Singular Calabi-Yau threefolds: Springer resolution and quiver gauge theories

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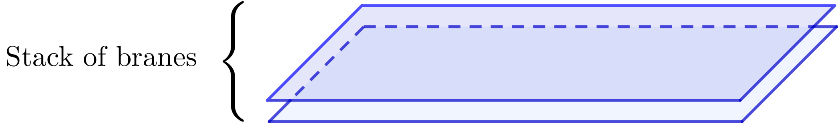
Outline

- F-theory and D7 brane backgrounds
- Higgs mechanism and unconventional Higgs VEVs
- Relation with ADE singularities in Calabi-Yau spaces
- Springer resolution
  - A_n cases
  - Reid Pagoda
  - D_4
- Motivation: gauge theory of branes on ADE singularities

D7-brane backgrounds

Consider type IIB limit of F-theory:

- Non-abelian gauge symmetries are realized stacking D7-branes on top of each other



Example: $A_1 = SU(2)$ gauge group

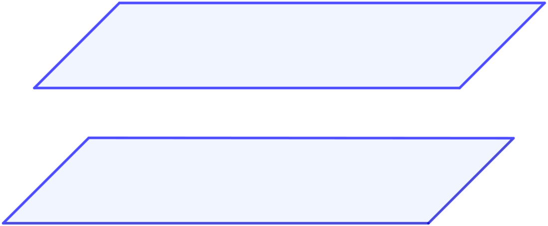
Higgsing

Give a VEV to the Higgs field in the Cartan of $SU(2)$:

$$< \Phi >= \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$$

with v a **constant** on the branes worldvolume. Physically:

$$< \Phi >= \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$$



- Symmetry broken to $U(1)$

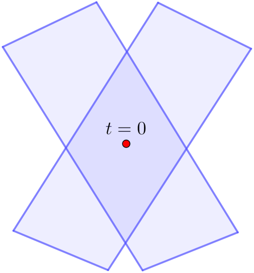
Higgsing

Let $<\Phi>$ depend on the brane worldvolume coordinates, e.g.:

$$<\Phi>=\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$$

Physically:

$$<\Phi>=\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$$



- $t \neq 0$ symmetry broken to $U(1)$
- on $t = 0$ enhancement to $SU(2)$

Non-standard Higgs

Take another case:

$$< \Phi >= \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

- Eigenvalues are $\pm\sqrt{t}$, they are not single-valued, they are **monodromic**
- Symmetry is broken to $U(1)$ everywhere
- These backgrounds are also called **T-branes** [Cecotti, Córdova, Heckman, Vafa, 2010]

Relation with geometry

What is the corresponding geometry? How to extract the corresponding Calabi-Yau space?

For the A_n case the procedure is known:

$$x^2 + y^2 = \det\left(z\mathbb{1} - <\Phi>\right)$$

• Non-monodromic case:

$$\Longrightarrow \text{Conifold: } x^2 + y^2 = z^2 + t^2$$

i.e. a **deformation** of the A_1 singularity. A 2-cycle can be blown up.

• Monodromic case:

$$\Longrightarrow x^2 + y^2 = z^2 + t$$

No 2-cycle can be blown up, due to monodromy.

ADE singularities

What happens for more general gauge groups, i.e. Higgs fields not valued in the Cartan of some ADE gauge group?

A systematic procedure can be found using the so-called **Grothendieck-Springer resolution**

Levi subalgebras

All the Higgs vevs (monodromic and non-monodromic) can be classified using **Levi subalgebras** of $\mathfrak{g}$.

Theorems by Springer and Grothendieck show that (suppressing all technical details):

Levi subalgebras of $\mathfrak{g}$ *are in correspondence with (partial) resolutions of singular ADE Calabi-Yau threefolds.*

Grothendieck-Springer resolution

Example: consider $A_2 = SU(3)$ and choose a Levi subalgebra:

$$L = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

L is in correspondence with the **complete resolution** of the deformed $SU(3)$ singularity, with equation:

$$x^2 + y^2 + z^3 - z(t_1^2 + t_2^2 + t_1 t_2) - t_1 t_2 (t_1 + t_2) = 0$$

with $t_1(t)$ and $t_2(t)$.

Grothendieck-Springer partial resolution

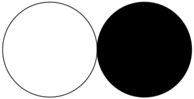
Example: if we choose a different Levi subalgebra:

$$L_p = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

L_p is in correspondence with the **partial resolution** of the deformed $SU(3)$ singularity, in which only one two-cycle is resolved, with equation:

$$x^2 + y^2 + z^3 - z(\sigma_1^2 - \sigma_2) - \sigma_1\sigma_2 = 0$$

with $\sigma_1(t)$ and $\sigma_2(t)$.

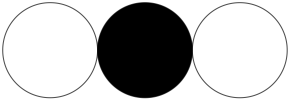


Reid pagoda

Other example: the Reid pagoda can be seen as the (monodromic) deformation of a A_3 singularity:

$$x^2 + y^2 + z^4 + t^2 = 0$$

Only the central node can be resolved.



The corresponding Levi subalgebra giving rise to this geometry is:

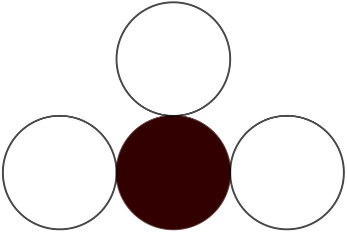
$$L_{pagoda} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

A new D_4 singular threefold

Until now, essentially only one example of singular threefold with D_4 singularity where only the central node can be resolved is known.

It is **Laufer's case** [Cachazo, Katz, Vafa, 2001], with equation:

$$x^2 + y^3 + tz^2 + t^3 y = 0$$



A new D_4 singular threefold

With the previously mentioned techniques we have found a new singular threefold with these properties:

$$x^2 + zy^2 - z^3 - \frac{a^3t^3}{16} + \frac{1}{8}a^2bt^4 - \frac{9}{16}a^2t^2z - \frac{1}{16}ab^2t^5 - \frac{3}{8}abt^3z - \frac{3}{2}atz^2 - \frac{1}{16}b^2t^4z - \frac{1}{2}bt^2z^2 = 0$$

with a and b constants.

The corresponding Levi subalgebra of $SO(8)$ is:

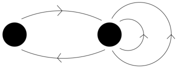
$$L_{D_4} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & * \\ 0 & 0 & * & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & * & * \\ 0 & 0 & * & 0 & 0 & 0 & * & * \end{pmatrix}$$

Gauge theory of D5 branes

Why do we care about these setups?

We can wrap D5 branes around the resolved 2-cycles:

$\longrightarrow \mathcal{N} = 1$ theory with superpotential, described by a **quiver**



- Understanding these theories can be useful for large N dualities, as in [Cachazo, Katz, Vafa, 2001]

Using their techniques we have obtained a superpotential for our D_4 example:

$$W = \frac{4}{3a}\left(X^3 + Y^3 + Z^3\right) - T\left(X + Y + Z + \frac{\sqrt{b}}{4}T\right)$$

with X, Y, Z, T chiral superfields.

Recap and outlook

- Monodromic Higgses correspond to geometries where only some 2-cycles can be resolved
- The Grothendieck-Springer resolution allows to describe these geometries precisely
- Superpotentials for the theory of D5 branes wrapped on the resolved cycles can in some cases be computed using known techniques

Looking at the future:

- Consider further examples in the ADE classification
- Apply the formalism to concrete T-brane settings

Thank you!