ABCDEFG of Gauge and Painlevé

Fran Globlek

Supervisors: Giulio Bonelli
Alessandro Tanzini

Avogadro 2020
Gauge

- more precisely: 4d $\mathcal{N} = 2^*$ G-SYM on self-dual $\Omega$-background
- we will pass to pure SYM limit and decouple the hyper
- after a twist, PI localizes on instantons
- zero modes $\rightarrow$ integration over moduli space of instantons
- solution is schematically

$$Z(\vec{a}, \epsilon; \Lambda) = Z_{\text{classical}}(\vec{a}, \epsilon; \Lambda) Z_{1 \text{ loop}}(\vec{a}, \epsilon) \left( \sum_{n \geq 0} \Lambda^{2kh^y} Z_k(\vec{a}, \epsilon) \right)$$

$$Z_k(\vec{a}, \epsilon) = \int_{\mathcal{M}_{G,k}} 1$$

- $\epsilon \rightarrow 0$ asymptotics give the SW prepotential $F_{SW}$,

$$Z \sim \exp\left(-\frac{1}{\epsilon^2} F_{SW} + \ldots\right)$$
Integrability...

- old story: the SW solution of the IR theory is a complex integrable system (DW), base is C.B., fibers are $Jac(SW\text{ curve})$
- the SW curve = the spectral curve of the classical integrable system (something to do with Lax pairs)
- for 1 adjoint hyper: elliptic Calogero-Moser on $L^G$
- big concept: the tau function $\tau$

$$\frac{\partial}{\partial t_i} \log \tau = H_i$$

note how all the Hamiltonians have to Poisson commute

- in gauge theory, $\tau$ is $\Theta$ on Jacobian
...and beyond: Painlevé

- some integrable systems can be lifted: think about turning on RG flow
- simplest example leads to $P_{III}(D_8)$

$$y'' = \frac{(y')^2}{y} - \frac{y}{z} + \frac{1}{z} (\alpha y^2 + \beta), \alpha \cdot \beta \neq 0$$

- what is the Painlevé property?
- question: why is it good to not have movable branch cuts?
- ubiquitous functions describing many universal properties
- transcendent: solutions in terms of known functions generically impossible
what happens to gauge/integrable system correspondence? promotion!

most concisely, for $G = SU(2)$

$$\tau_{III}(\eta, a/\epsilon; t) = Z^{N\cdot O}(\eta, a, \epsilon; t) = \sum_{n \in 2\mathbb{Z}} e^{2\pi i n \cdot \eta} Z(a/\epsilon + n, t)$$

where $t = (\frac{\Lambda}{\epsilon})^4$

this means we have to solve an ODE with a prescribed Ansatz to get the full partition function, fixing only the asymptotics
Extending of the correspondence

- and for $G$ any simple Lie group? yes!
- systems of equations which generalize Painlevé to higher rank
- we obtain more $\tau$ functions, labelled by the center $Z(G)$, related to each other by shifts
- 1-loop determinants? from functional equations solved by Barnes’ $G$ functions
- instantons $Z_k$? from recursion relations (not blowup - same $\Omega$-background)
- for cognoscenti: no residues needed + a new way to obtain exceptional instantons
for example, $C_3 = Sp(3)$

\[-2t^{13/4}y_1y_1'y_1'' + 2t^{17/4}y_1(y_1')^2y_1'' - 3t^{21/4}y_1y_1'(y_1'')^2 - 2t^{25/4}y_1y_1(3)y_1y_1'' - 4t^{17/4}y_1^2y_1(3)y_1' + 4t^{21/4}y_1y_1(3)(y_1')^2 - t^{21/4}y_1^2y_1(4)y_1' + t^{25/4}y_1y_1(4)(y_1')^2 + 2t^{17/4}y_1^2(y_1'')^2 + t^{25/4}y_1(y_1'')^3 - t^{25/4}y_1^2y_1(4)y_1'' + t^{25/4}y_1^2(y_1(3))^2 - t^2y_1^2(y_2')^2 + t^2y_1^2y_2y_2'' + ty_1^2y_2y_2' = 0\]

and then

\[y_1 = Z_{C_3}^{N,O}(%(\vec{\eta}, \vec{a}/\epsilon); t^{1/2})\]

\[y_2 = Z_{C_3}^{N,O}(%(\vec{\eta}, \vec{a}/\epsilon + (1/2, 1/2, 1/2); t^{1/2})\]

▶ note: it’s not actually this scary, I made it worse to intimidate you
FURTHER TOPICS

- rigid surface operators and higher-form symmetries
- connection of its 5d lift to ”ABJM” matrix models via TS/ST correspondence
- product groups i.e. linear quivers
- connection to other backgrounds via blowup equations \((c = -2)\)

Thank you!