \frac{du}{u} + d \right) \varphi \\ &= \int_{\mathcal{C}} u (\omega + d) \varphi = \int_{\mathcal{C}} u \nabla_{\omega} \varphi = 0 \quad \mathcal{P}_{\omega} = \{z \mid z \text{ is a pole of } \omega\} \end{aligned}$$

rewriting Integration by Parts Identities as

$$\int_{\mathcal{C}} u (\varphi + \nabla_{\omega} \xi) = \int_{\mathcal{C}} u \varphi \quad \Rightarrow \quad \varphi \sim \varphi + \nabla_{\omega} \xi$$

**Equivalence class between forms defines the Twisted cohomology group.**

$$\omega \langle \varphi | \equiv \{ \varphi | \nabla_\omega \varphi = 0 \} / \{ \nabla_\omega \xi \} = H_\omega^n$$

**Key relation between IBPs and Twisted Cohomology**

[Aomoto (1975)]

$$\nu = \dim(H_\omega^n)$$

[Lee, Pomeransky (2013)]

$$\begin{aligned} &= \chi(X) = (-1)^n (n+1 - \chi(\mathcal{P}_\omega)) \\ &= \{\# \text{ of solutions of } \omega = 0\} \end{aligned}$$

**Contours have similar structure**

$$\int_{\mathcal{C}} u\varphi = \int_{\mathcal{C} + \partial_\omega g} u\varphi \quad \Rightarrow \quad [\mathcal{C}] = H_n^\omega \text{Twisted Homology group}$$

Feynman integrals are **pairing**

Dual integrals

$$\langle \varphi | \mathcal{C} \rangle = \underbrace{\int_{\mathcal{C}} u(z)}_{\text{cycle}} \overbrace{\varphi(z)}^{\text{cocycle}}$$

$$\begin{aligned} [\mathcal{C} | \varphi \rangle &= \int_{\mathcal{C}} u^{-1}(z) \varphi(z) = \int_{\mathcal{C}} u^{-1} (\varphi + \nabla_{-\omega} \xi) \\ &\Rightarrow |\varphi \rangle = H_{-\omega}^n , \quad [\mathcal{C}] = H_n^{-\omega} \end{aligned}$$

# Twisted intersection number

Seeking relations between integrals (forms)  $\Rightarrow$  **Intersection number**

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)^n} \int \iota(\varphi_L) \wedge \varphi_R$$

[Mastrolia, Mizera (2018)]

IBP built in naturally within such formalism.

[Frellesvig, Gasparotto, Laporta, Mandel,

Mastrolia, L.M., Mizera (2019)]

$$\langle \varphi_L + \nabla_\omega \xi | \varphi_R \rangle = \langle \varphi_L | \varphi_R + \nabla_{-\omega} \xi \rangle = \langle \varphi_L | \varphi_R \rangle$$

Define basis of independent form and dual form

$$\langle e_1 |, \quad \langle e_2 |, \quad \dots, \quad \langle e_\nu |, \quad | h_1 \rangle, \quad | h_2 \rangle, \quad \dots, \quad | h_\nu \rangle$$

Build the matrix

$$\mathbf{M} = \begin{pmatrix} \langle \varphi_L | \varphi_R \rangle & \langle \varphi_L | h_1 \rangle & \langle \varphi_L | h_2 \rangle & \dots & \langle \varphi_L | h_\nu \rangle \\ \langle e_1 | \varphi_R \rangle & \langle e_1 | h_1 \rangle & \langle e_1 | h_2 \rangle & \dots & \langle e_1 | h_\nu \rangle \\ \langle e_2 | \varphi_R \rangle & \langle e_2 | h_1 \rangle & \langle e_2 | h_2 \rangle & \dots & \langle e_2 | h_\nu \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu | \varphi_R \rangle & \langle e_\nu | h_1 \rangle & \langle e_\nu | h_2 \rangle & \dots & \langle e_\nu | h_\nu \rangle \end{pmatrix} \equiv \begin{pmatrix} \langle \varphi_L | \varphi_R \rangle & \mathbf{A}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

# Master Decomposition Formula

$\langle \varphi_L |$  depends on the basis element

$$\det \mathbf{M} = \det \mathbf{C} \left( \langle \varphi_L | \varphi_R \rangle - \mathbf{A}^T \mathbf{C}^{-1} \mathbf{B} \right) = 0$$

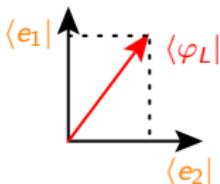
↓

$$\langle \varphi_L | \varphi_R \rangle = \mathbf{A}^T \mathbf{C}^{-1} \mathbf{B} = \sum_{i,j=1}^{\nu} \langle \varphi_L | h_j \rangle (\mathbf{C}^{-1})_{ji} \langle e_i | \varphi_R \rangle$$

Since  $|\varphi_R\rangle$  is arbitrary

$$\langle \varphi_L | = \sum_{i,j=1}^{\nu} \langle \varphi_L | h_j \rangle (\mathbf{C}^{-1})_{ji} \langle e_i | \stackrel{\mathbf{C}_{ij} = \delta_{ij}}{=} \sum_{i=1}^{\nu} \langle \varphi_L | h_i \rangle \langle e_i | = \sum_{i=1}^{\nu} c_i j_i$$

Direct projection



# Univariate computation

$$\begin{aligned}\langle \varphi_L | \varphi_R \rangle &= \frac{1}{(2\pi i)} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{\text{Res}} \left( (\nabla_\omega^{-1} \varphi_L) \varphi_R \right) = - \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{\text{Res}} \left( \varphi_L \nabla_{-\omega}^{-1} \varphi_R \right) \\ &= \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{\text{Res}} (\psi_p \varphi_R) \stackrel{dLog}{=} \sum_{p \in \mathcal{P}_\omega} \frac{\underset{z=p}{\text{Res}} (\varphi_L) \underset{z=p}{\text{Res}} (\varphi_R)}{\underset{z=p}{\text{Res}} (\omega)}\end{aligned}$$

$$\psi_p = \nabla_\omega^{-1} \varphi_L \quad \Rightarrow \quad (d + \omega) \psi_p = \varphi_L$$

only **local** solution to  $\psi_p$  needed  $\Rightarrow$  power series ansatz

$$\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)} \tau^j + \mathcal{O}(\tau^{\max+1})$$

$\psi_p$  obtained by **pattern matching**

Sanity check:

$$\langle \nabla_\omega \xi | \varphi_R \rangle = \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{\text{Res}} (\xi \varphi_R)$$

# Univariate computation

$$\begin{aligned} \langle \varphi_L | \varphi_R \rangle &= \frac{1}{(2\pi i)} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_\omega} \text{Res}_{z=p} \left( (\nabla_\omega^{-1} \varphi_L) \varphi_R \right) = - \sum_{p \in \mathcal{P}_\omega} \text{Res}_{z=p} \left( \varphi_L \nabla_{-\omega}^{-1} \varphi_R \right) \\ &= \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{\text{Res}} (\psi_p \varphi_R) \stackrel{dLog}{=} \sum_{p \in \mathcal{P}_\omega} \frac{\text{Res}_{z=p}(\varphi_L) \text{Res}_{z=p}(\varphi_R)}{\text{Res}_{z=p}(\omega)} \end{aligned}$$

$$\psi_p = \nabla_\omega^{-1} \varphi_L \quad \Rightarrow \quad (d + \omega) \psi_p = \varphi_L$$

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Sanity check:

$$\langle \nabla_\omega \xi | \varphi_R \rangle = \sum_{p \in \mathcal{P}_{\xi \varphi_R}} \text{Res}_{z=p} (\xi \psi_R)$$

# Univariate computation

$$\begin{aligned} \langle \varphi_L | \varphi_R \rangle &= \frac{1}{(2\pi i)} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{\text{Res}} \left( (\nabla_\omega^{-1} \varphi_L) \varphi_R \right) = - \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{\text{Res}} \left( \varphi_L \nabla_{-\omega}^{-1} \varphi_R \right) \\ &= \sum_{p \in \mathcal{P}_\omega} \underset{z=p}{\text{Res}} (\psi_p \varphi_R) \stackrel{dLog}{=} \sum_{p \in \mathcal{P}_\omega} \frac{\underset{z=p}{\text{Res}} (\varphi_L) \underset{z=p}{\text{Res}} (\varphi_R)}{\underset{z=p}{\text{Res}} (\omega)} \end{aligned}$$

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Sanity check:

$$\langle \nabla_\omega \xi | \varphi_R \rangle = \sum_{p \in \mathcal{P}_{\xi \varphi_R}} \underset{z=p}{\text{Res}} (\xi \varphi_R) = 0$$

# Reduction on the Maximal Cut

**Maximal Cut**  $\Rightarrow$  univariate integral representation



$$\omega = d \log u = 0 \quad \text{2 sols.} \Rightarrow \text{2 MIs} \quad \nu = 2$$

$$u = \left( \frac{1}{4} z^2 (s - 2z - 1)(s - 2z + 3) \right)^{\frac{d-5}{2}}$$

The MIs chosen as

$$J_1 = I_{1,1,1,1,1,1,1;0} = \langle e_1 | \mathcal{C} \rangle = \langle 1 | \mathcal{C} \rangle \quad \& \quad J_2 = I_{1,1,1,1,1,1,1;-1} = \langle e_2 | \mathcal{C} \rangle = \langle z | \mathcal{C} \rangle$$

**Decompose**

$$I_{1,1,1,1,1,1,1;-2} = \langle \varphi | \mathcal{C} \rangle = \langle z^2 | \mathcal{C} \rangle$$

**Compute**

$$\langle \varphi | e_i \rangle \quad i = 1, 2 \quad \& \quad \mathbf{C}_{ij} = \langle e_i | e_j \rangle \quad i, j = 1, 2$$

Plug in the **Master Decomposition Formula**

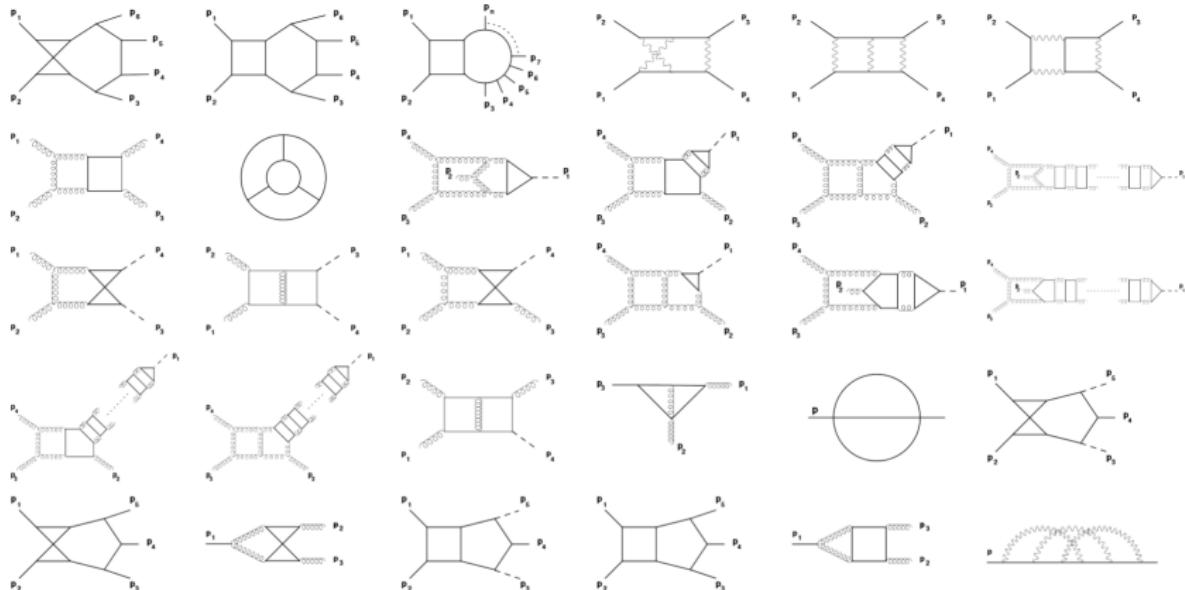
$$c_i = \sum_{j=1}^2 \langle \varphi | e_j \rangle (\mathbf{C}^{-1})_{ji} \quad c_1 = -\frac{(d-4)(s-1)(s+3)}{4(2d-7)} \quad c_2 = \frac{(3d-11)(s+1)}{2(2d-7)}$$

agreement with SYS

# Examples

$\mathcal{O}(30)$  examples checked on the maximal cut

[Frellesvig, Gasparotto, Laporta, Mandal,  
Mastrolia, L.M., Mizera (2019)]



# Multivariate :: Fibration

Univariate: **known**; Multivariate?

$$\int u(z) \varphi_L(z) \stackrel{?}{=} \int dz_n \int dz_{n-1} \cdots \int dz_1 u(z_1, \dots, z_n) \varphi_L(z_1, \dots, z_n)$$
$$\int dz_n \varphi_L^{(n)}(z_n) u^{(n)}(z_n)$$

[Mizera (2019)]  
[Frellesvig, Gasparotto, Mandal,  
Mastrolia, L.M., Mizera (2019)]

**Integration** connects them. How does it translate to our formalism?

$$\int u(\mathbf{z}) \varphi_L(\mathbf{z}) = \int dz_n \cdots \int dz_2 \underbrace{\int dz_1 u(z_1, \dots, z_n) \hat{\varphi}_L(z_1, \dots, z_n)}_{\exists \nu_1 \text{ MI in } z_1}$$
$$= \int dz_n \cdots \int dz_3 \underbrace{\int dz_2 \sum_i c_i^{(1)}(z_n, \dots, z_2) J_i^{(1)}(z_n, \dots, z_2)}_{\vdots \quad \exists \nu_2 \text{ MI in } z_2}$$
$$= \int dz_n \sum_i c_i^{(n)}(z_n) J_i^{(n)}(z_n)$$

# Multivariate :: The algorithm

**Goal:**  $\langle \varphi_L | \varphi_R \rangle$

[Mizera (2019)]  
 [Frellesvig, Gasparotto, Mandal,  
 Mastrolia, L.M., Mizera (2019)]

## Inputs

$\langle \varphi_L | , |\varphi_R \rangle$  n-forms,  
 $\omega = \sum_i^n \omega_i$  connection,  
 $\nu_{n-1}$  number master  $(n-1)$ -forms  
 $\langle e_i^{(n-1)} | , |h_j^{(n-1)} \rangle$  inner basis

$\Rightarrow$

$\mathbf{C}_{ij}^{(n-1)} = \langle e_i^{(n-1)} | h_j^{(n-1)} \rangle$  metric matrix,  
 $\langle \varphi_{L,i}^{(n)} | , |\varphi_{R,i}^{(n)} \rangle$  projected form,  
 $\Omega_{ij}^{(n)}$  projected connection

## Get

## Compute

see also [Matsumoto (1998)]  
 [Matsubara et al. (2019)]

[Weinzierl (2020)]

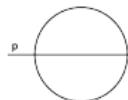
$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}_{\Omega^{(n)}}} \underset{z_n=p}{\text{Res}} \left( \psi_{p,i}^{(n)} \varphi_{R,j}^{(n)} \mathbf{C}_{ij}^{(n-1)} \right)$$

$$\partial_{z_n} \psi_i^{(n)} + \hat{\Omega}_{ij}^{(n)} \psi_j^{(n)} = \hat{\varphi}_{L,i}^{(n)}$$

**Terminating conditions:**  $\Omega^{(1)} = \omega_1$ ,  $\mathbf{C}^{(0)} = \mathbf{1}$ ,  $\varphi_{L,R}^{(1)} = \varphi_{L,R}$

# Multivariate :: Sunrise

**Maximal Cut**  $\Rightarrow$  2 variables integral representation



$$u = (z_1 z_2 (1 - z_1 - z_2))^\gamma$$

**Decompose**  $I_{1,1,1,0,-1} = \langle \varphi | \mathcal{C} \rangle = \langle z_2 | \mathcal{C} \rangle$  on **MIs**  $J_1 = I_{1,1,1,0,0} = \langle e^{(12)} | \mathcal{C} \rangle = \langle 1 | \mathcal{C} \rangle$

**Compute**  $\langle z_2 | 1 \rangle$  &  $\mathbf{C} = \langle 1 | 1 \rangle$

## Inputs

$$\langle \varphi_L | = \langle 1 | , \quad | \varphi_R \rangle = | 1 \rangle ,$$

$$\omega = \sum_i^n \omega_i , \quad \nu_1 = 1$$

$$\langle e^{(1)} | = \langle z_1 | , \quad | h^{(1)} \rangle = | z_1 \rangle$$

## Get

$$\mathbf{C}^{(1)} = \frac{\gamma(z_2 - 1)^4}{8(2\gamma - 1)(2\gamma + 1)},$$

$$\langle \varphi_L^{(2)} | = \langle -\frac{2}{z_2 - 1} | , \quad | \varphi_R^{(2)} \rangle = | -\frac{2}{z_2 - 1} \rangle$$

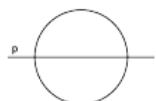
$$\hat{\Omega}^{(2)} = \frac{(3\gamma + 2)z_2 - \gamma}{(z_2 - 1)z_2}$$

## Compute

$$\langle 1 | 1 \rangle = \frac{\gamma^2}{3(3\gamma - 2)(3\gamma - 1)(3\gamma + 1)(3\gamma + 2)}$$

# Multivariate :: Sunrise

**Maximal Cut**  $\Rightarrow$  2 variables integral representation



$$u = (z_1 z_2 (1 - z_1 - z_2))^\gamma$$

**Decompose**  $I_{1,1,1,0,-1} = \langle \varphi | \mathcal{C} \rangle = \langle z_2 | \mathcal{C} \rangle$  on **MIs**  $J_1 = I_{1,1,1,0,0} = \langle e^{(12)} | \mathcal{C} \rangle = \langle 1 | \mathcal{C} \rangle$

**Compute**  $\langle z_2 | 1 \rangle$  &  $\mathbf{C} = \langle 1 | 1 \rangle$

$$\langle z_2 | 1 \rangle = \frac{\gamma^2}{9(3\gamma - 2)(3\gamma - 1)(3\gamma + 1)(3\gamma + 2)} \quad \langle 1 | 1 \rangle = \frac{\gamma^2}{3(3\gamma - 2)(3\gamma - 1)(3\gamma + 1)(3\gamma + 2)}$$

Plug in the **Master Decomposition Formula**

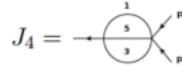
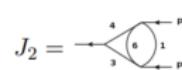
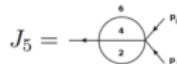
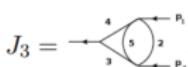
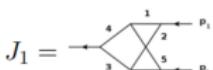
$$c_i = \sum_{i=1}^{\nu} \langle \varphi | e_j \rangle (\mathbf{C}^{-1})_{ji} = \frac{\langle z_2 | 1 \rangle}{\langle 1 | 1 \rangle} \Rightarrow c = \frac{1}{3}$$

# Multivariate :: Non Planar Triangle

$$= \int \frac{u(z) f(z)}{z_1 z_2 z_3 z_4 z_5 z_6} d^7 z$$

**5 MIs**

$$\left\{ \begin{array}{l} \nu_{(1,2,3,4,5,6)} = 1 \\ \nu_{(1,3,4,6)} = 1 \\ \nu_{(2,3,4,6)} = 1 \\ \nu_{(2,4,6)} = 1 \\ \nu_{(1,3,5)} = 1 \end{array} \right.$$



$$= c_1 \cdot \text{Diagram J1} + c_2 \cdot \text{Diagram J2} + c_3 \cdot \text{Diagram J3} + c_4 \cdot \text{Diagram J4} + c_5 \cdot \text{Diagram J5}$$

# Multivariate :: Non Planar Triangle

$$= \int \frac{u(z) z_7}{z_1 z_2 z_3 z_4 z_5 z_6} d^7 z$$

**Cut** $\{1,3,5\}$

$$= \int u_{135} \varphi_{135} \quad u_{135} = z_2^{\rho_2} z_4^{\rho_4} z_6^{\rho_6} u(0, z_2, 0, z_4, 0, z_6, z_7)$$

$$\hat{\varphi}_{135} = \frac{z_7}{z_2 z_4 z_6}$$

$$\varphi_{135} = \hat{\varphi}_{135} dz_2 \wedge dz_4 \wedge dz_6 \wedge dz_7$$

Number of MIs

$$\nu_{(2467)} = 2$$

$$\nu_{(246)} = 3$$

$$\nu_{(24)} = 2$$

$$\nu_{(2)} = 2$$

Inner basis

$$\hat{e}_1^{(2467)} = \frac{1}{z_2 z_4 z_6} \quad \hat{e}_2^{(2467)} = 1$$

$$\hat{e}_1^{(246)} = z_6 \quad \hat{e}_2^{(246)} = z_4 \quad \hat{e}_3^{(246)} = z_2$$

$$\hat{e}_1^{(24)} = z_4 \quad \hat{e}_2^{(24)} = z_2$$

$$\hat{e}_1^{(2)} = 1 \quad \hat{e}_2^{(2)} = z_2$$

# Multivariate :: Non Planar Triangle

$$= \int \frac{u(z) z_7}{z_1 z_2 z_3 z_4 z_5 z_6} d^7 z$$

**Cut**<sub>{1,3,5}</sub>

$$u_{135} = z_2^{\rho_2} z_4^{\rho_4} z_6^{\rho_6} u(0, z_2, 0, z_4, 0, z_6, z_7)$$

$$\hat{\varphi}_{135} = \frac{z_7}{z_2 z_4 z_6}$$

$$\varphi_{135} = \hat{\varphi}_{135} dz_2 \wedge dz_4 \wedge dz_6 \wedge dz_7$$

$$= c_1 + c_4$$

$$c_i = \sum_{j=1}^2 \langle \varphi_{135} | h_j^{(2467)} \rangle \left( \mathbf{C}_{(2467)}^{-1} \right)_{ji} \Rightarrow c_1 = -\frac{s}{2}, \quad c_2 = \frac{(d-3)(3d-10)(3d-8)}{2(d-4)^3 s^2}$$

# Summary and Outlook

## Giving a new perspective

- Direct decomposition in integral basis and direct construction of system of differential equations
- Algebra of Feynman integrals controlled by intersection number
- Intersection number: Scalar product/Projection between Feynman integrals
- useful for both Physics and Mathematics



## A lot of new possibilities

- study of **Differential Equations** for Feynman integrals
- application to different representations
- Combine with Finite Fields
- alternatives algorithm for the multivariate intersection number and for the MI reduction
- Quadratic relations  $\Leftrightarrow$  **Riemann twisted Period Relation**

December 18 - 20, 2019

**MathemAmplitudes 2019**  
Intersection Theory & Feynman Integrals

University of Padova  
Archivio Antico, Palazzo Bo  
Aula Rossina, Dipartimento di Fisica e Astronomia

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<http://indico.infn.it/event/836413/>

[see also [indico.cern.ch/event/836413/](http://indico.cern.ch/event/836413/)]



Thank you for your attention

# Multivariate :: Fibration

$$\int u(\mathbf{z}) \varphi_L(\mathbf{z}) = \int dz_n \sum_i c_i^{(n)}(z_n) J_i^{(n)}(z_n) = \int dz_n \sum_i \varphi_{L,i}^{(n)}(z_n) \int u(\mathbf{z}) e_i^{(n-1)}(\mathbf{z})$$

↓

[Mizera (2019)]  
[Frellesvig, Gasparotto, Mandal,  
Mastrolia, L.M., Mizera (2019)]

$$\langle \varphi_L | = \sum_{i=1}^{\nu_{n-1}} \langle \varphi_{L,i}^{(n)} | \wedge \langle e_i^{(n-1)} |$$

$\varphi_{L,i}^{(n)}$  coefficient of the reduction

$$\langle \varphi_{L,i}^{(n)} | = \sum_j \langle \varphi_L | h_j^{(n-1)} \rangle \left( \mathbf{C}_{(n-1)}^{-1} \right)_{ji}$$

Same decomposition on the dual basis

$$|\varphi_R\rangle = \sum_{i=1}^{\nu_{n-1}} |\varphi_{R,i}^{(n)}\rangle \wedge |h_i^{(n-1)}\rangle \quad \Rightarrow \quad |\varphi_{R,i}^{(n)}\rangle = \sum_j \left( \mathbf{C}_{(n-1)}^{-1} \right)_{ij} \langle e_j^{(n-1)} | \varphi_R \rangle$$

# Multivariate :: Computation

We are interested in

[Mizera (2019)]  
 [Frellesvig, Gasparotto, Mandal, Mastrolia,  
 L.M., Mizera (2019)]

$$\begin{aligned}
 \langle \varphi_L | \varphi_R \rangle &= \frac{1}{(2\pi i)^2} \int_X \iota(\varphi_L) \wedge \varphi_R \\
 &= \frac{1}{(2\pi i)} \int_{X_2} \left( \iota(\varphi_{L,i}^{(2)}) \wedge \varphi_{R,j}^{(2)} \underbrace{\overbrace{\frac{1}{(2\pi i)} \int_{X_1} \iota(e_i^{(1)}) \wedge h_j^{(1)}}^{\langle e_i^{(1)} | h_j^{(1)} \rangle = C_{ij}^{(1)}}} \right) \\
 &= \frac{1}{(2\pi i)} \int_{X_2} \left( \iota(\varphi_{L,i}^{(2)}) \wedge \varphi_{R,j}^{(2)} \ C_{ij}^{(1)} \right) \\
 &= \sum_{p \in \mathcal{P}_{\Omega^{(2)}}} \text{Res}_{z_2=p} \left( \psi_{p,i}^{(2)} \ \hat{\varphi}_{R,j}^{(2)} \ C_{ij}^{(1)} \right) \\
 \Rightarrow \quad &\frac{1}{(2\pi i)^n} \int \iota(\varphi_L) \wedge \varphi_R = \sum_{p \in \mathcal{P}_{\Omega^{(n)}}} \text{Res}_{z_n=p} \left( \psi_{p,i}^{(n)} \ \varphi_{R,j}^{(n)} \ C_{ij}^{(n-1)} \right)
 \end{aligned}$$

$\psi_i^{(n)}$  generalization of univariate case

$$\partial_{z_n} \psi_i^{(n)} + \psi_j^{(n)} \hat{\Omega}_{ji}^{(n)} = \hat{\varphi}_{L,i}^{(n)}$$

# Multivariate :: Connection

New **connection** arise:  $\Omega_{ij}^{(n)}$ . In the 2 variables case:

$$\begin{aligned}\int \varphi(z_1, z_2) u(z_1, z_2) &= \sum_i^{\nu_1} \int dz_2 \varphi_i^{(2)}(z_2) \int dz_1 e_i^{(1)}(z_1, z_2) u(z_1, z_2) \\ &= \sum_i^{\nu_1} \int dz_2 \varphi_i^{(2)}(z_2) u_i(z_2)\end{aligned}$$

**total derivative** is

$$\begin{aligned}0 &= \sum_i^{\nu_1} \int d \left( \varphi_i^{(2)}(z_2) u_i(z_2) \right) = \sum_i^{\nu_1} \int \left( d\varphi_i^{(2)}(z_2) u_i(z_2) + \varphi_i^{(2)}(z_2) du_i(z_2) \right) \\ &= \sum_i^{\nu_1} \int \left( d\varphi_i^{(2)}(z_2) \delta_{ij} + \varphi_i^{(2)}(z_2) \Omega_{ij}^{(2)}(z_2) \right) u_j(z_2)\end{aligned}$$

# Multivariate :: Connection

$$\begin{aligned} \mathrm{d}u_i(z_2) &= \mathrm{d}_{z_2} \int e_i^{(1)}(z_1, z_2) u(z_1, z_2) \\ &= \int \left( \mathrm{d}_{z_2} e_i^{(1)}(z_1, z_2) + \frac{\mathrm{d}_{z_2} u(z_1, z_2)}{u(z_1, z_2)} \wedge e_i^{(1)}(z_1, z_2) \right) u(z_1, z_2) \\ &= \int u(z_1, z_2) (\mathrm{d}_{z_2} + \omega_2 \wedge) e_i^{(1)}(z_1, z_2) \Rightarrow \Omega_{ij}^{(2)}(z_2) \int e_i^{(1)}(z_1, z_2) u(z_1, z_2) \end{aligned}$$

$\Omega_{ij}^{(2)}(z_2)$  **projection** of the outer connection

$$\Omega_{ij}^{(2)}(z_2) = \sum_k \langle (\mathrm{d}_{z_2} + \omega_2 \wedge) e_i^{(1)} | h_k^{(1)} \rangle \left( \mathbf{C}_{(1)}^{-1} \right)_{kj}$$

⇓

$$\hat{\Omega}_{ij}^{(n)}(z_2) = \sum_k \langle (\mathrm{d}_{z_n} + \omega_n \wedge) e_i^{(\mathbf{n}-\mathbf{1})} | h_k^{(\mathbf{n}-\mathbf{1})} \rangle \left( \mathbf{C}_{(\mathbf{n}-\mathbf{1})}^{-1} \right)_{kj}$$