LECTURE III

2D - CFT

RECAP

GAUGE FIXING DIFF × WEYL

$$\boxed{S_{WS} = S_{matter}(X) + S_{ghost}(b, c)}$$

BRST SYMMETRY $\delta_B = [Q_B + \bar{Q}_B, \cdot\,]$

$Q_B^2 = 0 \iff C_{matter} + C_{ghost} = 0 \quad (D = 26)$

PHYSICAL STATE CONDITIONS

WORLD-SHEET		SPACE-TIME
"PHYSICAL STATE"	$Q_B \vert \Psi \rangle = 0$	EQUATION OF MOTION
EQUIVALENCE RELATION	$\vert \Psi \rangle \sim \vert \Psi \rangle + Q_B \vert \Lambda \rangle$	GAUGE INVARIANCE
		($b_0 \vert \Psi \rangle = 0$ GAUGE FIXING)

$$S = \frac{1}{2\pi}\int d^2w \left(\frac{\partial X_\mu \bar{\partial} X^\mu}{\alpha'} + b\bar{\partial}c + \bar{b}\partial\bar{c} \right)$$

CONFORMAL SYMMETRY

$$w \to f(w) \qquad \bar{w} \to \bar{f}(\bar{w})$$

- GAUGE REDUNDANCY

$$dw\,d\bar{w} \to |f'(w)|^2 dw\,d\bar{w}$$

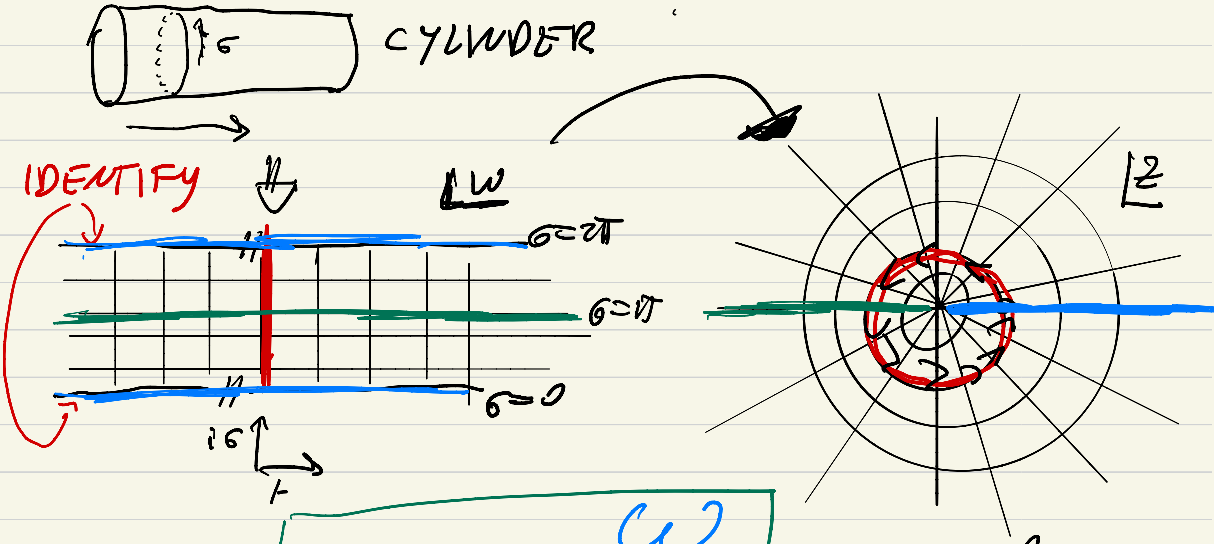
- WEYL SYMMETRY (GAUGE)

$$dw\,d\bar{w} \xrightarrow[\text{WEYL}]{} |f'(w)|^{-2} dw\,d\bar{w}$$

→ CONFORMAL SYMMETRY IS A GAUGE SYMMETRY

⇒ NEED ONLY CONFORMAL INVARIANT "OBJECTS"

CYLINDER $\longrightarrow$ COMPLEX PLANE



$$\boxed{z = e^{w}}$$

$$w = \ln z$$

$$X(w, \bar{w}) = X(\ln z, \ln \bar{z})$$

$$\partial X(w) = \left(\frac{dz}{dw}\right) \partial X(z) \qquad \partial_w X\, dw = \partial_z X\, dz$$

$$b_{ww}(w) = \left(\frac{dz}{dw}\right)^2 b(z) \qquad b(w)\, dw^2 = b(z)\, dz^2$$

$$c^w(w) = \left(\frac{dz}{dw}\right)^{-1} c(z) \qquad c(w)\, dw^{-1} = c(z)\, dz^{-1}$$

PRIMARY FIELD OF WEIGHT $(h, \bar{h})$

$\phi(w, \bar{w}) (dw)^h (d\bar{w})^{\bar{h}}$ is INVARIANT

$$\boxed{\phi(w, \bar{w}) = \left(\frac{dw}{dz}\right)^h \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}} \phi(z, \bar{z})}$$

$\partial X \to (1, 0)$ $\qquad$ $\bar{\partial} X \to (0, 1)$

$b \to (2, 0)$ $\qquad$ $\bar{b} \to (0, 2)$

$c \to (-1, 0)$ $\qquad$ $\bar{c} \to (0, -1)$

$$\phi^{(h)}_{cyl}(w) = \sum_r \phi_r e^{-rw}$$

$$\phi^{(h)}_{\mathbb{C}}(z) = \left(\frac{dw}{dz}\right)^h \phi^{(h)}_{cyl}(w(z)) =$$

$$\boxed{\phi^{(h)}_{\mathbb{C}}(z) = \sum_r \phi_r z^{-r-h}}$$

Ex

$$j^\mu(z) \equiv i\sqrt{\frac{2}{\alpha'}}\,\partial X^\mu = \sum_{m\in\mathbb{Z}} \alpha_m^\mu z^{-m-1}$$

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\,p^\mu$$

$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1}$$

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2}$$

INFINITESIMAL CONFORMAL TRANSFORMATIONS

$$z \to z + \varepsilon(z) = \left(1 + \underline{\varepsilon(z)\partial_z}\right) z$$

$$\varepsilon(z) = \sum_{n \in \mathbb{Z}} \varepsilon_n z^{-n+1}$$

$$z \longrightarrow \boxed{e^{\varepsilon(z)\partial_z} z \equiv f(z)}$$

$$\varepsilon(z)\partial_z = -\sum_{n \in \mathbb{Z}} \varepsilon_n \ell_{-n} \equiv \delta_\varepsilon$$

$$\ell_{-n} \equiv -z^{n+1}\partial_z$$

EXERCISE 1

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}$$

EXERCISE 2

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{\varepsilon_1 \partial \varepsilon_2 - \varepsilon_2 \partial \varepsilon_1}$$

PRIMARY FIELD UNDER INF. CONF. TRANSF

$$\phi(z) \to \phi'(z') = \left(\frac{dz}{dz'}\right)^h \phi(z)$$

$$z'(z) = z + \epsilon(z) \qquad \frac{\partial z'}{\partial z} = 1 + \partial\epsilon$$

$$\phi'(z+\epsilon(z)) = (1 - h\partial\epsilon)\,\phi(z)$$

$$\phi'(z) = (1 - h\partial\epsilon)\,\phi(z-\epsilon) =$$

$$= \phi(z) + \delta_\epsilon \phi(z)$$

$$\boxed{\delta_\epsilon \phi(z) = -\left(\epsilon\partial\phi + h\partial\epsilon\right)(z)}$$

RADIAL ORDERING

cylinder : t-ordering $\quad w = t + i\sigma$

$z = e^{w}$

$\Downarrow$

$|z|$-ordering

$$R\left(\phi_1(z)\,\phi_2(w)\right) = \begin{cases} \phi_1(z)\,\phi_2(w) & |z| > |w| \\ (-1)^{\varepsilon}\,\phi_2(w)\,\phi_1(z) & |z| < |w| \end{cases}$$

GRASSMANNALITY OF ϕ_1 and ϕ_2

EQUAL RADIUS GRADED COMMUTATOR

$$\left[\phi_1(z), \phi_2(w)\right]_{|z|=|w|} =$$

GRASSMANNALITY

$$= \lim_{\varepsilon \to 0^+} \phi_1(z)\,\phi_2(w)\Big|_{|z|=|w|+\varepsilon} \mp \phi_2(w)\,\phi_1(z)\Big|_{|z|=|w|-\varepsilon}$$

ENERGY-MOMENTUM TENSOR

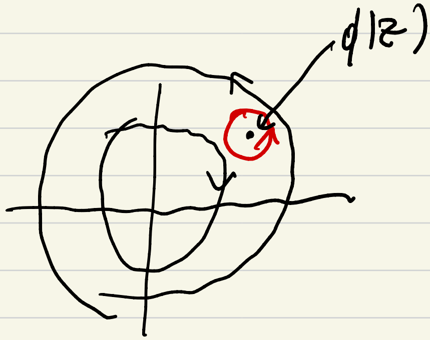
$$T_{\alpha\beta} = \frac{1}{4\pi} \frac{\delta S}{\delta g^{\alpha\beta}}$$

$$T_{zz} = T(z) \qquad T_{\bar z \bar z} = \bar T(\bar z)$$

$$T_{zz} \,,\; T_{\bar z \bar z} \,,\; \cancel{T_{z \bar z}}$$

Diff : $\partial_{\bar z} T_{zz} = 0 \qquad \partial_z T_{\bar z \bar z} = 0$

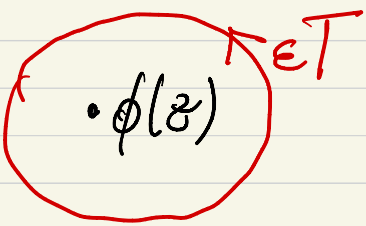
Weyl : $T_\alpha{}^\alpha = 0 \rightarrow T_{z\bar z} = 0$



$$\delta_\varepsilon = -[T_\varepsilon, \cdot\,]$$

$$T_\varepsilon = \oint_0 \frac{dw}{2\pi i} \varepsilon(w) T(w)$$

$$\delta_\varepsilon \phi(z) = -\oint_z \frac{dw}{2\pi i} \varepsilon(w) T(w) \phi(z)$$



$\Rightarrow$ ONLY SENSIBLE TO THE SHORT-DISTANCE BEHAVIOUR OF $T(w)\phi(z)$

OPERATOR PRODUCT EXPANSION

$$\phi_i^{(h_i)}(z)\,\phi_j^{(h_j)}(w) = \sum_k \underline{C_{ij}{}^{k}}\,\frac{\phi_k^{(h_k)}(w)}{(z-w)^{h_i+h_j-h_k}}$$

STRUCTURE CONSTANTS

$T(z)\,\phi^{(h)}(z)$, $\underline{\phi^{(h)} \text{ e PRIMARY}}$

$$\delta_\varepsilon \phi(z) = -\oint_z \frac{dw}{2\pi i}\,\varepsilon(w)\,T(w)\,\phi(z) =$$

$$= -(\varepsilon\partial\phi + h\,\partial\varepsilon\,\phi)(z)$$

$$\boxed{T(w)\,\phi^{(h)}(z) = \frac{h\,\phi(z)}{(w-z)^2} + \frac{\partial\phi(z)}{w-z} + \text{regular}}$$

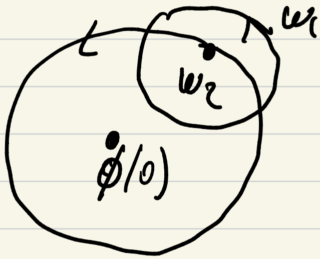
↳ DEFINITION OF A PRIMARY FIELD

$$T(z)\, T(w) = ?$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_1 \partial \epsilon_2 - \epsilon_2 \partial \epsilon_1} \qquad \phi^{(4)}(0)$$

$$\delta_\epsilon = \oint \frac{dw}{2\pi i}\, \epsilon(w)\, T(w)$$

$$\oint_0 \frac{dw_2}{2\pi i} \oint_{w_2} \frac{dw_1}{2\pi i}\, \epsilon_1(w_1)\, \epsilon_2(w_2)\, T(w_1)\, T(w_2)\, \phi(0)$$



$$T(w_1)\, T(w_2) = \frac{\# 1}{(w_1-w_2)^4} + \frac{2T(w_1)}{(w_1-w_2)^2} + \frac{\partial T(w_2)}{w_1 - w_2} + \text{reg.}$$

$$\boxed{\# = \frac{c}{2}}$$

$c \to$ CENTRAL CHARGE OF THE CFT

$\hookrightarrow$ DATA OF THE CFT

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$L_n = \oint_0 \frac{dz}{2\pi i} z^{n+1} T(z)$$

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} \underbrace{(n^3-n)}_{0 \;\; n=\pm 1, 0} \delta_{n+m}$$

$$L_{-1}, L_0, L_1 \to SL(2,\mathbb{C})$$

$\Rightarrow$ QUASI-PRIMARY FIELD

(T is quasi primary)

EXERCISE

$$[L_n, \phi^{(h)}(z)] = z^n\left(z\partial + (n+1)h\right)\phi(z)$$

$$[L_n, \phi_m] = \left(n(h-1) - m\right)\phi_{n+m}$$

HILBERT SPACE OF A CFT

① $SL(2,\mathbb{C})$-invariant vacuum

$|0\rangle$ $\qquad \lim_{w\to 0} T(w)|0\rangle$ = WELL-DEFINED

$\lim_{w\to 0} \hat{\#}\frac{1}{w^n}|0\rangle$ $\quad n>0$

$\hat{\#}|0\rangle = 0$

$$L_n|0\rangle = 0 \qquad n \geq -1$$

$$L_n^\dagger = L_{-n} \longrightarrow \langle 0|L_n = 0 \qquad n \leq 1$$

L_{-1}, L_0, L_1 annihilate $|0\rangle$, $\langle 0|$

$|0\rangle, \langle 0|$ $\quad SL(2,\mathbb{C})$ INVARIANT VACUUM

(12) $\phi^{(n)}(z)$ $\qquad \phi^{(n)}(z)\Big|_{z\to 0}|0\rangle$ WELL-DEFINED

$$\phi_n|0\rangle = 0 \qquad n \geq -h+1$$

$$\langle 0|\phi_n = 0 \qquad n \leq h-1$$

$\rightarrow$ c_{-1}, c_0, c_1

$\Downarrow$

ZERO MODES

(3) STATE - OPERATOR CORRESPONDENCE

$$\phi(z) \longrightarrow |\phi\rangle = \lim_{z\to 0} \phi(z)|0\rangle_{SL(2,\mathbb{C})}$$

$$L_{-1} = \partial_z = \ell_{-1}$$

$$\phi(z) = e^{zL_{-1}}\phi(0)\,e^{-zL_{-1}}$$

$$\phi(z)|0\rangle_{SL(2,\mathbb{C})} = e^{zL_{-1}}\phi(0)\underbrace{e^{-zL_{-1}}}_{1}|0\rangle_{SL(2,\mathbb{C})} =$$

$$= e^{zL_{-1}}\phi(0)|0\rangle = e^{zL_{-1}}|\phi\rangle$$

$$|\phi\rangle \rightarrow e^{zL_{-1}}|\phi\rangle = \phi(z)|0\rangle$$

PRIMARIES AND DESCENDENTS

$$\phi^{(h)}(z) \rightarrow |\phi^{(h)}\rangle = \phi^{(h)}(0)|0\rangle_{SL(2,\mathbb{C})}$$

$T-\phi$ OPE $\rightarrow \begin{cases} L_0|\phi^{(h)}\rangle = h|\phi\rangle \\ L_n|\phi^{(h)}\rangle = 0 & n>0 \end{cases}$

L_{-n} $n>0$ $\rightarrow$ RAISING OPERATORS ACTING ON "VACUA" GIVEN BY PRIMARY STATES

$$\mathrm{Span}\left\{L_{-k_1}\cdots\cdots L_{-k_n}|\phi^{(h)}\rangle\right\} \rightarrow \text{VERMA MODULE}$$

$\Downarrow$

DESCENDENTS OF $\phi^{(h)}$

$$\rightarrow L_0 = h + \sum_{i=1}^{n} k_i$$

BOSONIC STRING CFT

matter and ghost

2- DECOUPLED CFT's

$$X: \quad T^{(X)} = -\frac{1}{\alpha'} :\partial X_\mu \partial X^\mu:$$

$$bc: \quad T^{(bc)} = -2 :b\partial c:(z) - :\partial b c:(z)$$

$: \; : \rightarrow$ NORMAL ORDERING ON $|0\rangle_{SL(2,\mathbb{C})}$

$\rightarrow$ "bc" CFT is UNIVERSAL

$\rightarrow$ X - CFT is more "accidental"

$\hookrightarrow$ A CHOSEN BACKGROUND OF (BOSONIC) STRING THEORY

$$A(z)B(w) = \underbrace{A(z)\,B(w)} + :A(z)B(w):$$

CONTRACTION $\Downarrow$ THE SINGULAR PART OF THE OPE

NORMAL-ORDERED PRODUCT $\Downarrow$ REGULAR PART

$$:AB:(w) = \lim_{z \to w} :A(z)B(w): = \oint_w \frac{dz}{2\pi i} \frac{A(z)B(w)}{z-w}$$

$\to$ IN CASE OF FREE CFT'S (LIKE X, bc)

$\to$ EQUIVALENT TO OSCILLATOR NORMAL ORDERING ON $|0\rangle_{SL(2,\mathbb{C})}$

GHOST CFT

$$S=\frac{1}{2\pi}\int d^2z\,\left(b\bar{\partial}c+\bar{b}\partial\bar{c}\right)$$

$b(z)$ $(2,0)$

$c(z)$ $(-1,0)$

$$b(z)\,c(w)=\frac{1}{z-w} \qquad [b_n,c_m]=\delta_{n+m}$$

$$T^{(bc)}=-2:b\partial c:(z)-:\partial b\,c:(z)$$

EXERCISE

$$\frac{c}{2}$$

$$T^{bc}(z)\,T^{bc}(w)=\frac{-13}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w} \qquad c=-26$$

$$j_{gh}(z)=-:bc:(z)=\sum_{n\in\mathbb{Z}} j_n^{gh}\,z^{-n-1}$$

$$Q_{gh}=j_0^{(gh)}=\oint\frac{dz}{2\pi i}\,j_{gh}(z)$$

$$T(z)\,j_{gh}(w)=\frac{3}{(z-w)^3}+\frac{j_{gh}(w)}{(z-w)^2}+\frac{\partial j_{gh}(w)}{z-w}$$

Primaries

$\boxed{c \,,\, c\partial c \,,\, c\partial c \partial^2 c \,,}$ - - - - (RAISING #gh)

$b \,,\, b\partial b \,,\, b\partial b \partial^2 b \,,$ - - - - (LOWERING #gh)

$|0\rangle_{SL(2,\mathbb{C})} \longleftrightarrow 1$

$|\downarrow\rangle = c(0)|0\rangle \longleftrightarrow c(z)$

$c_1|0\rangle$

$|\uparrow\rangle = \partial c c(0)|0\rangle \longleftrightarrow \partial c c(z)$

$c_0 c_1|0\rangle$

$|\hat{0}\rangle = \frac{1}{2}\partial^2 c \partial c c(0)|0\rangle \longleftrightarrow \frac{1}{2}\partial^2 c \partial c c(z)$

$c_{-1} c_0 c_1 |0\rangle$

EXERCISE

- $\langle 0| c_{-1} c_0 c_1 |0\rangle = 1$
- $c(z) = \sum_{n \in \mathbb{Z}} c_n z^{-n+1}$

① $\langle \frac{1}{2}\partial^2 c \partial c c(z) \rangle = \langle 0| \frac{1}{2}\partial^2 c \partial c c(z)|0\rangle =$
$= 1$

② $\langle c(z_1) c(z_2) c(z_3) \rangle = z_{12} z_{13} z_{23}$

$z_{ij} = z_i - z_j$

BRST-CHARGE

$$Q_B = \sum_{n\in\mathbb{Z}} c_{-n} L_n^{(m)} + \frac{1}{2} :c_{-n} L_n^{(gh)}: =$$

$$= \oint \frac{dz}{2\pi i} \Big(\underbrace{c T^{(m)}(z) + \frac{1}{2} :c T^{(gh)}:(z)}_{\tilde{J}_B(z)} \Big)$$

$$j_B(z) = \tilde{J}_B(z) + \frac{3}{2}\partial^2 c \rightarrow$$ PRIMARY OF $h=1$

① $$[Q_B, X^\mu(z)] = \oint_z \frac{dw}{2\pi i} j_B(w) X^\mu(z) = c\,\partial X^\mu$$

② $$[Q_B, c(z)] = c\,\partial c(z)$$

③ $$[Q_B, b(z)] = T^X(z) + T^{(gh)}(z) = T^{(TOT)}(z)$$

$$j_B(z)\, j_B(w) = \cdots + \frac{c^{(m)}-26}{12}\,\frac{1}{z-w}\,\partial^3 c\, c(w) + \cdots$$

$$Q^2 \Rightarrow \oint_z \frac{dw}{2\pi i}\, j_B(w)\, j_B(z) = 0 \qquad c^{(m)}=26$$

$j_B(z)$ is BRST-EXACT

$$j_B(z) = -\left[Q_B,\, j_{gh}(z)\right]$$

$$\Downarrow$$

$$\left[Q_{gh},\, Q_B\right] = +Q_B$$

COHOMOLOGY OF Q_B

$gh=0 \qquad 1$

$$\boxed{gh=1 \qquad c\,V^{(m)}_{h=1}(z)} \rightarrow \text{PHYSICAL FIELDS}$$

$gh=2 \qquad c\partial c\, V^{(m)}_{h=1}(z)$

$gh=3 \qquad \frac{1}{2}\partial^2 c\,\partial c\, c(z)$

$$\int \frac{\mathcal{D}h\,\mathcal{D}X}{V_{\text{gauge}}}\; \underbrace{V_1 \cdots V_n}_{\to\ \text{GAUGE INVARIANT OPERATORS}}\; e^{-S[h,X]}$$

$gh=1$ elements of the cohomology are the first set of GAUGE-INVARIANT OPERATORS

$$c\,V_1^{(m)}(z) \rightarrow \text{NON-INTEGRATED VERTEX OPERATORS (CHIRAL PART)}$$

$$Q_B=0$$

FULL CLOSED STRING THEORY

$$c\,\bar c\, V^{(m)}_{(1,1)}(z,\bar z) \rightarrow \begin{array}{l} Q_B=0 \\ \bar Q_B=0 \end{array} \quad \boxed{Q_B+\bar Q_B=0}$$

→ ANOTHER SET OF GAUGE-INVARIANT OPERATORS

$$\underset{(-1)\,(1)}{c\,V_1^{(m)}} \longrightarrow \underbrace{\int dz}_{(-1)}\, V_1^{(m)}(z)$$

$$Q = cT + \tfrac{1}{2}\, cT^{gh}$$

$$\int dz\, V_1^{(m)}(z) \Rightarrow \boxed{\text{DIMENSION} = 0}$$

$$\text{III}$$

$$\mathbb{V} \rightarrow \text{INTEGRATED VERTEX OPERATOR}$$

$$[Q_B, V_1^{(m)}(z)] = \oint_z \frac{dw}{2\pi i}\, c\, \underbrace{T^{(m)}(w)\, V_1^{m}}(z) =$$

$$= \oint_z \frac{dw}{2\pi i}\, c(w) \left(\frac{V_1^{m}(z)}{(w-z)^2} + \frac{\partial V_1^{m}(z)}{w-z} \right) =$$

$$= \partial \left(c V_1^{(m)} \right)$$

$$[Q_B, \mathbb{V}] = 0 = \int dz\, \partial \left(c V_1^{(m)} \right) = 0$$

UP TO POSSIBLE BOUNDARY TERMS

(1) $c V_1^{(m)}(z)$ → NON INTEGRATED
V. O. $gh=1$

(2) $\int dz\, V_1^{(m)}(z)$ → INTEGRATED
V.O. $gh=0$

⇒ STRING'S AMPLITUDES USE
<u>BOTH</u>

$$\int \mathcal{D}b\,\mathcal{D}c \,\langle \mu, b\rangle\, V_i \cdots V_i\, e^{b_{\alpha\beta}(P, C)^{\alpha\beta}}$$