# ABCDEFG of Gauge and Painlevé


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# Gauge

- more precisely: 4d $\mathcal{N}=2^*$ $G$-SYM on self-dual $\Omega$-background
- we will pass to pure SYM limit and decouple the hyper
- after a twist, PI localizes on instantons
- zero modes $\rightarrow$ integration over moduli space of instantons
- solution is schematically

$$Z(\vec{a},\epsilon;\Lambda) = Z_{\text{classical}}(\vec{a},\epsilon;\Lambda) Z_{1\text{ loop}}(\vec{a},\epsilon)\left(\sum_{n\geq 0}\Lambda^{2kh^\vee} Z_k(\vec{a},\epsilon)\right)$$

$$Z_k(\vec{a},\epsilon) = \int_{\overline{\mathcal{M}}_{G,k}} 1$$

- $\epsilon \rightarrow 0$ asymptotics give the SW prepotential $F_{SW}$, $Z \sim \exp\left(-\frac{1}{\epsilon^2}F_{SW} + \ldots\right)$

# Integrability...

- old story: the SW solution of the IR theory is a complex integrable system (DW), base is C.B., fibers are *Jac*(SW curve)
- the SW curve = the spectral curve of the classical integrable system (something to do with Lax pairs)
- for 1 adjoint hyper: elliptic Calogero-Moser on ${}^{L}G$
- big concept: the tau function $\tau$

$$\frac{\partial}{\partial t_i} \log \tau = H_i$$

  note how all the Hamiltonians have to Poisson commute
- in gauge theory, $\tau$ is $\Theta$ on Jacobian

# ...AND BEYOND: PAINLEVÉ

- some integrable systems can be lifted: think about turning on RG flow
- simplest example leads to $PIII(D_8)$

$$y'' = \frac{(y')^2}{y} - \frac{y}{z} + \frac{1}{z}(\alpha y^2 + \beta), \alpha \cdot \beta \neq 0$$

- what is the Painlevé property?
- question: why is it good to not have movable branch cuts?
- ubiquitous functions describing many universal properties
- transcendent: solutions in terms of known functions generically impossible

# Gauge/Painlevé

- what happens to gauge/integrable system correspondence? promotion!
- most concisely, for $G = SU(2)$

$$\tau_{PIII}(\eta, a/\epsilon; t) = Z^{N.O.}(\eta, a, \epsilon; t) = \sum_{n \in 2\mathbb{Z}} e^{2\pi i n \cdot \eta} Z(a/\epsilon + n, t)$$

  where $t = \left(\frac{\Lambda}{\epsilon}\right)^4$
- this means we have to solve an ODE with a prescribed Ansatz to get the full partition function, fixing only the asymptotics

# Extending of the correspondence

- and for $G$ any simple Lie group? yes!
- systems of equations which generalize Painlevé to higher rank
- we obtain more $\tau$ functions, labelled by the center $Z(G)$, related to each other by shifts
- 1-loop determinants? from functional equations solved by Barnes' G functions
- instantons $Z_k$? from recursion relations (not blowup - same $\Omega$-background)
- for cognoscenti: no residues needed $+$ a new way to obtain exceptional instantons

- for example, $C_3 = Sp(3)$

$$
\begin{aligned}
&- 2t^{13/4}y_1^2 y_1' y_1'' + 2t^{17/4} y_1 \left(y_1'\right)^2 y_1'' - 3t^{21/4} y_1 y_1' \left(y_1''\right)^2 \\
&- 2t^{25/4} y_1 y_1^{(3)} y_1' y_1'' - 4t^{17/4} y_1^2 y_1^{(3)} y_1' + 4t^{21/4} y_1 y_1^{(3)} \left(y_1'\right)^2 \\
&- t^{21/4} y_1^2 y_1^{(4)} y_1' + t^{25/4} y_1 y_1^{(4)} \left(y_1'\right)^2 + 2t^{17/4} y_1^2 \left(y_1''\right)^2 \\
&+ t^{25/4} y_1 \left(y_1''\right)^3 - t^{25/4} y_1^2 y_1^{(4)} y_1'' + t^{25/4} y_1^2 \left(y_1^{(3)}\right)^2 \\
&- t^2 y_1^2 \left(y_2'\right)^2 + t^2 y_1^2 y_2 y_2'' + t y_1^2 y_2 y_2' = 0
\end{aligned}
$$

and then

$$
\begin{aligned}
y_1 &= Z_{C_3}^{N.O.}(\vec{\eta}, \vec{a}/\epsilon; t^{1/2}) \\
y_2 &= Z_{C_3}^{N.O.}(\vec{\eta}, \vec{a}/\epsilon + (1/2, 1/2, 1/2); t^{1/2})
\end{aligned}
$$

- note: it's not actually this scary, I made it worse to intimidate you

# Further topics

- rigid surface operators and higher-form symmetries
- connection of its 5d lift to "ABJM" matrix models via TS/ST correspondence
- product groups i.e. linear quivers
- connection to other backgrounds via blowup equations ($c = -2$)

Thank you!