BPS quivers of five-dimensional SCFTs, Topological Strings and q-Painlevé equations

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Based on 2007.11596, in collaboration with G. Bonelli, A. Tanzini

5d gauge theories are not UV free, but can have UV completion as interacting fixed points

5d $\mathcal{N} = 1$ SCFTs constructed and classified from string theory realization :

M-theory/Topological Strings on open CY_3

[Seiberg, Morrison, Intriligator 1996,1997]

Discrete (cluster) integrable systems defined from geometry of CY_3

[Goncharov Kenyon 2013]

[Fock Marshakov 2016]

- q-difference equations for the partition function of the theory
- Time evolution generates the spectrum of BPS states (in appropriate regions of moduli space)

Bilinear equations: differential vs difference equations

Differential equation (4d on $\mathbb{R}^4$, IIA on CY_3)
$$Z^D_{4d}(\tau,a)\frac{d^2}{d\tau^2}Z^D_{4d}(a)-\left(\frac{d}{d\tau}Z^D_{4d}(a)\right)^2=-e^{i\pi\tau}Z^D_{4d}(a+{}^1/_2)Z^D_{4d}(a-{}^1/_2)$$
Discrete equation (5d on $\mathbb{R}^4\times S^1_R$, Topological String/M-theory on CY_3)
$$Z^D_{5d}(\tau+Rg_s,a)Z^D_{5d}(\tau-Rg_s,a)=Z^D_{5d}(\tau,a)^2-R^2e^{i\pi\tau}Z^D(\tau,a+{}^1/_2)Z^D_{5d}(\tau,a-{}^1/_2)$$

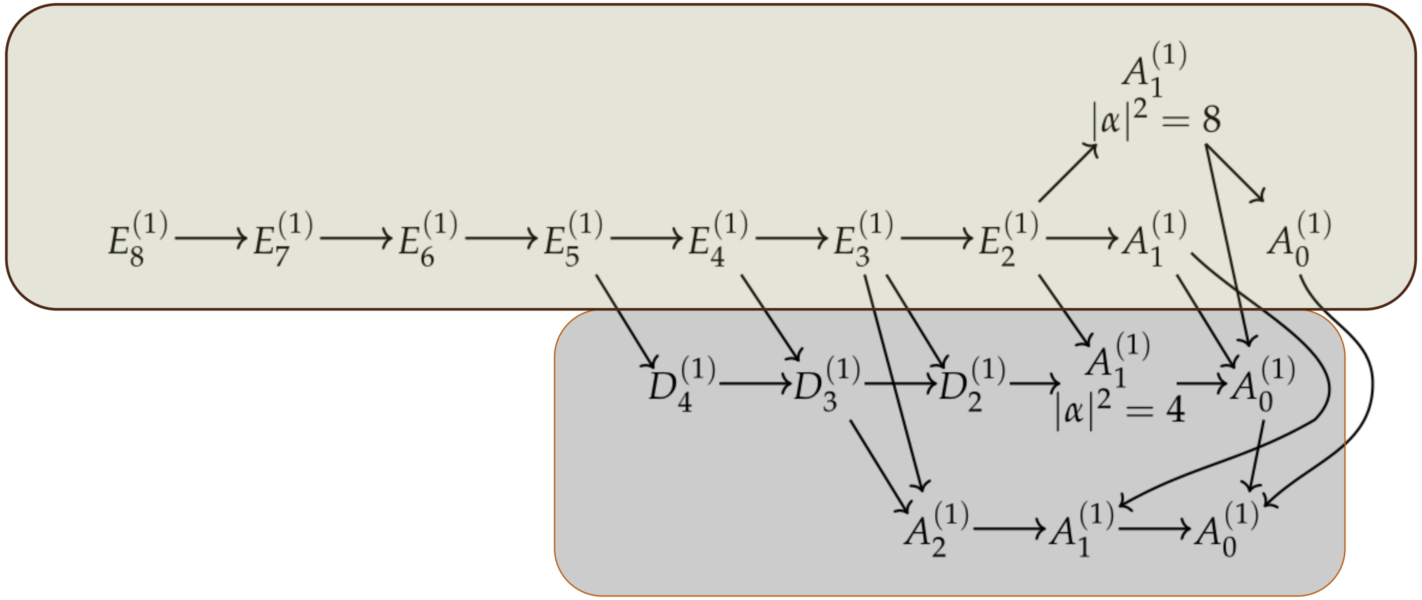
$$\tau_1(qt)\tau_1(q^{-1}t)=\tau_1(t)^2-z^{{}^1/_2}\tau_3(t)^2$$

$$q=e^{Rg_s}$$

$$t=R^4e^{2\pi i\tau}$$

Discrete integrable equations and 5d SCFTs

5d uplift of 4d gauge theories [Bershtein Shchechkin Gonin 2016-2020, Jimbo Nagoya Sakai 2017-2020]
Cluster Integrable Systems [Bershtein Gavrylenko Marshakov 2018,2019]
Quantum Curves [Bonelli Grassi Tanzini 2019]

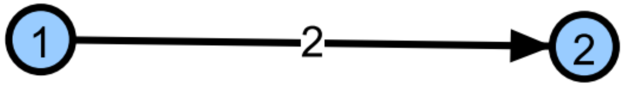


- Classification of discrete Painlevé equations by Sakai based on spaces of initial condition [Sakai 2001]
- Lower part of the diagram: discrete versions of the Painlevé equations for 4d $SU(2)$ gauge theories (see talk by Fran)
- Upper part of the diagram: q-difference Painlevé equations satisfied by 5d partition functions
- Same diagram as string theory classification of 5d SCFTs

Main tool: BPS quivers & cluster algebras

- BPS quiver encodes BPS states and their Dirac pairing
- Quiver obtained from the CY_3 "engineering" the theory
  $\vec{\gamma} \rightarrow$ BPS charges

4d SYM



$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

E/M charges

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- $\vec{\tau} \rightarrow$ Cluster variables: 5d partition functions on $\mathbb{R}^4_\hbar \times S^1_R$
  [Bershtein Gavrylenko Marshakov 2018]
- $\vec{y} \rightarrow$ Coefficients: masses, gauge couplings, 5d radius, string coupling

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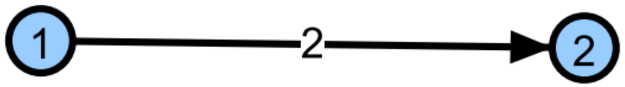

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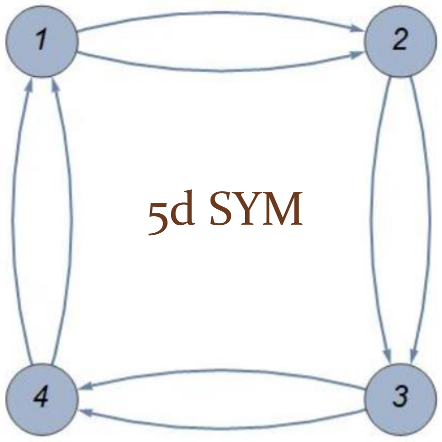

$$B = \begin{pmatrix} 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \end{pmatrix}$$

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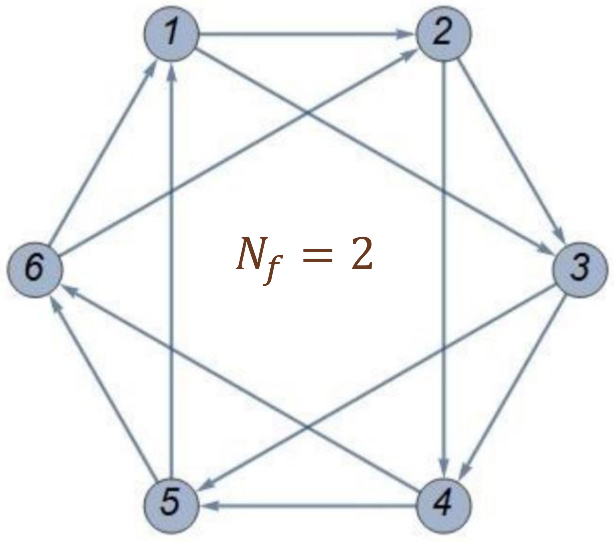

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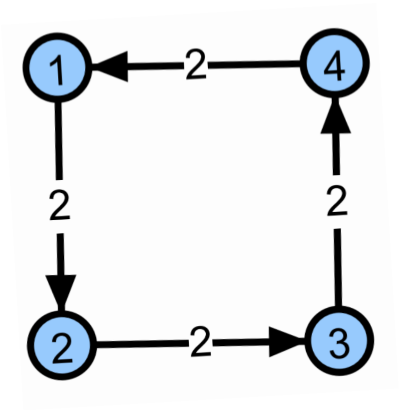
$$B = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

Example: 5d SYM



$\gamma_1, \gamma_2, \gamma_3, \gamma_4$

$\tau_1, \tau_2, \tau_3, \tau_4$

- 5d dualities can be described by quiver mutations [Closset Del Zotto 2019]
- When two quivers are mutation equivalent, they describe different BPS spectra, but these are simply different phases of the same UV theory

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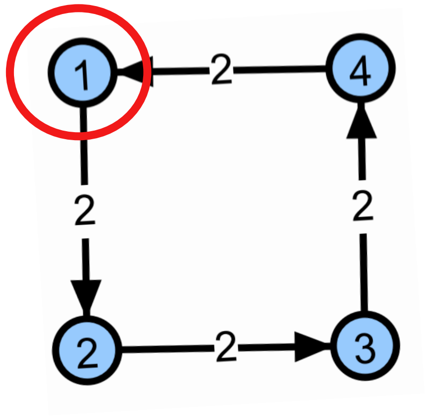

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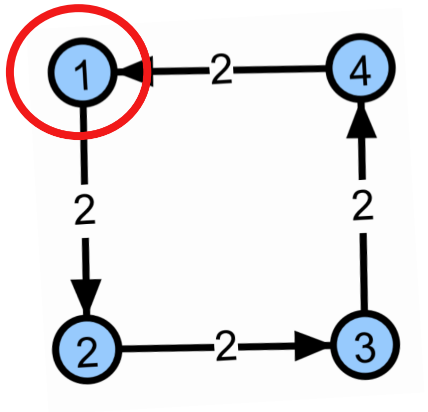

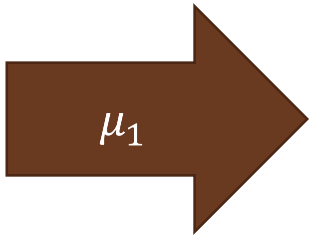

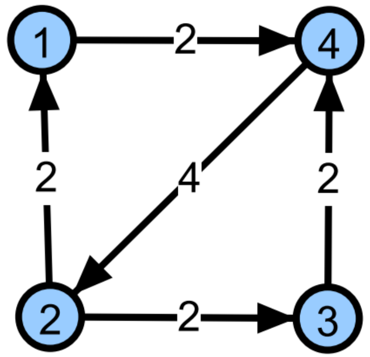

$\gamma_1, \gamma_2, \gamma_3, \gamma_4$

$\tau_1, \tau_2, \tau_3, \tau_4$

$-\gamma_1, 2\gamma_1 + \gamma_2, \gamma_3, \gamma_4$

$$\frac{\tau_2^2 + y_1 \tau_4^2}{\tau_1}, \tau_2, \tau_3, \tau_4$$

q-Painlevé and 5d partition functions

- Study self-dualities of the theory: sequences of quiver mutations and permutations leaving the quiver invariant: $T = (1,2)(3,4)\mu_1\mu_3$
- These give discrete flows generating q-Painlevé equations, and the pure SYM partition function is their tau function
  [Bershtein Gavrylenko Marshakov 2018,2019]
- The time, as happened in the 4d case, is the coupling constant, so that the solutions are expressed in the usual instanton series

$$\tau_1(qt)\tau_1(q^{-1}t) = \tau_1(t)^2 + \sqrt{t}\,\tau_3(t)^2$$

$$Z = Z_{cl}Z_{1-loop}Z_{inst} \qquad Z_{inst} = \sum_{n=0}^{\infty} t^n Z_n$$

$$\tau_1(t) = \sum_n s^n Z(uq^n, t) \quad \tau_3(t) = \sum_n s^{n+\frac{1}{2}} Z\left(uq^{n+\frac{1}{2}}, t\right)$$

$$u = e^{Ra}$$

$$t = R^4 e^{2\pi i \tau}, q = e^{Rg_s} \quad T(t) = qt \quad (\tau_{5d} \to \tau_{5d} + Rg_s)$$

Multiple time flows and $N_f = 2$ theory

[Bonelli FDM Tanzini 2020]

- $N_f = 2$: many discrete flows [Tsuda 2005, Joshi Nakazono Shi 2015]
- $T_1 = (3,6)\mu_6\mu_3(2,5)\mu_5\mu_2(1,2,3,4,5,6), \quad T_2, T_3, T_4$
- Many flows: generic behavior of the q-Painlevé geometries: every flow gives a set of equations
- Flows are affine Weyl translations acting on root systems: partition function as vector on the root lattice
- Spectrum of 5d theories from q-Painlevé flows, we reproduce [Closset Del Zotto 2019] for the pure gauge theory spectrum, proposal from other spectra from q-Painlevé flows

Outlook and Generalizations

- Some of the extra flows lead in 4d limit lead to nonlagrangian (Argyres-Douglas) theories: partition function by $R \to 0$?
- Quantum cluster integrable system $\to$ equations for refined Topological Strings partition functions on the CY_3
- Relation between cluster IS and exponential networks [Banerjee Longhi Romo 2019-2020]
- Kontsevich Sobeilman wall-crossing invariants for 5d theories?
- Relation to quantum curves? [Grassi Hatsuda Mariño 2016]

Thank you!