Lecture II :

BRST quantization of BOSONIC STRING

- Quantize the space of solutions

$$S_m[X] + S_{gh}[b,c] \Rightarrow \begin{cases} \partial\bar\partial X^\mu = 0 \\ \partial\bar b = \bar\partial b = 0 \\ \partial\bar c = \bar\partial c = 0 \end{cases}$$

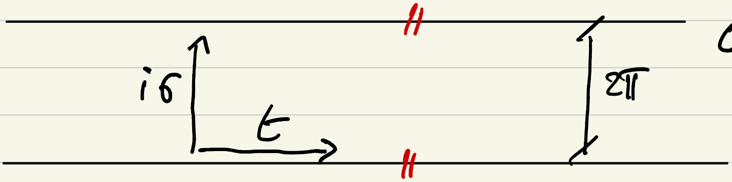
(*) BULK + BOUNDARY

MATTER

① CLOSED STRING

$$\partial\bar\partial X^\mu(w,\bar w) = 0$$

$$X^\mu(w,\bar w) = X_L^\mu(w) + X_R^\mu(\bar w)$$



$w = t + i\sigma$

$\sigma \in [0, 2\pi)$

$$X^\mu(w+2\pi i, \bar w - 2\pi i) = X^\mu(w,\bar w)$$

$$X_L^\mu(w) = \frac{1}{2}x_0^\mu - i\frac{\alpha'}{2}p^\mu(t+i\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{nw}$$

$$X_R^\mu(\bar w) = \frac{1}{2}x_0^\mu - i\frac{\alpha'}{2}p^\mu(t-i\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar\alpha_n}{n}e^{n\bar w}$$

MOMENTUM DENSITY

$$P_\mu \equiv \frac{\partial\mathcal{L}}{\partial\partial_\tau X^\mu} = -i\frac{\partial\mathcal{L}}{\partial\partial_t X^\mu} =$$

$\tau = -it$

$\partial_\tau = i\partial_t$

$$= \frac{i}{2\pi\alpha'}(\partial X_\mu + \bar\partial X_\mu)$$

$\partial = \partial_w$

$\bar\partial = \partial_{\bar w}$

CANONICAL QUANTIZATION

$$[X^\mu(t,\sigma), P_\nu(t,\sigma')] = i\,\eta^\mu{}_\nu\,\delta(\sigma-\sigma')$$

$$\left.\begin{aligned} &[x_0^\mu, p^\nu] = i\,\eta^{\mu\nu} \\ &[\alpha_m^\mu, \alpha_n^\nu] = m\,\eta^{\mu\nu}\delta_{m+n} \\ &[\bar\alpha_m^\mu, \bar\alpha_n^\nu] = m\,\eta^{\mu\nu}\delta_{m+n} \\ &[\alpha, \bar\alpha] = 0 \end{aligned}\right\} \rightarrow \infty \text{ HARMONIC OSCILLATOR}$$

$$\begin{cases} a_m = \frac{1}{\sqrt{m}}\,\alpha_m \\ a_m^\dagger = \frac{1}{\sqrt{m}}\,\alpha_{-m} \end{cases}$$

<u>X is REAL</u>

$$\Rightarrow \alpha_m^* = \alpha_{-m}$$

$$\Downarrow$$

$$\alpha_m^\dagger = \alpha_{-m}$$

<u>matter Hilbert space</u>

$$\text{Span}\left\{\alpha_{-m_1}^{\mu_1}\cdots\alpha_{-m_k}^{\mu_k}\,\bar\alpha_{-n_1}^{\nu_1}\cdots\bar\alpha_{-n_q}^{\nu_q}\,|0,p\rangle\right\}$$

$$\Downarrow$$

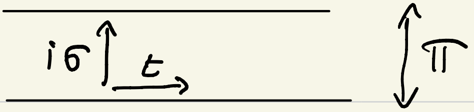
$$t^{\mu_1\dots\mu_k,\nu_1\dots\nu_q}(p)$$

$$\hat p^\mu|0,p\rangle = p^\mu|0,p\rangle$$

$$\alpha_n|0,p\rangle = 0 \quad n>0$$

$$\bar\alpha_n|0,p\rangle = 0 \quad n>0$$

OPEN STRINGS



$w = t + i\sigma$

$$\begin{cases} \partial\bar{\partial} X^\mu(w,\bar{w}) = 0 \\ \delta X_\mu \, \partial_\sigma X^\mu = 0 \Big|_{\sigma=0,\pi} \end{cases}$$

NEUMANN B.C. $\partial_\sigma X^\mu \big|_{\partial\Sigma} = 0 \Rightarrow$ NO FLOW OF MOMENTUM OFF THE BOUNDARY

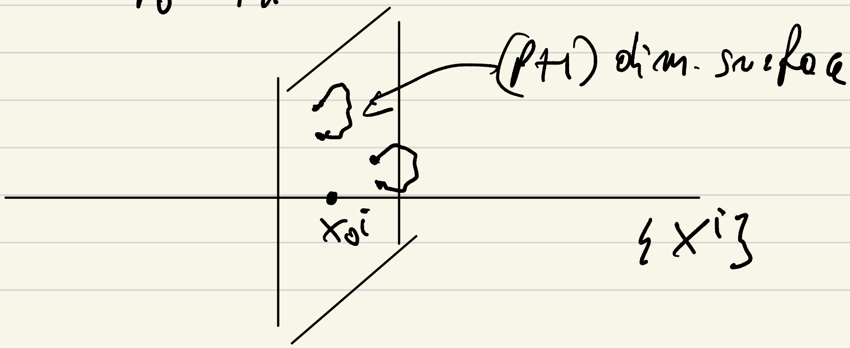
DIRICHLET B.C. $\delta X^\mu \big|_{\partial\Sigma} = 0$ → FIXED END POINTS

$X^\mu(t, \sigma=0) = x_0^\mu$

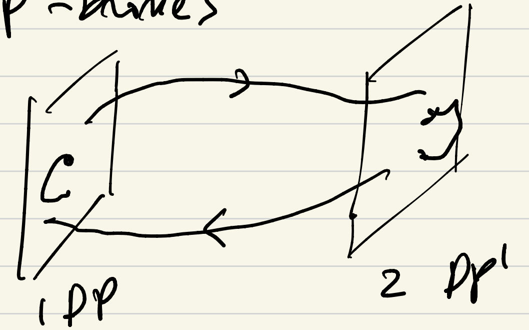
Dp-brane

→ CONFIGURATION in which p+1 directions are N and the remaining are D

$$\begin{cases} \partial_\sigma X^\mu \big|_{\sigma=0,\pi} = 0 & \mu = 0, \dots, p \\ X^i \big|_{\sigma=0,\pi} = x_0^i & i = p+1, \dots, D-1 \end{cases}$$



→ An open string can stretch between 2 Dp-branes



$$\Rightarrow \underbrace{\left(\begin{array}{c|c} 11 & 12 \\ \hline 21 & 22 \end{array}\right)}_{\text{CHAN-PATON MATRIX}}$$

Main object for the open string is still given by the chiral field

$$X_L(w) = \frac{x_L}{2} - i\alpha' p_L w + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{nw}$$

$$X_R(\bar{w}) = \frac{x_R}{2} - i\alpha' p_R \bar{w} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar{\alpha}_n}{n}e^{nw}$$

N-N b.c. $\Rightarrow X_L(w) = X_R(\bar{w})\Big|_{w=\bar{w}}$

$$\Rightarrow x_L = x_R = \frac{x_0}{2}$$

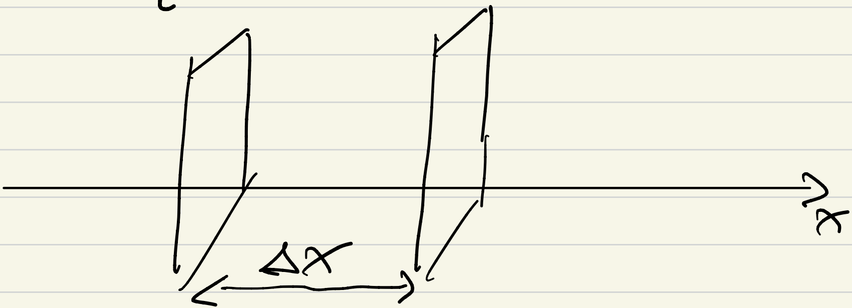
$$\Rightarrow p_L = p_R = \frac{p}{2}$$

$$\Rightarrow \bar{\alpha}_n = \alpha_n$$ ⇒ 1 SET OF OSCILLATORS!

$$X^{NN}(w,\bar w) = x_0 + i\alpha' p^\mu\left(\frac{w+\bar w}{2}\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}\left(e^{nw}+e^{n\bar w}\right)$$

D-D b.c.

$$\begin{cases} \sigma = 0 \to x_0 \\ \sigma = \pi \to x_0 + \Delta x \end{cases}$$



$$X_L(w) = -X_R(\bar w)\Big|_{w=\bar w}$$

$$X^{DD}(w,\bar w) = x_0 + \frac{\Delta x}{\pi}\overbrace{\frac{w-\bar w}{2i}}^{\sigma} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}\left(e^{nw}-e^{-nw}\right)$$

N-D Boundary conditions

$\Longrightarrow$ HALF-INTEGER OSCILLATOR MODES

$\Longrightarrow$ OPEN STRINGS $\Rightarrow$
- DIFFERENT BOUNDARY CONDITIONS → (CHAN-PATON MATRIX)
- ONLY ONE SET OF OSCILLATORS.

GHOST - THEORY

$$S_{bc} = \frac{1}{2\pi}\int d^2w \left(b\bar{\partial}c + \bar{b}\partial\bar{c}\right)$$

EOM

$$\bar{\partial}c = \partial\bar{c} = 0$$
$$\bar{\partial}b = \partial\bar{b} = 0$$

CLOSED STRING

$\sigma \in (0, 2\pi)$

$$c(w) = \sum_{n\in\mathbb{Z}} c_n e^{-nw} \; ; \; \bar{c}(\bar{w}) = \sum_{n\in\mathbb{Z}} \bar{c}_n e^{-n\bar{w}}$$

$$b(w) = \sum_{n\in\mathbb{Z}} b_n e^{-nw} \; ; \; \bar{b}(\bar{w}) = \sum_{n\in\mathbb{Z}} \bar{b}_n e^{-n\bar{w}}$$

$b \rightarrow$ "momentum" of c $\quad$ $\bar{b} \rightarrow$ "momentum" of $\bar{c}$

$$[b(t+i\sigma), c(t+i\sigma')]_+ = \delta(\sigma-\sigma')$$

$$\begin{cases} [b_n, c_m] = \delta_{n+m} \\ [\bar{b}_n, \bar{c}_m] = \delta_{n,+m} \\ [b, \bar{c}] = [\bar{b}, c] = 0 \end{cases}$$

$$c_{-n} = c_n^{\dagger}$$
$$b_{-n} = b_n^{\dagger}$$

$[\;,\;]$ = GRADED COMMUTATOR

$[b, b]$ = comm

$[f, b]$ = comm

$[f, f] = \{\;,\;\}$

GHOST VACUUM

$$|\downarrow\rangle \Big/ \quad c_n|\downarrow\rangle = 0 \quad n>0 \quad \langle\downarrow|c_{-n}=0$$

$$b_n|\downarrow\rangle = 0 \quad n>0 \quad \langle\downarrow|b_{-n}=0$$

$$[c_0, b_0] = 1 \qquad c_0|\downarrow\rangle \equiv |\uparrow\rangle$$

$$c_0^2 = 0 = b_0^2 \qquad b_0|\uparrow\rangle = |\downarrow\rangle \rightarrow \boxed{b_0|\downarrow\rangle = 0}$$

$$\langle\downarrow|\downarrow\rangle = 0 = \langle\downarrow|[c_0, b_0]|\downarrow\rangle =$$

$$\langle\uparrow|\uparrow\rangle = 0$$

$$\boxed{\langle\downarrow|\uparrow\rangle = \langle\uparrow|\downarrow\rangle = \langle\downarrow|c_0|\downarrow\rangle = 1}$$

$$|\downarrow\rangle = c_1|0\rangle \rightarrow |0\rangle = b_{-1}|\downarrow\rangle$$

↳ CONFORMAL VACUUM

$SL(2,\mathbb{C})$-INVARIANT VACUUM

GHOST HILBERT SPACE

- **GHOST NUMBER** = # of ghosts (c) − # of anti ghosts (b)

$$Q_{gh} \equiv - \sum_{k \geq 2} b_{-2} c_k + \sum_{k \geq -1} c_{-k} b_k$$

$$Q_{gh} |\downarrow\rangle = +1 |\downarrow\rangle$$

$$Q_{gh} |\uparrow\rangle = +2 |\uparrow\rangle$$

$$\langle \downarrow | \uparrow \rangle = 1$$

$$Q_{gh} |0\rangle = 0$$

$$Q_{gh} |\hat{0}\rangle = 3 |\hat{0}\rangle \qquad |\hat{0}\rangle \equiv c_{-1} c_0 c_1 |0\rangle =$$

$$= c_{-1} c_0 |\downarrow\rangle =$$

$$= c_{-1} |\uparrow\rangle$$

$$\boxed{\langle \downarrow | \uparrow \rangle = \langle \hat{0} | 0 \rangle = \langle 0 | c_{-1} c_0 c_1 | 0 \rangle = 1}$$

Hilbert space

$$\left(b_{-m_1} \cdots b_{-n_k} c_{-m_1} \cdots c_{-m_q} \right) |\downarrow\rangle \otimes (\text{Right moving part})$$

$$H^{TOT} = H^{matter} \otimes H^{ghost}$$

→ SPACE OF ALL POSSIBLE (UNPHYSICAL) CONFIGURATIONS OF THE STRING.

BRST-QUANTIZATION

$$\delta_B X^\mu = c\,\partial X^\mu + \bar{c}\,\bar{\partial} X^\mu$$

$$\delta_B c = c\,\partial c \qquad \delta_B \bar{c} = \bar{c}\,\bar{\partial}\bar{c}$$

$$\delta_B b = -\frac{1}{\alpha'}\partial X\cdot\partial X + (2b\,\partial c + (\partial b)\,c)$$

$$\delta_B \bar{b} = -\frac{1}{\alpha'}\bar{\partial} X\cdot\bar{\partial} X + (2\bar{b}\,\bar{\partial}\bar{c} + (\bar{\partial}\bar{b})\,\bar{c})$$

δ_B → NOETHER CHARGE ⇒ Q_B

CANONICAL QUANTIZATION

$$\delta_B = [Q_B, \cdot\,]$$

$$\delta_B^2 = 0 \implies \boxed{Q_B^2 = 0}\;!$$ → DELICATE

hol, $\overline{\text{hol}}$ $\qquad \delta_B = [Q_B + \bar{Q}_B, \cdot\,]$

$$\delta_B \phi = [Q_B, \phi] + [\bar{Q}_B, \phi]$$

$$\overset{(-)}{Q}_B = \sum_{m\in\mathbb{Z}} \overset{(-)}{c}_m \overset{(-)}{L}{}^{(m)}_{-m} + \frac{1}{2}\sum_{m\in\mathbb{Z}} \underline{:\overset{(-)}{c}_{-m} \overset{(-)}{L}{}^{(gh)}_{m}:}$$

$$L^{(m)}_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\alpha^\mu_{m-k}\,\alpha^\nu_k:\,\eta_{\mu\nu} \qquad \underbrace{\alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}}\,p^\mu}_{\text{CLOSED STRING}}$$

$$L^{(gh)}_m = \sum_{k\in\mathbb{Z}} (m-k)\,\underline{:b_{m+k}c_{-k}:}$$

NORMAL ORDERING

- CREATION - ANNIHILATION N.O.

MATTER: $|0,p\rangle$ $\quad \alpha_{n>0}$ ARE ANNIHILATORS

GHOST: $\boxed{\text{N.O. on } |0\rangle = b_{-1}|\downarrow\rangle}$

$$|0\rangle \begin{cases} c_m|0\rangle = 0 & m \geq 2 \\ b_m|0\rangle = 0 & m \geq -1 \end{cases}$$

$L_n^{(m)}, L_n^{(gh)}$ ARE VIRASORO OPERATORS

EXERCISE 1

SPACE-TIME DIMENSION

$$[L_n^{(m)}, L_m^{(m)}] = (n-m) L_{n+m}^{(m)} + \frac{D}{12}(n^3-n)\delta_{n+m}$$

EXERCISE 2

$$[L_n^{(gh)}, L_m^{(gh)}] = (n-m) L_{n+m}^{(gh)} + \frac{-26}{12}(n^3-n)\delta_{n+m}$$

$\Rightarrow$ CENTRAL CHARGE c $\begin{cases} c = D \text{ (MATTER)} \\ c = -26 \text{ (GHOST)} \end{cases}$

$$L_n^{(TOT)} = L_n^{(m)} + L_n^{(gh)} \Rightarrow \boxed{c^{TOT} = D - 26}$$

$$\boxed{[Q_B, b_n] = L_n^{TOT}}$$

$$\boxed{[Q_B, Q_B] = \sum_{m\in\mathbb{Z}} \frac{\boxed{D-26}}{12}(m^3-m)\, c_{-m} c_m}$$

$$Q_B^2 \longleftrightarrow C_{TOT}$$

$$C_{TOT} = 0 \Longrightarrow \boxed{Q_B^2 = 0}$$

$$Q_B^2 = 0 \Longrightarrow C_{TOT} = 0$$

$$\boxed{[L_n^{TOT}, L_m^{TOT}]} = [[Q, b_n], L_m^{TOT}] =$$

$$= [Q, [b_n, L_m^{TOT}]] = (n-m)[Q, b_{n+m}] = (b)$$

$$[b_n, L_m^{TOT}] = [b_n, L_m^{gh}] = (n-m)\, b_{n+m}$$

$$\boxed{C(b) = (n-m)\, L_{n+m}^{TOT}} \qquad \boxed{C_{TOT} = 0}$$

$$\boxed{Q_B^2 = 0 \Longleftrightarrow C_{TOT} = 0 \quad (D = 26)}$$

PHYSICAL STATES

$\delta_{GF} \langle f | i \rangle = 0$ (AMPLITUDES DON'T DEPEND ON THE GAUGE FIXING)

$$\Downarrow$$

$\langle f | [Q_B, \cdot] | i \rangle = 0$ $(Q_B^\dagger = Q_B)$

$$\Rightarrow \boxed{Q_B|i\rangle = 0, \; Q_B|f\rangle = 0} \quad \boxed{Q_B^2 = 0}$$

① $Q_B|phys\rangle = 0$ $(\mathrm{Ker}\, Q_B)$

② $|phys\rangle \sim |phys\rangle + Q_B|\Lambda\rangle$ $(\mathrm{Im}\, Q_B)$

$$\boxed{\text{COHOMOLOGY of } Q_B \equiv \frac{\mathrm{Ker}\, Q_B}{\mathrm{Im}\, Q_B}}$$

$$\boxed{\text{PHYSICAL STATES} \longleftrightarrow \text{COHOMOLOGY OF } Q_B}$$

COHOMOLOGY OF Q_B

① $|0\rangle$ → conformal vacuum
SL(2,ℂ) vacuum

$\overset{\shortparallel}{|0,p=0\rangle} \otimes |0\rangle$

$Q_B|0\rangle = 0$ $|0\rangle \neq Q_B|\Lambda\rangle$

$L_m^{(m)}|0\rangle = 0$ $\langle 0|L_m^{(m)} = 0$ $\boxed{m=-1,0,1}$

$L_m^{(gh)}|0\rangle = 0$ $\langle 0|L_m^{(gh)} = 0$ $[m=-1,0,1]$

$[L_i, L_j] = (i-j)L_{i+j}$ $i,j=-1,0,1$

$\Longrightarrow$ SL(2,ℂ)

L_{-1}, L_0, L_1 ⟶

gh=0 COHOMOLOGY

- Physical states are found at gh=1

$$|\Psi\rangle = c_1|\phi^{(matter)}\rangle$$

$$Q_B|\Psi\rangle \Longrightarrow \begin{cases} (L_0^{(m)}-1)|\phi^{(m)}\rangle = 0 \\ L_n|\phi^{(m)}\rangle = 0 \quad n>0 \end{cases} \quad (*1)$$

↳ PHYSICAL STATE CONDITION

$b_0|\Psi\rangle = 0 \to$ "Siegel gauge" condition

$$Q_B|\Psi\rangle = 0 \Longrightarrow \boxed{L_0^{TOT}|\Psi\rangle = 0} \quad (*2)$$

2D - CFT point of view

$(*1)$ $|\phi^{(m)}\rangle$ IS A CONFORMAL PRIMARY OF WEIGHT $h=1$ (WEIGHT IS EIGENVALUE OF L_0)

$(*2)$ $|\Psi\rangle = c_1|\phi^{(m)}\rangle$ IS A CONFORMAL PRIMARY OF WEIGHT $\boxed{h^{TOT}=0}$

COHOMOLOGY

$$\begin{tikzcd}
{|0\rangle_{SL(2,\mathbb{C})}} & gh=0 \arrow[ddd, leftrightarrow, bend left] \\
{c_1 |\underline{\phi^{(m)}}\rangle = |\Psi^1\rangle} & gh=1 \arrow[d, leftrightarrow, bend left] \\
{c_0 c_1 |\underline{\phi^{(m)}}\rangle = |\Psi^2\rangle} & gh=2 \\
{|\hat{0}\rangle.} & gh=3
\end{tikzcd}$$

↳ CONFORMAL PRIMARY OF WEIGHT 1

$$\langle \Psi^1 | \Psi^2 \rangle = \langle \phi_1^{(m)} | c_{-1} c_0 c_1 | \phi_2^{(m)} \rangle$$

↳ "NO GHOST THEOREM"

→ This inner product is POSITIVE DEFINITE [IN THE COHOMOLOGY]