# Free-fermion condition in the context of AdS/CFT

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**Integrable models** are characterized by a high amount of symmetry that usually makes them exactly solvable.

The Yang-Baxter equation signals the presence of integrable structures

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

$R_{ij} = R_{ij}(u_i, u_j)$.

The solution is called $R-$matrix and generates a tower of conserved charges.

### Definition (Integrable models)

Models characterized by an infinite tower of conserved charges $\mathbb{Q}_n$

$$[\mathbb{Q}_n, \mathbb{Q}_m] = 0, \quad n, m = 1, \ldots, \infty, \quad \mathbb{Q}_2 = \mathbb{H}.$$

*Examples*: Kepler's problem, Heisenberg spin chain, Hubbard model, AdS/CFT.

*Applications*: condensed matter, statistical physics, (quantum) field theory, string theory and quantum information theory.

## How to solve the Yang-Baxter equation?

- Algebraic approach: $R$-matrix exhibits some symmetries [Drinfeld, Faddeev, Kulish, Reshetikin, . . . ]
- Direct solution of YBE [R.S. Vieira]
- Our approach: Reduce YBE to coupled ordinary differential equations.

The transfer matrix

$$T(u,\theta) = \mathrm{tr}_0 \left( R_{0L}(u,\theta) \dots R_{01}(u,\theta) \right),$$

generates an infinite tower of commuting conserved charges

$$\log T(u,\theta) = \mathbb{Q}_1(\theta) + (u-\theta)\mathbb{Q}_2(\theta) + \frac{1}{2}(u-\theta)^2 \mathbb{Q}_3(\theta) + \dots$$

$R(u,u) = P$ *regular solution* $\rightarrow$ *nearest neighbour* spin chain

$\log T(\theta,\theta) = \mathbb{Q}_1(\theta) \sim \mathbb{P}$

$\mathbb{Q}_2 = \mathbb{H}(\theta) = \sum_{n=1}^{L} P_{n,n+1} \frac{d}{du} R_{n,n+1}(u,\theta)|_{u\rightarrow\theta} \equiv \sum_n \mathcal{H}_{n,n+1}(\theta)$

## Our approach

Bottom-up approach: **Boost automorphism mechanism**

***Recipe***:

1) Consider a general nearest-neighbour Hamiltonian $\begin{pmatrix} h_1(\theta) & 0 & 0 & h_8(\theta) \\ 0 & h_2(\theta) & h_6(\theta) & 0 \\ 0 & h_5(\theta) & h_3(\theta) & 0 \\ h_7(\theta) & 0 & 0 & h_4(\theta) \end{pmatrix}$.

2) Use the boost operator $\mathcal{B}[\mathbb{Q}_2]$ to generate the next conserved charge $\mathbb{Q}_3(\theta)$, $\quad \mathcal{B}[\mathbb{Q}_2] := \partial_\theta + \sum_{n=-\infty}^{\infty} n\mathcal{H}_{n,n+1}$, $\quad \mathbb{Q}_3 \sim [\mathcal{B}[\mathbb{Q}_2], \mathbb{Q}_2]$.

3) From $[\mathbb{Q}_2(\theta), \mathbb{Q}_3(\theta)] = 0$, solve a set of coupled first order, non-linear, differential equations for the entries of $\mathcal{H}_{i,i+1}$, so ($h_i(\theta)$, $1 \leq i \leq 8$).

4) Plug the solutions in the $\mathcal{H}_{i,i+1}(\theta)$.

We need to find $R$-matrix whose logarithmic derivative is $\mathcal{H}_{i,i+1}(\theta)$

5) Solve Sutherland equation $\left[R_{13}R_{23}, \mathcal{H}_{12}(u)\right] = \dot{R}_{13}R_{23} - R_{13}\dot{R}_{23}$ to find $R$.

Using this method, we classified *all possible regular* integrable spin-chains whose $\mathcal{H}$ and $R$-matrix are of the form

$$\mathcal{H}(\theta) = \begin{pmatrix} h_1(\theta) & 0 & 0 & h_8(\theta) \\ 0 & h_2(\theta) & h_6(\theta) & 0 \\ 0 & h_5(\theta) & h_3(\theta) & 0 \\ h_7(\theta) & 0 & 0 & h_4(\theta) \end{pmatrix}, \qquad R(u,v) = \begin{pmatrix} r_1(u,v) & 0 & 0 & r_8(u,v) \\ 0 & r_2(u,v) & r_6(u,v) & 0 \\ 0 & r_5(u,v) & r_3(u,v) & 0 \\ r_7(u,v) & 0 & 0 & r_4(u,v) \end{pmatrix}. \qquad (1)$$

We found two classes of models characterized by the following entries of the Hamiltonian:

A
- 6-Vertex $h_6 \neq 0$ and $h_7 = h_8 = 0$
- 8-Vertex $h_6 \neq 0$, $h_7 \neq 0$ $h_8 \neq 0$

A: Contains the usual XXZ and XYZ chain

B
- 6-Vertex $h_6 = h_7 = h_8 = 0$
- 8-Vertex $h_6 = 0$, $h_7 \neq 0$ $h_8 \neq 0$

B: Contains $AdS_2$ and $AdS_3$

$R$-matrices of all these models satisfy the condition: $\frac{(r_1 r_4 + r_2 r_3 - r_5 r_6 - r_7 r_8)^2}{r_1 r_2 r_3 r_4} = \text{const}$

const $\neq 0$ Baxter condition (class A models),

const $= 0$ **Free-fermion condition** (class B models).

For **class B models** 6-Vertex type

*Set-up*: 2 D space spanned by one boson $|\phi\rangle$ and one fermion $|\psi\rangle$

$$|\phi\rangle \equiv |0\rangle, \quad |\psi\rangle \equiv c^\dagger|0\rangle, \quad c|0\rangle = 0,$$

$$\{c, c^\dagger\} = 1, \quad \{c, c\} = \{c^\dagger, c^\dagger\} = 0, \quad c^\dagger c = n, \quad cc^\dagger = m.$$

$$R_{ij}^{(osc)}(u, v) = r_1 m_i m_j + r_2 n_i m_j + r_3 m_i n_j - r_4 n_i n_j - r_5 c_i c_j^\dagger + r_6 c_i^\dagger c_j$$

$$\mathcal{H}_{ij}^{(osc)} = P_{ij}\partial_u R_{ij}^{(osc)}(u, v)\Big|_{v=u} =$$
$$h_1 + (h_6 - h_1)n_j - (h_1 + h_5)n_i - (h_1 + h_4 - h_5 - h_6)n_i n_j + h_3 c_j^\dagger c_i + h_2 c_i^\dagger c_j.$$

Free-fermion condition on the level of the Hamiltonian:

$$h_1 + h_4 - h_5 - h_6 = 0.$$

How can we use the Free fermion condition to diagonalize the transfer matrix?

*Homogeneous spin-chain* of length $N$

Canonical transformation $c_k = \frac{1}{\sqrt{N}} \sum_n e^{2\pi i \frac{kn}{N}} \eta_n, \quad c_k^\dagger = \frac{1}{\sqrt{N}} \sum_n e^{-2\pi i \frac{kn}{N}} \eta_n^\dagger,$

$$\mathbb{H} = \sum_{n=1}^N \mathcal{H}_{i,i+1} = h_1 N + \sum_{n=1}^N \left[ (h_2 + h_3) \cos \frac{2\pi n}{N} + i(h_2 - h_3) \sin \frac{2\pi n}{N} - h_1 + h_4 \right] \eta_n^\dagger \eta_n,$$

$$T = -\exp\left[A + \sum_k B_k N_k\right], \quad A = \log(r_1^N - r_3^N), \qquad B_k = \log\left[\frac{r_2 + e^{\frac{2\pi i k}{N}} r_4}{r_1 - e^{\frac{2\pi i k}{N}} r_3}\right].$$

*Inhomogeneous spin-chain* of length $N$

$$c_i^\dagger = \sum_n \frac{f_i(v_n) \prod_{r=1}^{i-1} S_r(v_n)}{\sqrt{\sum_r f_r(v_n) f_r^*(v_n)}} \eta_n^\dagger, \qquad c_i = \sum_n \frac{f_i^*(v_n) \prod_{r=i}^{N} S_r(v_n)}{\sqrt{\sum_r f_r(v_n) f_r^*(v_n)}} \eta_n,$$

$$T = -\exp\left[A + \sum_{m=1}^N B_m N_m\right],$$

$$A = \log\left[\prod_{m=1}^N r_1(\theta, \theta_m) - \prod_{m=1}^N r_3(\theta, \theta_m)\right], \qquad B_m = \log\left[\frac{r_3(\theta, v_m)}{r_1(\theta, v_m)}\right].$$

Compact, elegant and useful form of the transfer matrix!

## Applications

- $AdS_3$: additive structure to the eigenvalues of the transfer matrix is manifest

## Summary and Conclusion

- Using the boost authomorphism mechanism we *classified all regular solutions of the YBE of 8-vertex type*
- The class of solutions that contains holographic integrable models satisfy the *free fermion conditions*
- Using the free fermion condition we gave the expression of the *transfer matrix* of 6 Vertex type models for arbitrary number of sites and inhomogeneities

*Thank you*