Mellin Transform in 1d CFTs

based on work in progress with L. Bianchi, G. Bliard and V. Forini

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Conformal Field Theory

$$O_i(x_i)O_j(x_j) \overset{x_i \to x_j}{=} \sum_k c_{ijk}\,(x_{ij}^2)^{\frac{\Delta_k - \Delta_i - \Delta_j}{2}}\left[O_k(x_j) + c\,x_{ij}^2\,\partial^2 O_k(x_j) + \ldots\right]$$

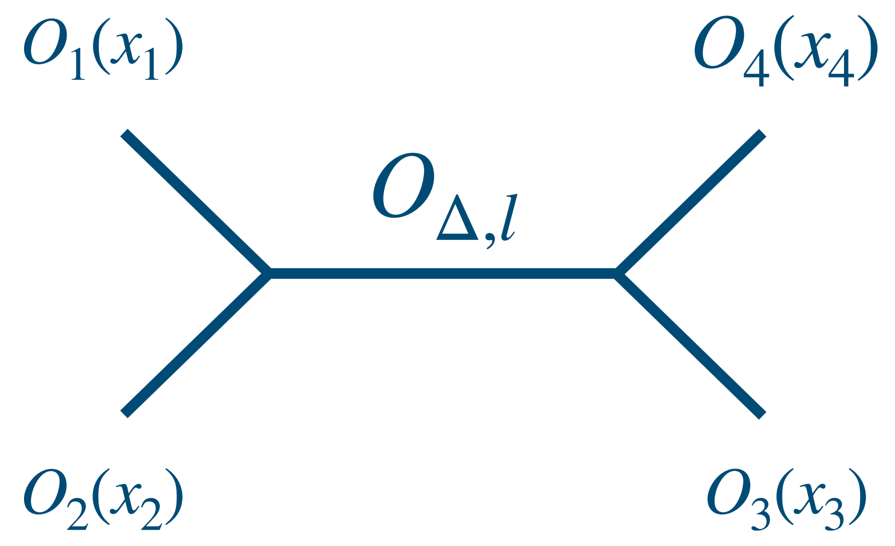
primary operators $O_{\Delta,l}$
their descendants
$\partial_{\mu_1}\ldots\partial_{\mu_n}O_{\Delta,l}$

$$\langle O_\Delta(x_i), O_\Delta(x_j)\rangle \to \Delta$$

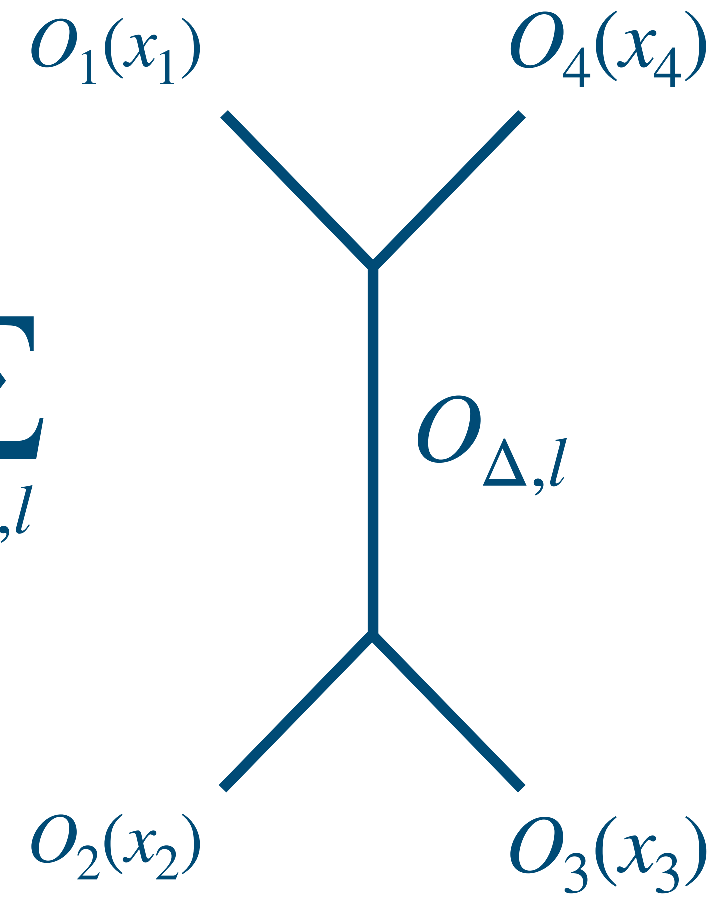
$$\langle O_i(x_1), O_j(x_2), O_k(x_3)\rangle \to c_{ijk}$$

$$\{\Delta_i, c_{ijk}\}$$

$+ \sum_{\Delta,l}$



$= \sum_{\Delta,l}$



DCFT in 1d

- Modifications which preserve part of the conformal symmetry
- Conformal defect as an external probe
- Every higher-d CFT is a 1d CFT
- Simpler but still constraining setting for testing ideas about higher-d CFT

→ AdS_2/CFT_1

1/2-BPS Wilson Line

[Giombi, Roiban & Tseytlin, 2017]

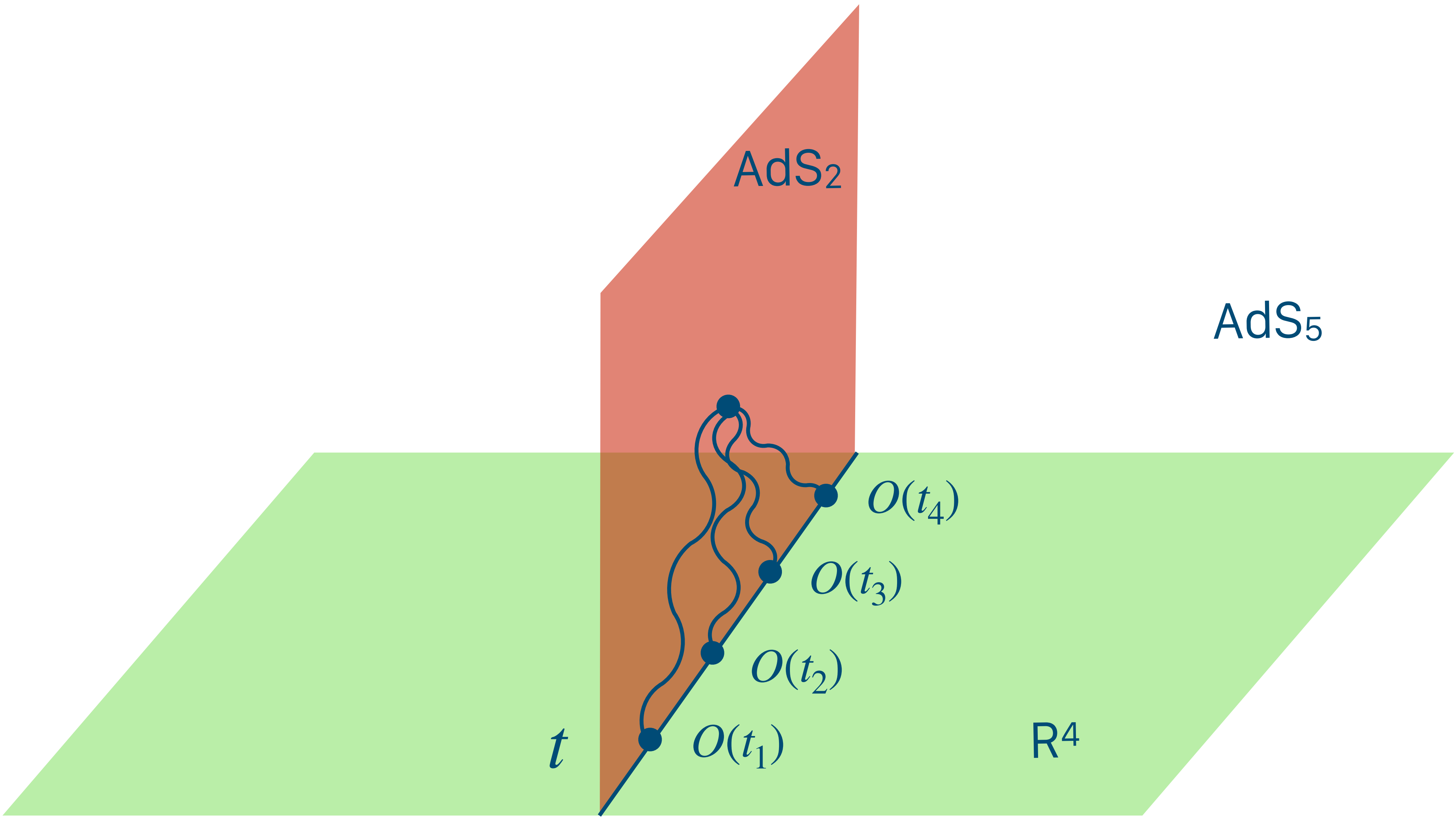
$AdS_5 \times S^5 \longleftrightarrow \mathcal{N} = 4$ SYM

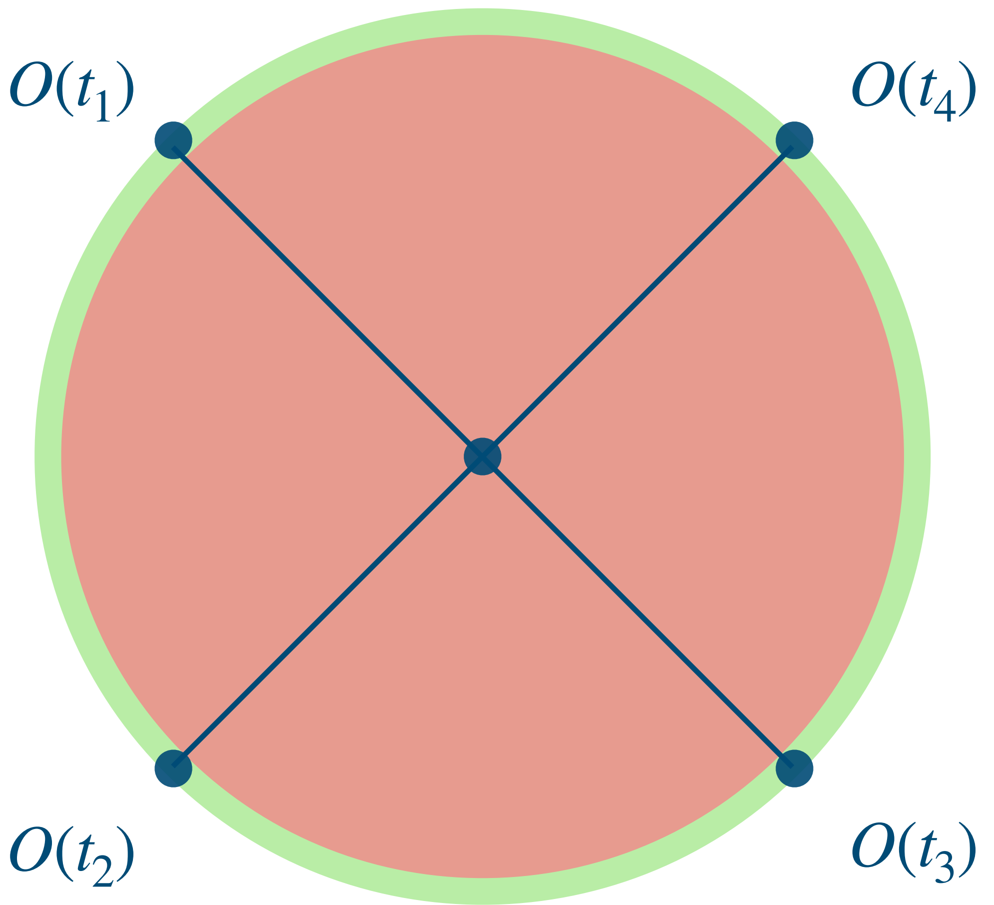
$$W = trPe^{\int dt(iA_t + \Phi^6)}$$

[Maldacena, 1998]

Symmetries: $SO(6)_R \longrightarrow SO(5)$

$SO(2,4) \longrightarrow SO(3) \times SO(2,1)$





$$\langle\langle O(t_1)O(t_2)\ldots O(t_n)\rangle\rangle = \langle X(t_1)X(t_2)\ldots X(t_n)\rangle_{AdS_2}$$

4-point functions computed expanding in $\dfrac{1}{\sqrt{\lambda}}$

via AdS_2 Witten diagram

Mellin Transform

[Mack, 2009]

Is there a "momentum space" representation of CFT correlators that simplifies these computations?

Yes, the Mellin Transform is a useful tool and it delivers a representation of conformal correlation functions that makes their scattering nature more transparent

- Simple representation
- Crossing-symmetric
- Meromorphic (correspondence poles-OPE operators)
- Polynomially bounded

$$\langle O_\Delta(x_1)\ldots O_\Delta(x_4)\rangle = \frac{1}{(x_{13}x_{24})^{2\Delta}}\int_{-i\infty}^{i\infty}\frac{d\gamma_{12}\gamma_{14}}{(2\pi i)^2}\,\underbrace{M(\gamma_{12},\gamma_{14})\Gamma^2(\gamma_{12})\Gamma^2(\gamma_{14})\Gamma^2(\Delta-\gamma_{12}-\gamma_{14})}_{\hat{M}(\ldots)}u^{-\gamma_{12}}v^{-\gamma_{14}}$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = \chi\bar{\chi} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-\chi)(1-\bar{\chi})$$

Mellin Transform

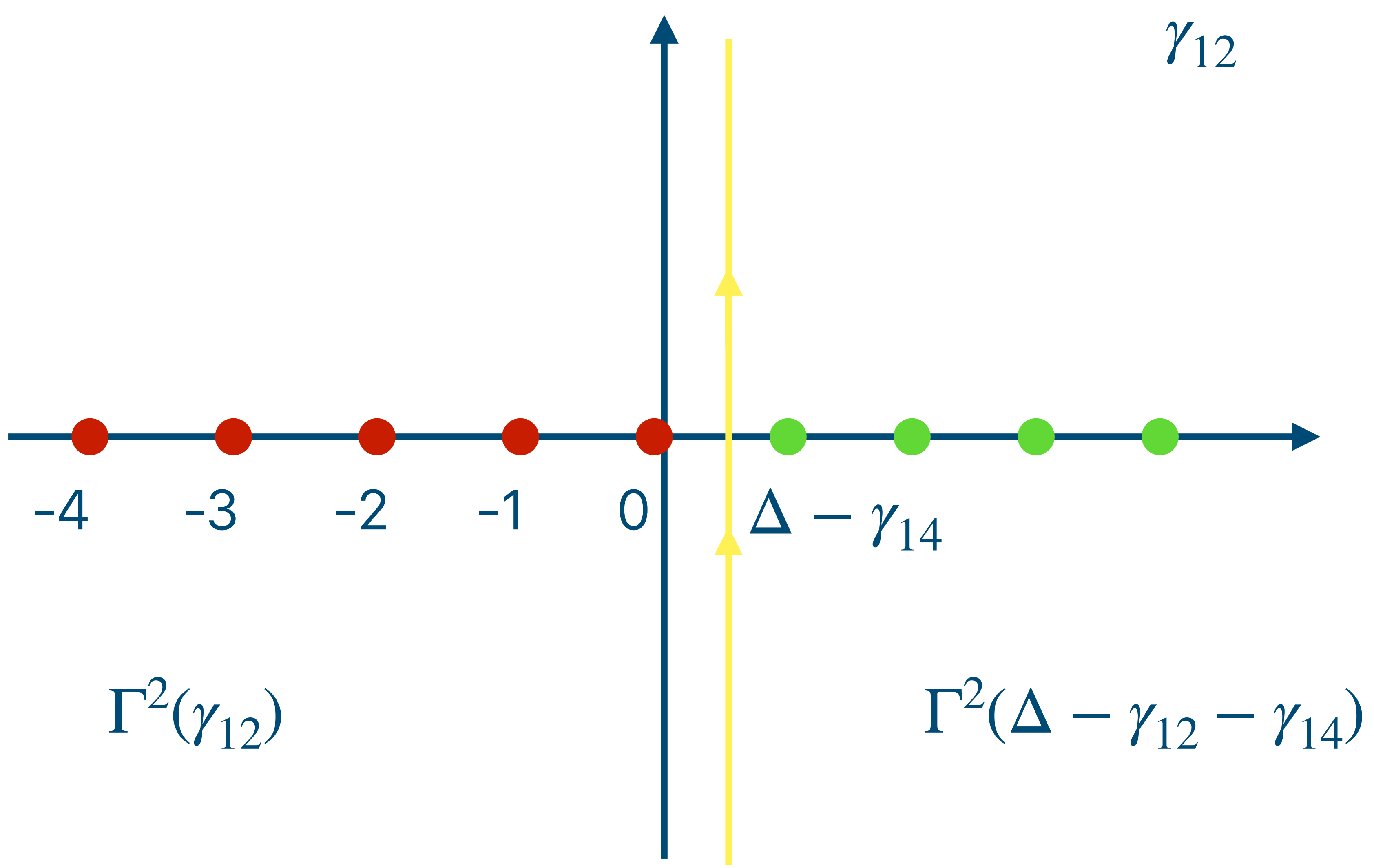
$$\sum_{j=1}^{4} \gamma_{ij} = 0, \qquad \gamma_{ij} = \gamma_{ji}, \qquad \gamma_{ii} = -\Delta_i$$

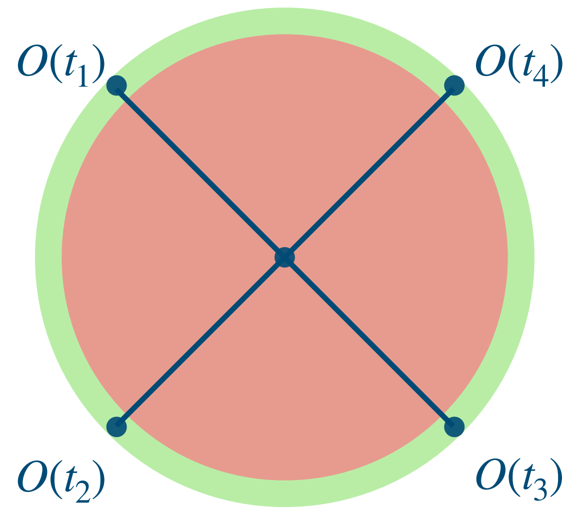
We solve for the constraints introducing some fictitious “momenta" variables $\gamma_{ij} = p_i \cdot p_j$

that obey “momentum conservation” and the “on-shell condition”

$$\sum_{i=1}^{4} p_i = 0 \qquad p_i^2 = -\Delta_i$$

→ Mandelstam variables





→ Contact Witten diagram corresponds to a constant Mellin amplitude

1d Mellin Transform

Kinematics of 2→2 scattering of identical scalar operators:

$$s = -(p_1+p_2)^2 = 2\Delta - \gamma_{12}$$
$$t = -(p_1+p_4)^2 = 2\Delta - \gamma_{14}$$
$$u = -(p_1+p_3)^2 = 2\Delta - \gamma_{13}$$

→ $s = 4E^2 \qquad t = 4m^2 - s \qquad u = 0$

→ Reducing the higher dimensional one by imposing the diagonal limit ($\chi = \bar{\chi}$) but hard integrals to perform and no guarantees that we get a simple Mellin amplitude

→ Try to define it from scratch

1. General form:
$$f(\chi) = \int_{-i\infty}^{+i\infty} ds\, \hat{M}(s)\, \frac{\chi^{s+A}}{(1-\chi)^{s+B}}$$

2. Constrain with crossing-symmetry:
$$f(\chi) = f(1-\chi) \qquad \hat{M}(s) = \hat{M}(4\Delta - s)$$

3. Constrain with the OPE

BUT not unique and simple representation of the correlator!

1d Mellin Transform

$$f(\chi) = \int_{-i\infty}^{+i\infty} ds\, \hat{M}(s) \frac{\chi^{s-1}}{(1-\chi)^s} \quad \longleftrightarrow \quad \hat{M}(s) = \int_0^1 d\chi\, f(\chi) \frac{\chi^{-s}}{(1-\chi)^{-s+1}}$$

- Simple representation ⟶ $\bar{D}$-functions ✓

$\bar{D}_{\Delta,\Delta,\Delta,\Delta}$

$$\bar{D}_{1,1,1,1} = -\frac{2log(1-\chi)}{\chi} - \frac{2log(\chi)}{1-\chi} \quad \longleftrightarrow \quad \hat{M}_{1,1,1,1} = 2\,\Gamma(s-1)\,\Gamma(-s)$$

$$\bar{D}_{2,2,2,2} = -\frac{2(\chi^2-\chi+1)}{15(1-\chi)^2\chi^2} + \frac{(2\chi^2-5\chi+5)log(\chi)}{15(\chi-1)^3} - \frac{(2\chi^2+\chi+2)log(1-\chi)}{15\chi^3} \quad \longleftrightarrow \quad \hat{M}_{2,2,2,2} = 2\,(2-s+s^2)\,\Gamma(s-3)\,\Gamma(-2-s)$$

$\bar{D}_{\Delta,\Delta,\Delta,\Delta}$ or $\bar{D}_{\Delta_1,\Delta_2,\Delta_1,\Delta_2}$

$$M(s) = 2\sum_{n=0}^{\Delta_1-1} (-1)^n \frac{\Gamma(\Delta_1)\Gamma(\Delta_2)}{\Gamma(\Delta_1-n)\Gamma(\Delta_2-n)\Gamma(n+1)} \Gamma^3(\Delta_1+\Delta_2-1-n)\,\Gamma[s-(\Delta_1+\Delta_2-1-n)]\,\Gamma[1-s-(\Delta_1+\Delta_2-1-n)]$$

- Crossing-symmetric: $s \to 1-s$ ✓
- Meromorphic ✓
- Polynomially bounded ✓

Conclusions and further developments

- 1d DCFT are an interesting playground, simple but still rich in structure, see in particular the CFT_1 defined by correlators of operator insertions on a 1/2 BPS Wilson line
- Higher dimensional experience teaches us that Mellin space is the natural framework in which to simplify computations and expressions of the correlators
- 1d reduction of the Mellin is subtle but we propose a consistent definition in 1d, that satisfies the expected properties

→ Test our definition on various CFT_1 extracting CFT data

→ What about higher loops? What about the flat-space limit? Are these systems integrable?

→ Can this simple Mellin language help us to answer these questions?

THANK YOU!