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Eight-dimensional ADHM construction and orbifold DT invariants

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Joint work with

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Self-duality in four-dimensional gauge theory, for a given Riemannian structure:

$$F = \star F,$$

$F = dA + A \wedge A$ being the curvature of the gauge bundle.

$$\Downarrow$$

In higher dimension no direct analogue of SD equations is available. One guess would be to search for solutions to

$$F \wedge T = \star F$$

with T a certain invariant closed form.

$$\Downarrow$$

On $8d$ Riemannian manifold there is no closed $SO(8)$-invariant four-form $\Rightarrow$ holonomy restricted to be contained in $Spin(7)$.

Any solutions to the eight-dimensional analogue of the self-duality equations is also a solution to the Yang-Mills equations, as

$$D_\mu F_{\mu\nu} = \frac{1}{2} T_{\mu\nu\lambda\rho} D_\mu F_{\lambda\rho} = \frac{1}{2} T_{\mu\nu\lambda\rho} D_{[\mu} F_{\lambda\rho]} = 0.$$

$$\Downarrow$$

Localised solution are true minima of the Yang-Mills action. However, in dimension greater than four, there is no localised solution (by Derrick's theorem).

classically the moduli space of solutions is empty

$$\Downarrow$$

The issue is solved by introducing non-commutativity, i.e. on the coordinates of $\mathbb{R}^8$ we impose the commutation relations

$$[x^\mu, x^\nu] = i\zeta(\omega^{-1})^{\mu\nu}$$

for some real parameter ζ and constant non-degenerate two-form ω.

Some spinorial notation

A concrete realisation of $SO(8)$ gamma matrices is given by

$$\Gamma^\mu = \begin{pmatrix} 0 & \Sigma^\mu \\ \overline{\Sigma}^\mu & 0 \end{pmatrix},$$

where $\Sigma^0 = \overline{\Sigma}^0 = \mathbb{1}_8$ and $\Sigma^i = -\overline{\Sigma}^i$ for $i = 1, \dots, 7$. We let $S_\pm$ be real Majorana-Weyl spinor representations of $Spin(8)$ of definite chirality. Together with the defining representation $\mathbb{R}^8$ they are interchanged by triality.

Two basis of real antisymmetric 8×8 matrices are given by

$$\Sigma^{\mu\nu} = \frac{1}{2}\Sigma^{[\mu}\Sigma^{\nu]}, \qquad \overline{\Sigma}^{\mu\nu} = \frac{1}{2}\overline{\Sigma}^{[\mu}\overline{\Sigma}^{\nu]}$$

so that we have Fierz identities

$$(\Sigma^{\mu\nu})_{\alpha\beta}(\Sigma^{\mu\nu})_{\gamma\delta} = (\overline{\Sigma}^{\mu\nu})_{\alpha\beta}(\overline{\Sigma}^{\mu\nu})_{\gamma\delta} = -8(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}).$$

ADHM construction

Fix $\psi_+ \in S_+$ normalised so that $\psi_+^T \psi = 1$, e.g. $\psi_+^\alpha = \delta_0^\alpha \Rightarrow$ This choice corresponds to the splitting $S_+ = 1 \oplus 7$ under $Spin(7) \subset Spin(8)$. We form the $Spin(7)$ tensor

$$T^{\mu\nu\rho\sigma} = \frac{1}{4} \psi_+^T \Gamma^{[\mu} \Gamma^\nu \Gamma^\rho \Gamma^{\sigma]} \psi_+$$

so that $T = \star T$. Then

$$T \wedge F = \star F \Longleftrightarrow (\Sigma^{\mu\nu})_{\alpha\beta} \psi_+^\beta F_{\mu\nu} = (\Sigma^{\mu\nu})_{\alpha 0} F_{\mu\nu}$$

We break the $Spin(7)$ symmetry to $Spin(6) \cong SU(4) \subset Spin(7)$ by choosing a complex structure on $\mathbb{R}^8$ induced by one of the symplectic forms $(\Sigma^{\mu\nu})_{A0} \Rightarrow$ This amounts to fixing one more normalised spinor χ_+, e.g. $\chi_+^\alpha = \delta_1^\alpha$.

Analogously to the four-dimensional ADHM construction, the main player is the $(8k+N)\times 8k$ matrix

hermitean $k \times k$ matrix

$$\Delta(x) = \begin{pmatrix} (B_\mu - x_\mu \mathbb{1}_k) \otimes \overline{\Sigma}^\mu \\ I^\dagger \otimes (\psi_+^\dagger + i\chi_+^\dagger) \end{pmatrix},$$

$N \times k$ matrix

Let $U(x)$ be an $(8k+N)\times N$ matrix satisfying

$$\Delta^\dagger(x)U(x) = 0, \qquad U^\dagger(x)U(x) = \mathbb{1}_N.$$

We use $U(x)$ to represent the connection $A_\mu \Rightarrow A_\mu = U^\dagger(x)\partial_\mu U(x)$.

$$\Downarrow$$

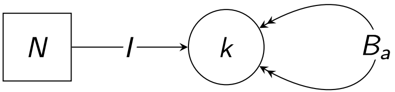
$$\Delta^\dagger(x)\Delta(x)\psi_+ = \psi_+ \otimes \mathsf{f}_k^{-1} \Rightarrow \left(\Delta^\dagger(x)\Delta(x)\right)_{A0} = 0,\ A = 1,\dots,7.$$

By using explicitly the choice of the complex structure induced by $\omega^1_{\mu\nu}$ we can rewrite (for $1 \leq a < b \leq 4$)

$$\left(\Delta^\dagger(x)\Delta(x)\right)_{A0} = 0 \Leftrightarrow \begin{cases} \sum_{1\leq a\leq 4} [B_a, B_a^\dagger] + II^\dagger = \zeta \mathbb{1}_k, \\ [B_a, B_b] - \frac{1}{2}\epsilon_{abcd}[B_c^\dagger, B_d^\dagger] = 0. \end{cases}$$

$$\Downarrow$$

Solutions to the eight-dimensional ADHM equations realise stable representations of the framed four-loop quiver



with relations

$$[B_a, B_b] = 0, \qquad 1 \leq a < b \leq 4.$$

Instantons and sheaves

From the previous discussion the moduli space of stable representations of numerical type (N, k) (i.e. $\mathrm{Quot}_{\mathbb{A}^4}(\mathscr{O}^N, k)$) is isomorphic to the moduli space $\mathscr{M}_{k,N}$ of solutions to the $8d$ self-duality equations.

$$\Downarrow$$

Proposition

Let $\mathrm{F}_{k,N}(\mathbb{P}^4)$ be the moduli space of framed torsion-free sheaves on $\mathbb{P}^4$ with Chern character $(N, 0, 0, 0, -k)$. There is a scheme-theoretic isomorphism

$$(\mathscr{M}_{k,N} \cong)\, \mathrm{Quot}_{\mathbb{A}^4}(\mathscr{O}^{\oplus N}, k) \cong \mathrm{F}_{k,N}(\mathbb{P}^4),$$

BPS-bound states counting reproduces DT theory on $\mathbb{C}^4$

Orbifolds of $\mathbb{C}^4$

Twisted representations of the Chan-Paton factors under G

$$\Downarrow$$

The ADHM datum $(W, V, B_1, \ldots, B_4)$ realising representations of the ADHM quiver gets decomposed in irreps of G

$$W = \bigoplus_r W_r \otimes \rho_r^\vee, \qquad V = \bigoplus_r V_r \otimes \rho_r^\vee.$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\dim W = \sum_r N_r = \sum_r \dim W_r \qquad \dim V = \sum_r k_r = \sum_r \dim V_r$$

The linear maps B_α are also decomposed in $B_\alpha^r : V_r \to V_{r+r_\alpha}$ according to the orbifold action on the coordinates of $\mathbb{C}^4$: $z_\alpha \mapsto r_\alpha z_\alpha$. The ADHM equations become

$$B_\beta^{r+r_\alpha} B_\alpha^r = B_\alpha^{r+r_\beta} B_\beta^r$$

Partition functions can be computed by SUSY localisation. We have

SUSY fixed points $\longleftrightarrow$ $G-$ coloured solid partitions

and

$$\mathsf{Z}^{\mathrm{orb}}_{\overline{N}}(\underline{q}) = \sum_{\overline{k}} \underline{q}^{\overline{k}} \mathsf{Z}^{\mathrm{orb}}_{\overline{N},\overline{k}} = \sum_{\overline{k}} \underline{q}^{\overline{k}} \sum_{u_* \in \mathfrak{M}_{\mathrm{sing}}} \text{JK-Res}_{u_*,\zeta}\, \chi^{\mathrm{orb}}_{\overline{k}} d^{\overline{k}} \overline{x}$$

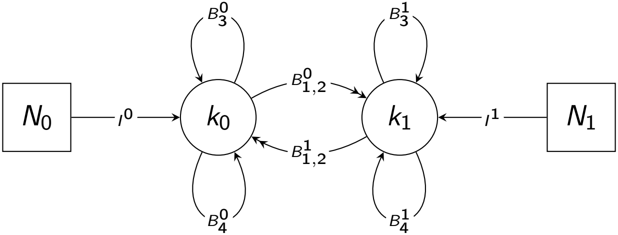
Mathematical interpretation: more difficult and not rigorous yet.

Conjecture/Slogan

The orbifold quiver gauge theory encodes the DT theory of the CY quotient stack $[\mathbb{C}^4/G]$.

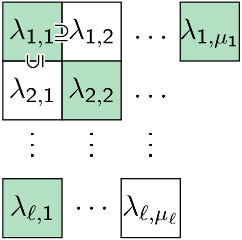
If there is a crepant resolution $X \to \mathbb{C}^4/G$, $X = G\text{-Hilb}(\mathbb{C}^4)$, counting substacks of $[\mathbb{C}^4/G]$ should be equivalent to DT theory on X.

Example: $\mathbb{C}^4/\mathbb{Z}_2 \cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$



SUSY-fixed locus is indexed by $\mathbb{Z}_2$-coloured solid partitions:

$p \in$ fixed locus $\longleftrightarrow$



Consider the case $N = 1$, $k = 2$. Fixed points labelled with the colouring corresponding to the orbifold action will then be

$$\sigma_1 = \boxed{[1^1]}\boxed{[1^1]}, \qquad \sigma_2 = \begin{array}{|c|}\hline [1^1] \\ \hline [1^1] \\ \hline \end{array}, \qquad \sigma_3 = \boxed{[2^1]}, \qquad \sigma_4 = \boxed{[1^2]}.$$

The orbifold partition function will be

$$\begin{aligned} Z^{D7,\text{orb}}(\epsilon; q_0, q_1) =& Z^{D7,\text{orb}}_{(1,0),(2,0)}(\epsilon; q_0, q_1) + Z^{D7,\text{orb}}_{(1,0),(1,1)}(\epsilon; q_0, q_1) \\ =& \frac{q_0^2 m \epsilon_{12}}{2\epsilon_3 \epsilon_{123}(\epsilon_{12} + 2\epsilon_3)} \left(\frac{(\epsilon_3 - m)(\epsilon_{12} - \epsilon_3)}{\epsilon_3} \right. \\ & \left. + \frac{(\epsilon_{123} + m)(2\epsilon_{12} + \epsilon_3)}{\epsilon_{123}} \right) \\ & + \frac{q_0 q_1 m \epsilon_{12}}{2\epsilon_3 \epsilon_{123}(\epsilon_2 - \epsilon_1)} \left(\frac{(2\epsilon_2 + \epsilon_3)(\epsilon_{13} - \epsilon_2)}{\epsilon_2} \right. \\ & \left. - \frac{(2\epsilon_1 + \epsilon_3)(\epsilon_{23} - \epsilon_1)}{\epsilon_1} \right). \end{aligned}$$

Thank You!