Eight-dimensional ADHM construction and orbifold DT invariants

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Self-duality in four-dimensional gauge theory, for a given Riemannian structure:

\[ F = \star F , \]

\[ F = dA + A \wedge A \] being the curvature of the gauge bundle.

\[ \Downarrow \]

In higher dimension no direct analogue of SD equations is available. One guess would be to search for solutions to

\[ F \wedge T = \star F \]

with \( T \) a certain invariant closed form.

\[ \Downarrow \]

On 8d Riemannian manifold there is no closed \( SO(8) \)-invariant four-form \( \Rightarrow \) holonomy restricted to be contained in \( Spin(7) \).
Any solutions to the eight-dimensional analogue of the self-duality
equations is also a solution to the Yang-Mills equations, as

\[ D_\mu F_{\mu\nu} = \frac{1}{2} T_{\mu\nu\lambda\rho} D_\mu F_{\lambda\rho} = \frac{1}{2} T_{\mu\nu\lambda\rho} D_{[\mu} F_{\lambda\rho]} = 0. \]

\[ \Downarrow \]

Localised solution are true minima of the Yang-Mills action. However, in dimension greater than four, there is no localised
solution (by Derrick’s theorem).

\[ \Downarrow \]

Classically the moduli space of solutions is empty

\[ \Downarrow \]

The issue is solved by introducing non-commutativity, i.e. on the
coordinates of \( \mathbb{R}^8 \) we impose the commutation relations

\[ [x^\mu, x^\nu] = i\zeta(\omega^{-1})^{\mu\nu} \]

for some real parameter \( \zeta \) and constant non-degenerate two-form \( \omega \).
A concrete realisation of $SO(8)$ gamma matrices is given by

$$\Gamma^\mu = \begin{pmatrix} 0 & \Sigma^\mu \\ \overline{\Sigma}^\mu & 0 \end{pmatrix},$$

where $\Sigma^0 = \overline{\Sigma}^0 = 1_8$ and $\Sigma^i = -\overline{\Sigma}^i$ for $i = 1, \ldots, 7$. We let $S_\pm$ be real Majorana-Weyl spinor representations of $Spin(8)$ of definite chirality. Together with the defining representation $\mathbb{R}^8$ they are interchanged by triality.

Two basis of real antisymmetric $8 \times 8$ matrices are given by

$$\Sigma^{\mu\nu} = \frac{1}{2} \Sigma^{[\mu} \Sigma^{\nu]}, \quad \overline{\Sigma}^{\mu\nu} = \frac{1}{2} \overline{\Sigma}^{[\mu} \overline{\Sigma}^{\nu]}$$

so that we have Fierz identities

$$(\Sigma^{\mu\nu})_{\alpha\beta}(\Sigma^{\mu\nu})_{\gamma\delta} = (\overline{\Sigma}^{\mu\nu})_{\alpha\beta}(\overline{\Sigma}^{\mu\nu})_{\gamma\delta} = -8(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}).$$
Fix $\psi_+ \in S_+$ normalised so that $\psi_+^T \psi = 1$, e.g. $\psi_+^\alpha = \delta_0^\alpha \Rightarrow$ This choice corresponds to the splitting $S_+ = 1 \oplus 7$ under $Spin(7) \subset Spin(8)$. We form the $Spin(7)$ tensor

$$T^{\mu\nu\rho\sigma} = \frac{1}{4} \psi_+^T \Gamma[\mu \Gamma_\nu \Gamma_\rho \Gamma_\sigma] \psi_+$$

so that $T = \star T$. Then

$$T \wedge F = \star F \iff (\Sigma^{\mu\nu})_{\alpha\beta} \psi_+^\beta F_{\mu\nu} = (\Sigma^{\mu\nu})_{\alpha 0} F_{\mu\nu}$$

We break the $Spin(7)$ symmetry to $Spin(6) \cong SU(4) \subset Spin(7)$ by choosing a complex structure on $\mathbb{R}^8$ induced by one of the symplectic forms $(\Sigma^{\mu\nu})_{\alpha 0} \Rightarrow$ This amounts to fixing one more normalised spinor $\chi_+$, e.g. $\chi_+^\alpha = \delta_1^\alpha$. 

Analogously to the four-dimensional ADHM construction, the main player is the $(8k + N) \times 8k$ matrix

\[
\Delta(x) = \begin{pmatrix}
\left(B_\mu - x_\mu \mathbb{1}_k \right) \otimes \overline{\Sigma}^\mu \\
I^\dagger \otimes \left(\psi^\dagger + i\chi^\dagger\right)
\end{pmatrix},
\]

where $\Delta(x)$ is a hermitean $k \times k$ matrix and $N \times k$ matrix.

Let $U(x)$ be an $(8k + N) \times N$ matrix satisfying

\[
\Delta(x)^\dagger U(x) = 0, \quad U(x)^\dagger U(x) = \mathbb{1}_N.
\]

We use $U(x)$ to represent the connection $A_\mu \Rightarrow A_\mu = U^\dagger(x) \partial_\mu U(x)$.

\[
\Delta(x)^\dagger \Delta(x) \psi_+ = \psi_+ \otimes f_k^{-1} \Rightarrow \left(\Delta(x)^\dagger \Delta(x)\right)_{A_0} = 0, \quad A = 1, \ldots, 7.
\]
By using explicitly the choice of the complex structure induced by \( \omega^{1}_{\mu\nu} \) we can rewrite (for \( 1 \leq a < b \leq 4 \))

\[
\left( \Delta^{\dagger}(x)\Delta(x) \right)_{A0} = 0 \iff \begin{cases} 
\sum_{1 \leq a \leq 4} [B_a, B_a^\dagger] + II^\dagger = \zeta \mathbb{1}_k, \\
[B_a, B_b] - \frac{1}{2} \epsilon_{abcd} [B_c^\dagger, B_d^\dagger] = 0.
\end{cases}
\]

\[\downarrow\]

Solutions to the eight-dimensional ADHM equations realise stable representations of the framed four-loop quiver

![Diagram](attachment:image.png)

with relations

\[ [B_a, B_b] = 0, \quad 1 \leq a < b \leq 4. \]
From the previous discussion the moduli space of stable representations of numerical type \((N, k)\) (i.e. \(\text{Quot}_{\mathbb{A}^4}(\mathcal{O}^N, k)\)) is isomorphic to the moduli space \(\mathcal{M}_{k,N}\) of solutions to the 8d self-duality equations.

\[
\downarrow
\]

**Proposition**

Let \(F_{k,N}(\mathbb{P}^4)\) be the moduli space of framed torsion-free sheaves on \(\mathbb{P}^4\) with Chern character \((N, 0, 0, 0, -k)\). There is a scheme-theoretic isomorphism

\[
(\mathcal{M}_{k,N} \cong) \text{Quot}_{\mathbb{A}^4}(\mathcal{O}^{\oplus N}, k) \cong F_{k,N}(\mathbb{P}^4),
\]

BPS-bound states counting reproduces DT theory on \(\mathbb{C}^4\)
Twisted representations of the Chan-Paton factors under $G$

\[ \Downarrow \]

The ADHM datum $(W, V, B_1, \ldots, B_4)$ realising representations of the ADHM quiver gets decomposed in irreps of $G$:

\[ W = \bigoplus_r W_r \otimes \rho_r^\vee, \quad V = \bigoplus_r V_r \otimes \rho_r^\vee. \]

\[ \Downarrow \]

\[ \dim W = \sum_r N_r = \sum_r \dim W_r \quad \Downarrow \quad \dim V = \sum_r k_r = \sum_r \dim V_r \]

The linear maps $B_\alpha$ are also decomposed in $B_\alpha^r : V_r \to V_{r+r_\alpha}$ according to the orbifold action on the coordinates of $\mathbb{C}^4$: $z_\alpha \mapsto r_\alpha z_\alpha$. The ADHM equations become

\[ B_\beta^{r+r_\alpha} B_\alpha^r = B_\alpha^{r+r_\beta} B_\beta^r \]
Partition functions can be computed by SUSY localisation. We have

\[
\text{SUSY fixed points } \leftrightarrow G - \text{ coloured solid partitions}
\]

and

\[
Z_{\overline{N}}^{\text{orb}}(q) = \sum_{\overline{k}} q^{\overline{k}} Z_{\overline{N},k}^{\text{orb}} = \sum_{\overline{k}} q^{\overline{k}} \sum_{u^* \in \mathcal{M}_{\text{sing}}} \text{JK-Res}_{u^*,\zeta} \chi_k^{\text{orb}} d^{\overline{k}} x
\]

Mathematical interpretation: more difficult and not rigorous yet.

**Conjecture/Slogan**

The orbifold quiver gauge theory encodes the DT theory of the CY quotient stack \([\mathbb{C}^4/G]\).

If there is a crepant resolution \(X \to \mathbb{C}^4/G\), \(X = G\text{-Hilb}(\mathbb{C}^4)\), counting substacks of \([\mathbb{C}^4/G]\) should be equivalent to DT theory on \(X\).
Example: $\mathbb{C}^4/\mathbb{Z}_2 \cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$

SUSY-fixed locus is indexed by $\mathbb{Z}_2$-coloured solid partitions:

$$p \in \text{fixed locus} \leftrightarrow \begin{array}{c}
\lambda_{1,1} \cup \lambda_{1,2} \\
\lambda_{2,1} \lambda_{2,2} \\
\vdots \\
\lambda_{\ell,1} \cdots \lambda_{\ell,\mu_\ell}
\end{array}$$

$$\lambda_{1,1} \cup \lambda_{1,2} \\
\lambda_{2,1} \lambda_{2,2} \\
\vdots \\
\lambda_{\ell,1} \cdots \lambda_{\ell,\mu_\ell}$$
Consider the case \( N = 1, k = 2 \). Fixed points labelled with the colouring corresponding to the orbifold action will then be

\[
\begin{align*}
\sigma_1 &= \begin{bmatrix} 1^1 \end{bmatrix} [1^1], \\
\sigma_2 &= \begin{bmatrix} 1^1 \\ 1^1 \end{bmatrix}, \\
\sigma_3 &= [2^1], \\
\sigma_4 &= [1^2].
\end{align*}
\]

The orbifold partition function will be

\[
Z_{D7, orb}^{D7}(\epsilon; q_0, q_1) = Z_{(1,0),(2,0)}^{D7, orb}(\epsilon; q_0, q_1) + Z_{(1,0),(1,1)}^{D7, orb}(\epsilon; q_0, q_1)
\]

\[
= \frac{q_0^2 m \epsilon_{12}}{2 \epsilon_3 \epsilon_{123} (\epsilon_{12} + 2 \epsilon_3)} \left( \frac{(\epsilon_3 - m)(\epsilon_{12} - \epsilon_3)}{\epsilon_3} + \frac{(\epsilon_{123} + m)(2 \epsilon_{12} + \epsilon_3)}{\epsilon_{123}} \right) + \frac{q_0 q_1 m \epsilon_{12}}{2 \epsilon_3 \epsilon_{123} (\epsilon_2 - \epsilon_1)} \left( \frac{(2 \epsilon_2 + \epsilon_3)(\epsilon_{13} - \epsilon_2)}{\epsilon_2} - \frac{(2 \epsilon_1 + \epsilon_3)(\epsilon_{23} - \epsilon_1)}{\epsilon_1} \right).
\]
Thank You!