SLIGHT CORRECTION FROM YESTERDAY

→ SBN'S CONJECTURES

D-$\bar{D}$

: ALL POSSIBLE LOWER DIMENSIONAL BRANES

3rd CONJECTURE

CLOSED STRING VACUUM

1st conj. : ENERGY $= -T_{D\bar{D}}$

2nd conj. : NO OPEN STRINGS EXCITATIONS

- Superstring :
  1) ✓ ANALYTICALLY PROVEN
  2) ✓
  3) → $\boxed{\text{still to prove!}}$

- Bosonic :
  1) ✓
  2) ✓ → ANALYTICALLY PROVEN
  3) ✓

ALL BRANES ARE UNSTABLE HERE

Lecture VIII

ELEMENTARY OVERVIEW ON STRINGS AMPLITUDES

Given a set of physical states

GAUGE INVARIANT
VERTEX OPERATORS

$$\{\underbrace{V_1 \cdots V_{n_c}}_{\text{CLOSED STRINGS}} ; \underbrace{\Phi_1 \cdots \Phi_{n_o}}_{\text{OPEN STRINGS}}\}$$

$$A(1, \ldots, n_c ; 1, \ldots, n_o) =$$

$$= g_s^{n_c + \frac{n_o}{2}} \sum_{g=0}^{\infty} g_s^{2(g-1)} \sum_{b=1}^{\infty} g_s^{b} A^{(g,b)}(V_1 \cdots V_{n_c} ; \Phi_1, \ldots, \Phi_{n_o})$$

WHERE:

$$A^{(g,b)}(V_1 \cdots V_{n_c} ; \Phi_1, \ldots, \Phi_{n_o}) = \overbrace{}^{\text{PHYSICAL VERTEX OPERATORS}}$$

$$= \int_{\mathcal{M}} dt^i \left\langle \prod_{i=1}^{m} \langle \partial_i \hat{h}(t), b \rangle \left[V_1 \cdots V_{n_c} ; \Phi_1, \ldots, \Phi_{n_o}\right] \right\rangle_{\Sigma_{b,g}}$$

$\mathcal{M}$ ↳ MODULI SPACE INTEGRAL

$$\underline{\#\text{moduli} - \#\text{CKV}} = -3\chi = 3(2g + b - 2)$$

$$\underset{\#\text{SCKs} - \#\text{smoduli}}{\underline{\text{TOTAL PICTURE}}} = -2\chi = +2(2g + b - 2)$$

TREE-AMPLITUDES (ORIENTED)

$\#_{M} - \#_{CKV} = -6$ (6 c ZERO MODES)

$P = -4$ $\langle e^{-2\phi} e^{-2\phi} \rangle_C$

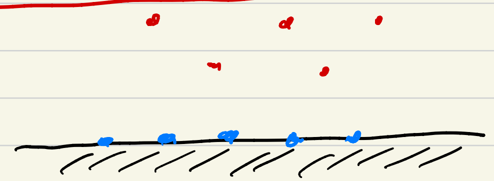
(1) SPHERE $g=0,\ b=0$

$\boxed{(2)\ \text{DISK}\quad g=0,\ b=1}$

$\#_{M} - \#_{CKV} = -3$ (3 c ZERO MODES)

$P = -2$ $\langle e^{-2\phi} \rangle_{UHP}$

UHP



$$t \to g_s^{n_c + \frac{n_o}{2}}\, g_s^{-2+1}\, A^{(g=0,b=1)}\left(V_1 \cdots V_{n_c};\ \Phi_1, \dots, \Phi_{n_o}\right)$$

Now we have to choose the gauge invariant operators to correctly saturate the ZERO MODES FROM THE (SUPER) GHOSTS

$$\langle e^{-2\phi}\ \tfrac{1}{2} c \partial c \partial^2 c \rangle = 1$$

→ GHOST $= 3$

→ PICT $= -2$

⇒ INTEGRATED / UN-INTEGRATED VERTEX OP's

+ ASSIGNEMENT OF PICTURE

Let's see some example

① VENEZIANO AMPLITUDE

→ OPEN BOSONIC STRINGS ON A D25-brane

→ 4-open-strings amplitude

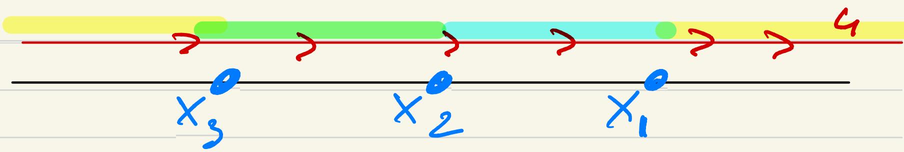
$$V_t^{(x)} = c\, e^{ip\cdot X}(x)$$ NON INTEGRATED

$$\mathbb{V}_t = \int_{-\infty}^{\infty} dx\, e^{ip\cdot X}(x)$$ INTEGRATED $\boxed{\alpha' p^2 = 1}$

- 3 NON INTEGRATED $\langle 0|c_{-1} c_0 c_1|0\rangle = 1$
- 1 INTEGRATED

$$A = g_s \, P\int_{-\infty}^{+\infty} dx \left\langle c\, e^{ip_1\cdot X}(x_1)\; c\, e^{ip_2\cdot X}(x_2)\; c\, e^{ip_3\cdot X}(x_3)\; e^{ip_4\cdot X}(x) \right\rangle_{UHP}$$

PATH-ORDERED INTEGRAL



$$+ (2 \leftrightarrow 3)$$

→ WE HAVE TO SUM OVER ALL POSSIBLE (UNPHYSICAL) ORDERINGS

GUIDED EXERCISE FOR THE VENEZIANO AMPLITUDE

① Send $(x_1, x_2, x_3) \to (\Lambda \to \infty, 1, 0)$

(NOTHING CHANGES $\Rightarrow$ CHECK IT!)

② Compute the correlator

$$\langle c\phi_1(\Lambda)\, c\phi_2(1)\, c\phi_3(0)\, \phi_4(x) \rangle \Big|_{\Lambda \to \infty} = \#$$

$$\phi_i(x) = e^{i p_i \cdot X}(x)$$

<u>Use</u> • $\langle c(x_1)\, c(x_2)\, c(x_3) \rangle = x_{12} x_{13} x_{23}$

• $\langle \prod_{i=1}^{N} e^{i p_i \cdot X}(x_i) \rangle = \delta(p_i) \prod_{i<j} (x_i - x_j)^{2\alpha' p_i \cdot p_j}$ $\boxed{x_i > x_j}$!

$$\# = |x|^{2\alpha' p_3 \cdot p_4} \, |1-x|^{2\alpha' p_2 \cdot p_4}$$

NOTICE ABSOLUTE VALUE!
$x \in (-\infty, +\infty)$ + PATH ORDERING

<u>USE</u> $\alpha' p_i^2 = 1$
$\sum_i p_i = 0$

③ Split the path-ordered exponential

$$\int_{-\infty}^{+\infty} \to \int_{-\infty}^{0} dx + \int_{0}^{1} dx + \int_{1}^{\infty} dx$$

and change variable to have all the 3 integrals in $(0,1)$

$$\int_{-\infty}^{+\infty} dx \to \int_0^1 dy \Big[y^{2\alpha' P_3\cdot P_4}(1-y)^{2\alpha' P_1\cdot P_4} + y^{2\alpha' P_3\cdot P_4}(1-y)^{2\alpha' P_2\cdot P_4} + y^{2\alpha' P_1\cdot P_4}(1-y)^{2\alpha' P_2\cdot P_4} \Big]$$

(4) Realize that the other ordering $(2\leftrightarrow 3)$ gives the SAME contribution

(5) Define the Mandelstam variables

$$S = -(P_1+P_2)^2 \; ; \; t = -(P_1+P_3)^2 \; ; \; u = -(P_1+P_4)^2$$

$$S+t+u = -\sum P^2 = -\frac{4}{\alpha'}$$

$$2\alpha' P_i \cdot P_j = -2 - \alpha' S_{ij} \qquad \begin{cases} S_{12} = S \\ S_{13} = t \\ S_{14} = u \end{cases}$$

⑥ Finally write

$$A(1,2,3,4) = 2 g_s^2 \, \delta(P) \left[I(s,t) + I(t,u) + I(u,s) \right]$$

$$I(a,b) = \int_0^1 dy \, y^{-2-\alpha' a} (1-y)^{-2-\alpha' b} =$$

$$= B(1-\alpha' a, 1-\alpha' b)$$

EULER BETA FUNCTION

$$= B(-\alpha_0(a), -\alpha_0(b))$$

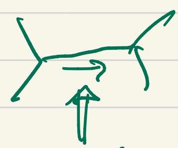
$$\alpha_0(x) = 1 + \alpha' x$$

REGGE-SLOPE

Also

$$I(s,t) = \frac{\Gamma(-\alpha_0(s))\,\Gamma(-\alpha_0(t))}{\Gamma(-\alpha_0(s) - \alpha_0(t))}$$

$S \leftrightarrow t$ CHANNEL DUALITY



ALL STRINGS STATES ARE EXCHANGED

② 3-GLUON SCATTERING (SUPERSTRING)

For simplicity let's do it on N D3-branes of Type IIB

$\rightarrow$ 3 OPEN STRING IN NS SECTOR ON THE DISK

- c-ZERO MODES: 3 NON integrated V.O.'s
- PICTURE : $(-1, -1, 0)$

$$\langle V_1^{(-1)}(x_1)\, V_2^{(0)}(x_2)\, V_3^{(-1)}(x_3) \rangle$$

$+ \ (1 \leftrightarrow 2)$ (inequivalent ordering)

$$V_i^{(-1)}(x) = A^{(i)}_{\mu_i}(p_i)\, c\, \psi^{\mu_i} e^{-\phi} e^{i p_i \cdot X}(x)$$

$$V_i^{(0)}(x) = X_0 V_i^{(-1)}(x) = \left[Q, \xi V_i^{(-1)}(x) \right] \quad [\alpha' = 1]$$

$$= A^{(i)}_{\mu_i}(p_i) \Big[c \left(j^{\mu_i} + (k \cdot \psi) \psi^{\mu_i} \right) - \underbrace{\eta e^{\phi} \psi^{\mu_i}}_{\text{DOES NOT PARTICIPATE!}} \Big] e^{i p_i \cdot X}(x)$$

- IT DOES NOT MATTER WHERE I DISTRIBUTE THE PICTURE:

$$\langle V_1^{(-1)}(x_1)\, \underline{V_2^{(0)}(x_2)}\, V_3^{(-1)}(x_3) \rangle =$$

$$= \langle V_1^{(-1)}(x_1) \left[Q, \xi V_2^{(-1)}(x_2)\right] V_3^{(-1)}(x_3) \rangle_{SHS} =$$

$$= \langle \xi V_1^{(-1)}(x_1) \left[Q, \xi V_2^{(-1)}(x_2)\right] V_3^{(-1)}(x_3) \rangle_{LHS}$$

$$= \langle \left[Q, \xi V_1^{(-1)}(x_1)\right] \xi V_2^{(-1)}(x_2)\, V_3^{(-1)}(x_3) \rangle_{LHS} =$$

$$= \langle \underline{V_1^{(0)}(x_1)}\, V_2^{(-1)}(x_2)\, V_3^{(-1)}(x_3) \rangle_{SHS}$$

(NEED ALL THE V's BRST INVARIANT!)

GUIDE THROUGH THE COMPUTATION

$$\langle V_{p_1}^{(-1)}(x_1)\, V_{p_2}^{(0)}(x_2)\, V_{p_3}^{(-1)}(x_3)\rangle =$$

$$= \underbrace{\mathrm{tr}\left[A_{\mu_1} A_{\mu_2} A_{\mu_3}\right]}_{\text{CHAN-PATON MATRICES}} \cdot \qquad \tilde{j}^{\mu} = j^{\mu} + (k\cdot\psi)\psi^{\mu}$$

$$\cdot \langle c\psi^{\mu_1} e^{-\phi} e^{ip_1\cdot X}(x_1)\; c\,\tilde{j}^{\mu_2} e^{ip_2\cdot X}(x_2)\; c\psi^{\mu_3} e^{-\phi} e^{ip_3\cdot X}(x_3)\rangle$$

Use the following correlators

$$\langle c(x_1)\, c(x_2)\, c(x_3)\rangle = x_{12}\, x_{13}\, x_{23}$$

$$\langle e^{-\phi}(x_1)\, e^{-\phi}(x_3)\rangle = \frac{1}{x_1 - x_3}$$

$$\langle \psi^{\mu}(x)\, \psi^{\nu}(y)\rangle = \frac{\eta^{\mu\nu}}{x-y}$$

$$\langle \psi^{\mu}(x)\, \psi^{\sigma}\psi^{\nu}(y)\, \psi^{\rho}(z)\rangle = \frac{\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\sigma\rho}}{(x-y)(y-z)}$$

$$\langle e^{ip_1\cdot X}\, e^{ip_2\cdot X}\, e^{ip_3\cdot X}\rangle = \delta(p_1+p_2+p_3) \qquad \Big|\; p_i^2 = 0, \quad p_1+p_2+p_3 = 0$$

$$\langle e^{ip_1\cdot X}(x_1)\, :j^{\nu} e^{ip_2\cdot X}(x_2):\, e^{ip_3\cdot X}(x_3)\rangle =$$

$$= \left(\frac{p_3^{\nu}}{x_2 - x_3} - \frac{p_1^{\nu}}{x_1 - x_2}\right)\delta(p_1+p_2+p_3)$$

• Use transversality conditions

$$A_{\mu_i}(p_i)\, p_i^{\mu_i} = 0 \Rightarrow \boxed{p_i^{\mu_i} = 0}$$

Finally (EXERCISE)

$$\langle V_{p_1}^{(-1)} V_{p_2}^{(0)} V_{p_3}^{(-1)} \rangle = \mathrm{tr}\left[A_{\mu_1}(p_1) A_{\mu_2}(p_2) A_{\mu_3}(p_3)\right] \cdot t^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)\, \delta(p_1+p_2+p_3)$$

where

$$t^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = \eta^{\mu_1\mu_2} p_2^{\mu_3} + \eta^{\mu_2\mu_3} p_3^{\mu_1} + \eta^{\mu_3\mu_1} p_1^{\mu_2}$$

• Other ordering $(2 \leftrightarrow 3)$ [IT DOESN'T MATTER WHERE IS THE PICTURE!]

$$\Rightarrow \mathrm{tr}\left[A_{\mu_1}(p_1) A_{\mu_3}(p_3) A_{\mu_2}(p_2)\right] t^{\mu_1\mu_3\mu_2}_{(p_1,p_3,p_2)}\, \delta(P)$$

• Compute

$$t^{\mu_1\mu_2\mu_3} + t^{\mu_1\mu_3\mu_2} =$$

$$= -\eta^{\mu_1\mu_2} \underset{0}{p_3^{\mu_3}} - \eta^{\mu_2\mu_3} \underset{0}{p_1^{\mu_1}} - \eta^{\mu_1\mu_3} \underset{0}{p_2^{\mu_2}}$$

So this finally gives

$$\langle V_1 V_2 V_3 \rangle + \langle V_1 V_3 V_2 \rangle =$$

$$= \mathrm{tr}\left[A_{\mu_1}\left[A_{\mu_2}, A_{\mu_3}\right]\right] t^{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)\, \delta(\underline{p})$$

→ this the amplitude obtained by the YM vertex

$$\boxed{\partial_\mu A_\nu [A^\mu, A^\nu]}$$

③ GLUON - GLUINO^2 COUPLING

let's stay in $D=10$ (i.e. D9 branes) for convenience

$$\langle V_R^{(-\frac{1}{2})} V_{NS}^{(-1)} V_R^{(-\frac{1}{2})} \rangle$$

$\Rightarrow$ TOTAL PICTURE IS $-\frac{1}{2}-\frac{1}{2}-1=-2$

$\Rightarrow$ NO NEED FOR PICTURE CHANGING

$\Rightarrow$ THIS IS REALLY AN ELEMENTARY VERTEX

$$V_R^{(-\frac{1}{2})}(x) = \lambda^{\alpha}(p)\, c\, S_{\alpha} e^{-\phi/2} e^{ip\cdot X}(x)$$

$$V_{NS}^{(-1)}(x) = A_{\mu}(p)\, c\, \psi^{\mu} e^{-\phi} e^{ip\cdot X}(x)$$

$$p^2=0 \qquad p_{\mu}A^{\mu}=0 \qquad p_{\mu}\gamma^{\mu}_{\alpha\beta}\lambda^{\beta}=0$$

• 1^{st} ordering

$$\langle V^{(-\frac{1}{2})}_{p_1}(x_1)\, V^{(-1)}_{p_2}(x_2)\, V^{(-\frac{1}{2})}_{p_3}(x_3)\rangle = (*)$$

$$= \mathrm{Tr}\left[\lambda^\alpha(p_1) A_\mu(p_2) \lambda^\beta(p_3)\right] \cdot$$

$$\cdot \langle c S_\alpha e^{-\phi/2} e^{ip_1\cdot X}(x_1)\, c\psi^\mu e^{-\phi} e^{ip_2\cdot X}(x_2)\, c S_\beta e^{-\phi/2} e^{ip_3\cdot X}(x_3)\rangle$$

Use

$$\langle e^{ip_1\cdot X} e^{ip_2\cdot X} e^{ip_3\cdot X}\rangle = \delta(p_1+p_2+p_3) \quad (p_i^2=0)$$

$$\langle S_\alpha e^{-\phi/2}(x_1)\, \psi^\mu e^{-\phi}(x_2)\, S_\beta e^{-\phi/2}(x_3)\rangle = \frac{\gamma^\mu_{\alpha\beta}}{x_{12} x_{13} x_{23}}$$

$$\langle c(x_1) c(x_2) c(x_3)\rangle = x_{12} x_{23} x_{13}$$

$$(*) = \mathrm{Tr}\left[\lambda^\alpha \gamma^\mu_{\alpha\beta} A_\mu \lambda^\beta\right] \delta(p_1+p_2+p_3)$$

• Other ordering

$$-\langle V^{(-\frac{1}{2})}_{p_1}(x_1)\, V^{(-\frac{1}{2})}_{p_3}(x_2)\, V^{(-1)}_{p_2}(x_3)\rangle =$$

GRASSMANALITY A_μ VS λ^α

$$= -\mathrm{Tr}\left[\lambda^\alpha \gamma^\mu_{\alpha\beta}\lambda^\beta A_\mu\right]\delta(p_1+p_2+p_3)$$

Summing the 2 orderings

$$\Rightarrow \mathrm{Tr}\left[\lambda^\alpha \gamma^\mu_{\alpha\beta}[\lambda^\beta, A_\mu]\right]$$

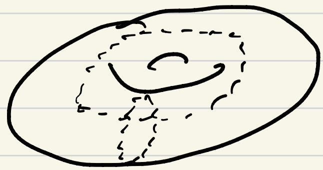
↳ SYM COUPLING

⇒ MORE TREE AMPLITUDES (ALSO WITH PURE-SPINORS)

→ O. SCHLOTTERER LECTURES @ GGI 2019 WORKSHOP ⇒ YOUTUBE
"STRING THEORY FROM A WORLD-SHEET PERSPECTIVE"

A LOOK AT 1-LOOP AMPLITUDES

- THE TORUS

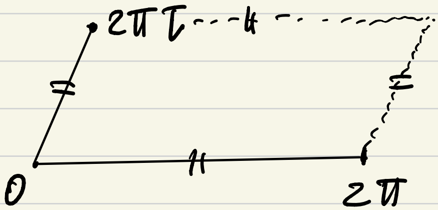


$\#_b - \#_c = 0$

2 moduli + 2 CKV → ROTATION OF THE TORUS AROUND THE TWO CYCLES

$b, \bar{b}$ $\quad$ $c, \bar{c}$

↳ $\tau = \mathrm{Re}\,\tau + i\,\mathrm{Im}\,\tau$



→ metric : $dz\,d\bar{z}$ but we have the identification

$$z \sim z + 2\pi n + 2\pi m \tau$$

Define $z = \sigma_1 + \tau \sigma_2$

$$\sigma_1 \in (0, 2\pi) \qquad \sigma_2 \in (0, 2\pi)$$

$$dz\,d\bar{z} = |d\sigma_1 + \tau\, d\sigma_2|^2$$

EXPLICIT DEPENDENCE ON THE MODULUS!

→ GLOBAL DIFFEOMORPHISMS

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \qquad \begin{matrix} ad - bc = 1 \\ a, b, c, d \in \mathbb{Z} \end{matrix}$$

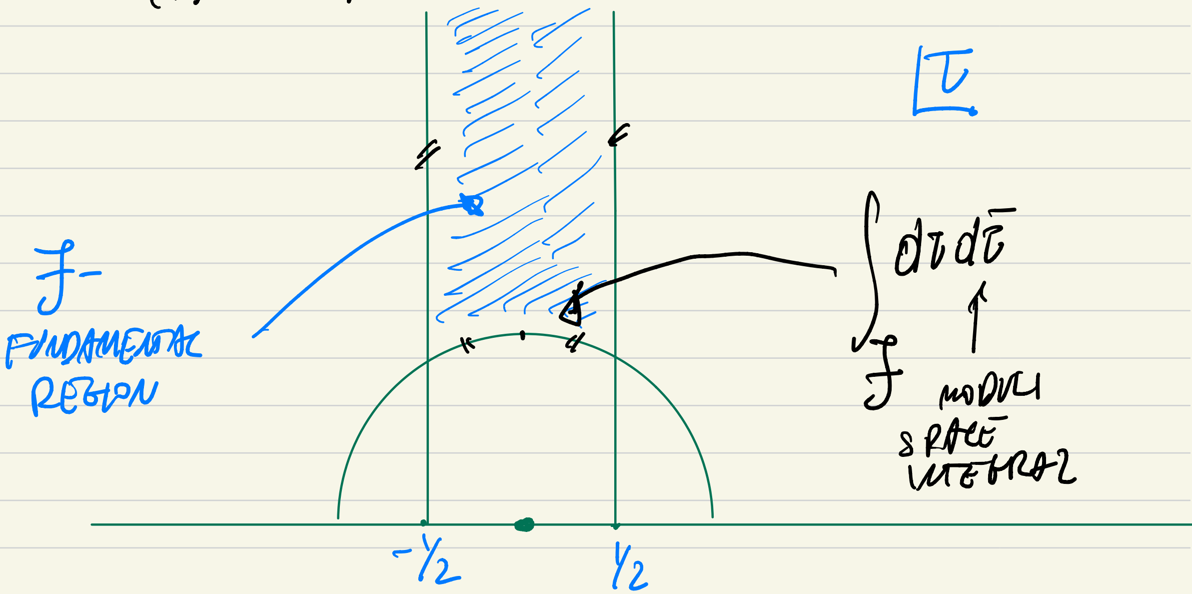
$$\Updownarrow$$

$$\tau \to \frac{d\tau + c}{b\tau + a}$$

→ IT GIVES THE SAME TORUS UP TO RESCALING

As a result

$$\tau \in \frac{UHP}{(P)SL(2,\mathbb{Z})} \rightarrow \begin{array}{l} S : \tau \to -\frac{1}{\tau} \\ T : \tau \to \tau + 1 \end{array}$$



Metric moduli (holomorphic quadratic differentials)

$$h + \delta_m h = dz\,d\bar z + \epsilon\, dz^2 + \bar\epsilon\, d\bar z^2$$

$$\tau + \delta\tau \Rightarrow \boxed{\epsilon = \frac{\delta\tau}{2i\,\mathrm{Im}\,\tau}}$$

$$\frac{\partial h}{\partial \tau} = \frac{1}{2i\,\mathrm{Im}\,\tau}$$
$$\frac{\partial h}{\partial \bar\tau} = \frac{1}{-2i\,\mathrm{Im}\,\tau}$$
$\rightarrow$ [CONSTANTS ON THE TORUS!]

$$\Rightarrow \left\langle \frac{\partial h}{\partial \tau}, b \right\rangle \left\langle \frac{\partial h}{\partial \bar\tau}, \bar b \right\rangle =$$

$$\frac{1}{2i\,\mathrm{Im}\tau}\int d^2z\, \underline{b(z)} \;\; \frac{1}{-2i\,\mathrm{Im}\tau}\int d^2w\, \underline{\bar b(\bar w)}$$

$\Rightarrow$ b-ghosts insertions

$\Rightarrow$ BUT $b, \bar b$ ZERO MODES ARE CONSTANT

$\int d^2z \to \mathrm{Im}\tau \quad \int d^2w \to \mathrm{Im}\tau$

AREA OF THE TORUS

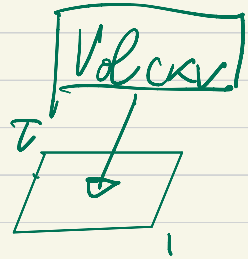
$$\boxed{\frac{1}{4} b_0 \bar b_0}$$

$\leftarrow$ b ZERO MODES ON THE TORUS

$\to$ The c-ghost zero modes can be similarly fixed

$$\to \frac{1}{\mathrm{Im}(\tau)} c_0 \bar c_0$$

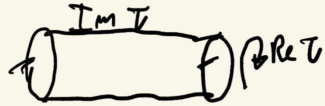
$\to$ AREA OF THE TORUS



Generic amplitude on the torus is thus

$$A(1, \ldots, n) = \int_{\mathcal{F}} \frac{d\tau\, d\bar\tau}{\underbrace{\mathrm{Im}\,\tau}_{\text{CKV VOLUME}}} \int d^2z_1 \cdots d^2z_n \left\langle b_0 \bar b_0 c_0 \bar c_0 V_1(z_1) \cdots V_n(z_n) \right\rangle_{T_\tau}$$

VACUUM AMPLITUDE



$$\mathcal{A}^{\text{vacuum}} = \int_{\mathcal{F}} \frac{d\tau\, d\bar{\tau}}{\operatorname{Im}\tau} \left\langle b_0 c_0 \bar{b}_0 \bar{c}_0 \right\rangle_{\text{TORUS}(\tau)} =$$

GHOST FERMION NUMBER

$$= \int \frac{d\tau\, d\bar{\tau}}{\operatorname{Im}\tau} \underbrace{\operatorname{Tr}\left[(-1)^{F_{gh}}\, b_0 c_0\, q^{L_0^{\text{TOT}}}\, \bar{b}_0 \bar{c}_0\, \bar{q}^{\bar{L}_0^{\text{TOT}}}\right]}_{\text{OPERATOR REALIZATION OF THE CFT CORRELATOR}}$$

$$\boxed{q = e^{2\pi i \tau}}$$

$$\operatorname{Tr} \to \mathcal{H}^{(\text{super})\text{-ghosts}}_{(bc,\, \beta\gamma)} \otimes \mathcal{H}^{(\text{super})\text{-matter}}_{(X,\, \psi)}$$

$(X^{\pm}, \psi^{\pm}), (X^{i}, \psi^{i})$

$\Rightarrow$ LIGHT-CONE REDUCTION

The TRACE OVER THE $\boxed{\text{SUPERGHOST SECTOR}}$ CANCELS THE TRACE OVER THE $\boxed{\text{LIGHT CONE DIRECTIONS}}$

Technical point: only one picture (say the natural) should be allowed to propagate in loops.

$$X^{\pm} = X^{0} \pm X^{D-1}$$

$$\psi^{\pm} = \psi^{0} \pm \psi^{D-1}$$

Finally giving a simpler result

$$A^{(\text{vacuum})} = \int_{\mathcal{F}} \frac{d\tau\, d\bar\tau}{(\operatorname{Im}\tau)^2}\ \underbrace{\operatorname{Tr}^{\perp}\left[q^{H}\,\bar q^{\bar H}\right]}_{\text{PARTITION FUNCTION OF THE TRANSVERSE } (X,\psi) \text{ CFT}}$$

$(\text{CKV})\cdot(\text{MOMENTUM INTEGRAL in } p^{\pm})$

$$H_R = L_0^{(R,\perp)} \qquad H_{NS} = L_0^{(NS,\perp)} - \frac{1}{2}$$

DUE TO THE DIFFERENT WEIGHTS OF THE VACUUM

- $\dfrac{d\tau\, d\bar\tau}{(\operatorname{Im}\tau)^2}$ IS $SL(2,\mathbb{Z})$ (MODULAR) INVARIANT

- $\Rightarrow$ $\boxed{\operatorname{Tr}^{\perp}\left[q^{H}\,\bar q^{\bar H}\right] \text{ MUST BE MODULAR INVARIANT!}}$

Type IIA/B give the SAME partition function

$$\left[\mathrm{Tr}_{NS}\left(P_{GSO^{(+)}_{NS}}\, q^{L_0^{NS}-\frac{1}{2}}\right)+\mathrm{Tr}_R\left(P_{GSO^{(+)}_R}\, q^{L_0^R}\right)\right]\cdot$$

$$\cdot\left[\mathrm{Tr}_{\overline{NS}}\left(\bar{P}_{GSO^{(+)}_{\overline{NS}}}\, \bar{q}^{\bar{L}_0^{NS}-\frac{1}{2}}\right)+\mathrm{Tr}_{\bar{R}}\left(\bar{P}_{GSO^{(\pm)}_{\bar{R}}}\, \bar{q}^{\bar{L}_0^R}\right)\right]$$

IIA/IIB

$$= Z(\tau,\bar{\tau})$$

$\Rightarrow$ THE PARTITION FUNCTION COUNTS THE BOSONIC AND FERMIONIC D.O.F. WITH GRASSMANALITY

$\Rightarrow$ WE EXPECT IT VANISHES DUE TO SPACE-TIME SUSY

Explicitly we get the celebrated

$$Z(\tau,\bar{\tau}) = \frac{1}{4}\underbrace{\frac{1}{(\mathrm{Im}\,\tau)^4}}_{\text{TRANSVERSE MOMENTA}}\underbrace{\frac{1}{|\eta(\tau)|^{24}}\Big|\underbrace{\theta_3^4(\tau)-\theta_4^4(\tau)}_{NS}-\underbrace{\theta_2^4(\tau)}_{R}\Big|^2}_{\text{TRANSVERSE } \partial X,\, \psi}$$

And indeed

$$Z(\tau,\bar\tau) \equiv 0 \quad \text{due to the}$$

RIEMANN IDENTITY

$$\boxed{\theta_3^4(\tau) - \theta_4^4(\tau) - \theta_2^4(\tau) = 0}$$

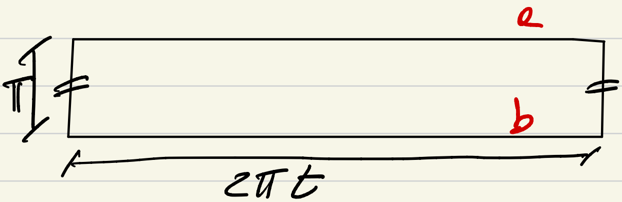
"AEQUATIO IDENTICA SATIS ABSTRUSA"

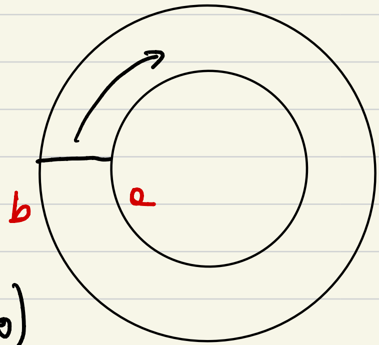
$\hookrightarrow$ IT IS A "CONSEQUENCE" OF 10-D SUPERSYMMETRY.

MODULAR INVARIANCE IS OK THANKS TO

T	S
$\theta_2(\tau+1) = e^{i\pi/4}\,\theta_2(\tau)$	$\theta_2\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\,\theta_4(\tau)$
$\theta_3(\tau+1) = \theta_4(\tau)$	$\theta_3\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\,\theta_3(\tau)$
$\theta_4(\tau+1) = \theta_3(\tau)$	$\theta_4\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\,\theta_2(\tau)$
$\eta(\tau+1) = e^{i\pi/12}\,\eta(\tau)$	$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\,\eta(\tau)$

OPEN STRING 1-LOOP VACUUM AMPLITUDE AND NO-FORCE CONDITION





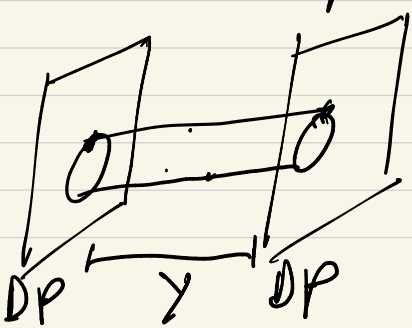
ANNULUS

1 REAL MODULUS $t \in (0,\infty)$

- 1 b, 1 c zero modes
- LIGHT-CONE REDUCTION

$$\mathcal{A}^{1\text{-loop}} = \int_0^\infty \frac{dt}{(2t)^2}\left[\mathrm{Tr}^{\perp}_{NS}\left(P_{GSO}\, e^{-2\pi t\left(L_0^{NS}-\frac{1}{2}\right)}\right) + \mathrm{Tr}^{\perp}_{R}\left(P_{GSO}\, e^{-2\pi t L_0^{R}}\right)\right]$$

Consider two parallel BPS-Dp branes

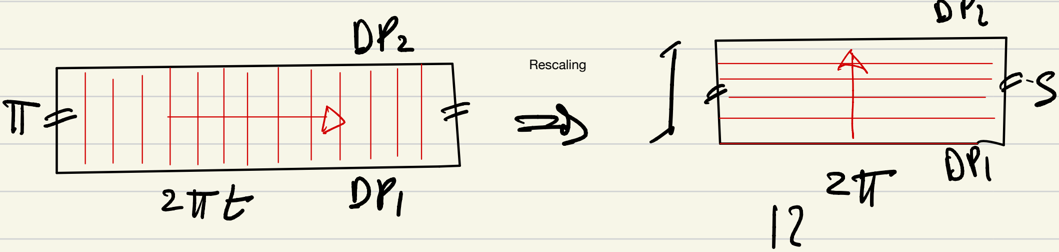


Explicit computation gives

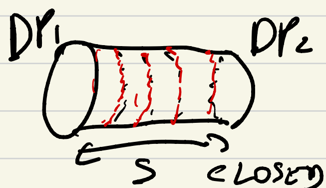
$$A^{1\text{-loop}} = V_{p+1} \int_0^\infty \frac{dt}{t}\, \underbrace{e^{-\frac{t}{2\pi\alpha'}Y^2}}_{\text{SEPARATION}} \underbrace{\frac{1}{(8\pi^2\alpha' t)^{\frac{p+1}{2}}}}_{\substack{\text{TRANSVERSE} \\ +\text{L.C. MOMENTA}}} \frac{1}{2} \underbrace{\left(\frac{\theta_3^4 - \theta_2^4 - \theta_4^4}{\eta^{12}}\right)(it)}_{\substack{\text{OSCILLATORS} \\ (=0!)}}$$

Polchinski: do the change of variable

$$s = \frac{\pi}{t}$$



OPERATORIALLY:



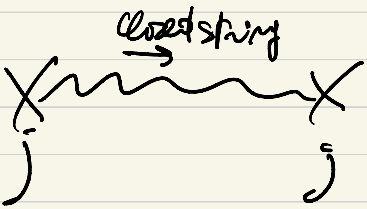
$$\mathrm{Tr}^{\perp}\left[e^{-2\pi t H_{\text{open}}}\right] = {}_{\perp}\langle DP_1 |\, e^{-\frac{\pi}{t} H_{\text{closed}}} \,| DP_2 \rangle_{\perp}$$

(TRANSVERSE) BOUNDARY STATES

The change of variable allows to

REINTERPRET THE LOOP VACUUM AMPLITUDE

AS A TREE LEVEL AMPLITUDE



j = SOURCE $\Rightarrow$ BOUNDARY STATE

$$A^{1\text{-loop}} = \frac{2V_{p+1}}{64(4\pi^2\alpha')^{p-4}} \int_0^\infty ds\, e^{-\frac{Y^2}{4\pi\alpha' s}} \frac{s}{(4\pi\alpha' s)^{\frac{9-p}{2}}} \left(\frac{\theta_3^4-\theta_4^4-\theta_2^4}{\eta^{12}}\right)(2is)$$

$s \to \infty$ = "LONG" STRING PROPAGATOR

$\rightarrow$ LOCALIZATION ON MASSLESS EXCHANGE

Taking only the LEADING TERM in $s \to \infty$ and integrating one finds

$$A^{1\text{-loop}} = \frac{\pi}{8}(4\pi^2\alpha')^{3-p} V_{p+1}\left(16_{NS} - 16_{R}\right)\frac{1}{\pi\alpha'} G_{9-p}(Y) + \dots$$

$$G_D(Y) \equiv \frac{1}{(D-2)\,Vol(S^{D-1})} \frac{1}{Y^{D-2}}$$

NEWTON/COULOMB POTENTIAL in D-TRANSVERSE DIMENSIONS

$\Rightarrow$ ATTRACTIVE POTENTIAL FROM NS-NS sector
(GRAVITON AND DILATON)

$\Rightarrow$ REPULSIVE POTENTIAL FROM R-R sector

$\Rightarrow$ NO NET FORCE $\Rightarrow$ BPS CONDITION!

- Matching with a FIELD THEORY MODEL where a GRAVITON and RR-FORM is EXCHANGED BETWEEN p-DIMENSIONAL SOURCES WITH GIVEN TENSION T_p and CHARGE DENSITY Q_p one finally finds

$\kappa_{10} = 8\pi^{7/2}\alpha'^2 g_s$

$$\boxed{T_p = Q_p = \frac{1}{g_s (2\pi)^p \alpha'^{\frac{p+1}{2}}} = \frac{\sqrt{2\pi}\left(2\pi\sqrt{\alpha'}\right)^{3-p}}{\left(\sqrt{2}\,\kappa_{10}\right)}}$$