What are $T\bar{T}$ deformations?

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Collaborators

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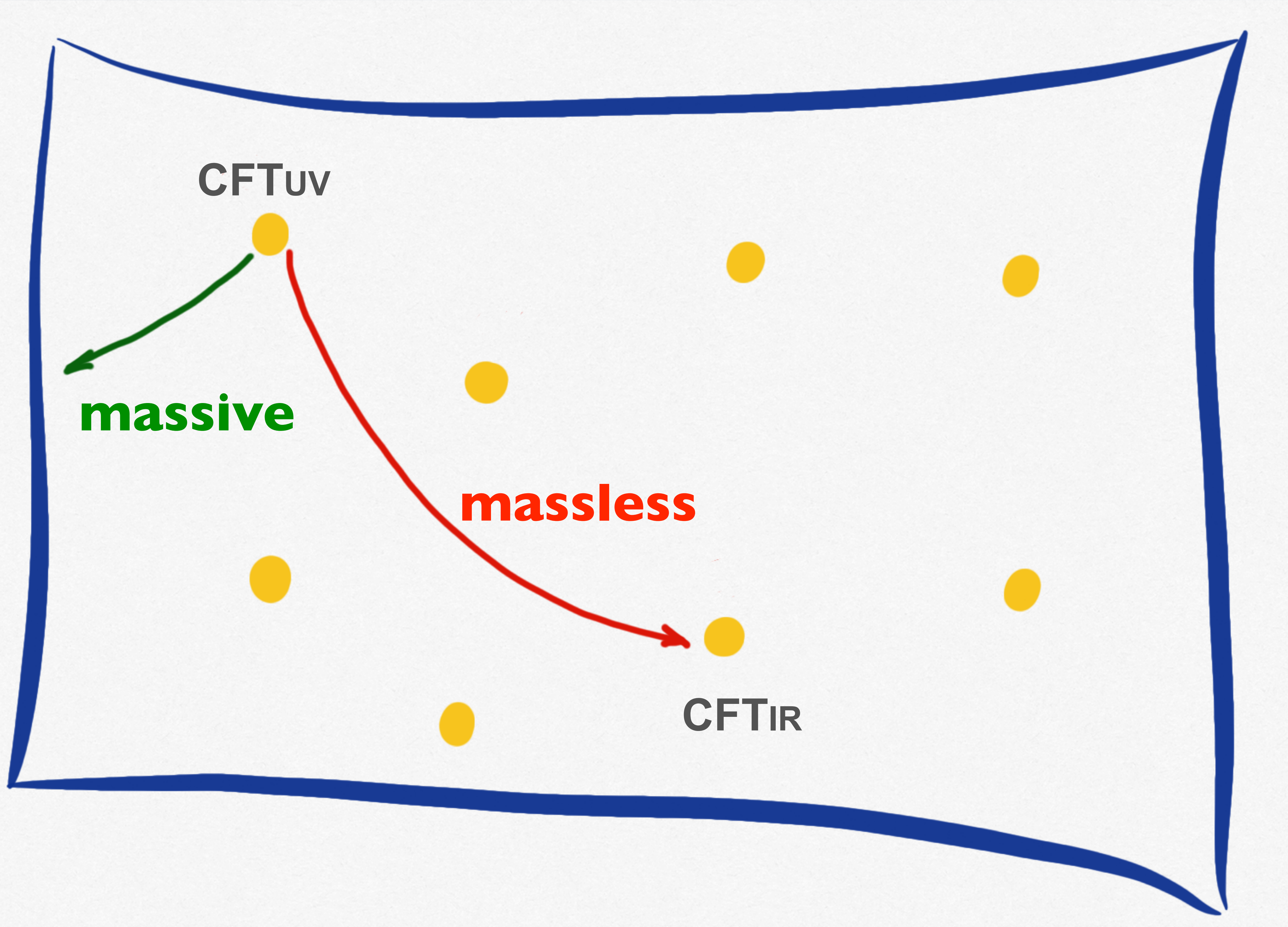
Main motivation:

Understanding the Space of Quantum Field Theories



CFT:

Operator and state content

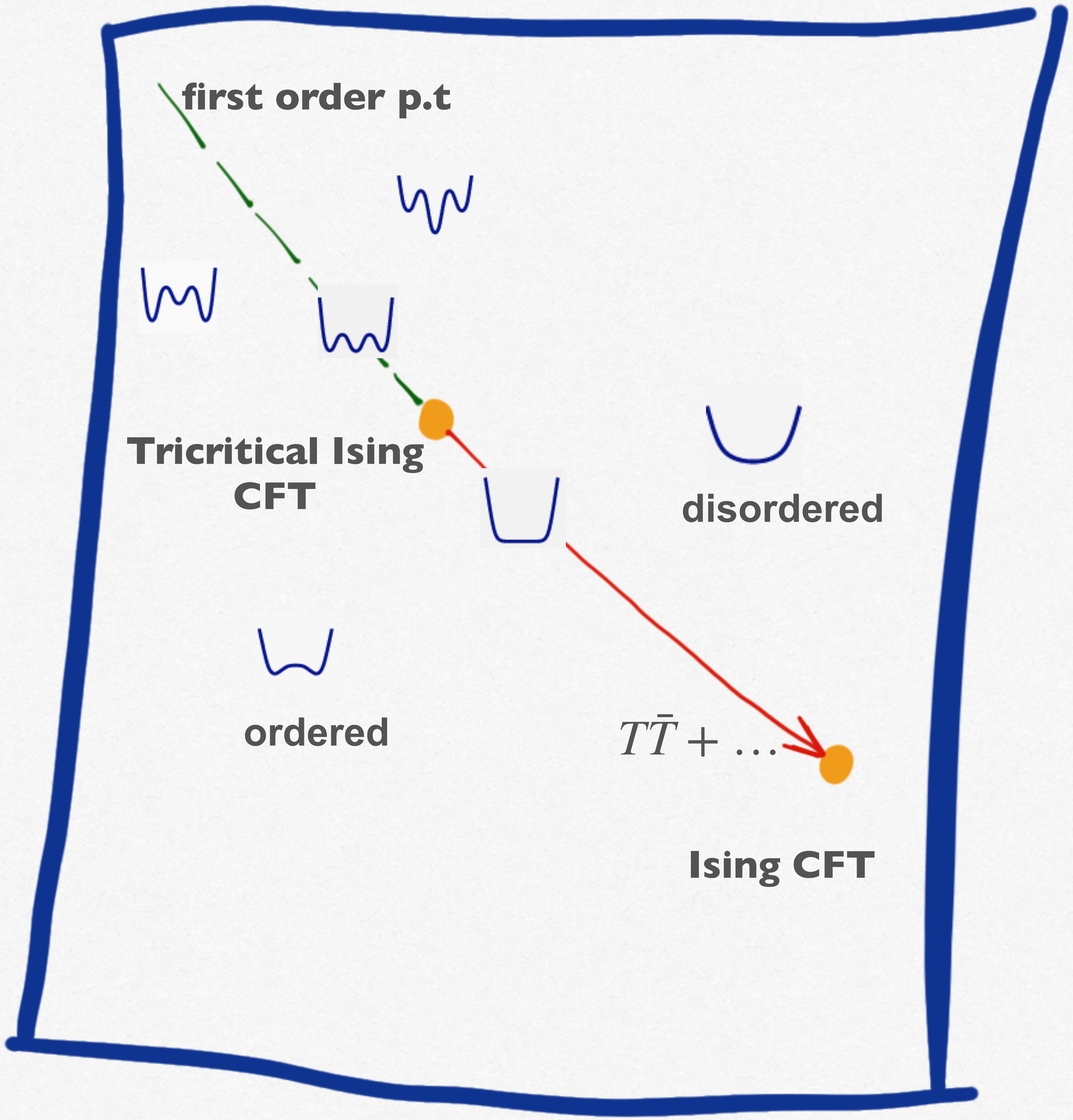
Critical exponents and correlation functions

Massive integrable CFT perturbations:

Exact S-matrix

Finite-Size spectrum (Thermodynamic Bethe ansatz)

Correlation Functions (Form-Factors)



Massless integrable CFT perturbations:

Exact S-matrix

Finite-Size spectrum
(Thermodynamic Bethe ansatz)

IR leading attracting operators

In a CFT

$$T_{xx} = -T_{yy} = -\frac{1}{2\pi}(\bar{T} + T)$$

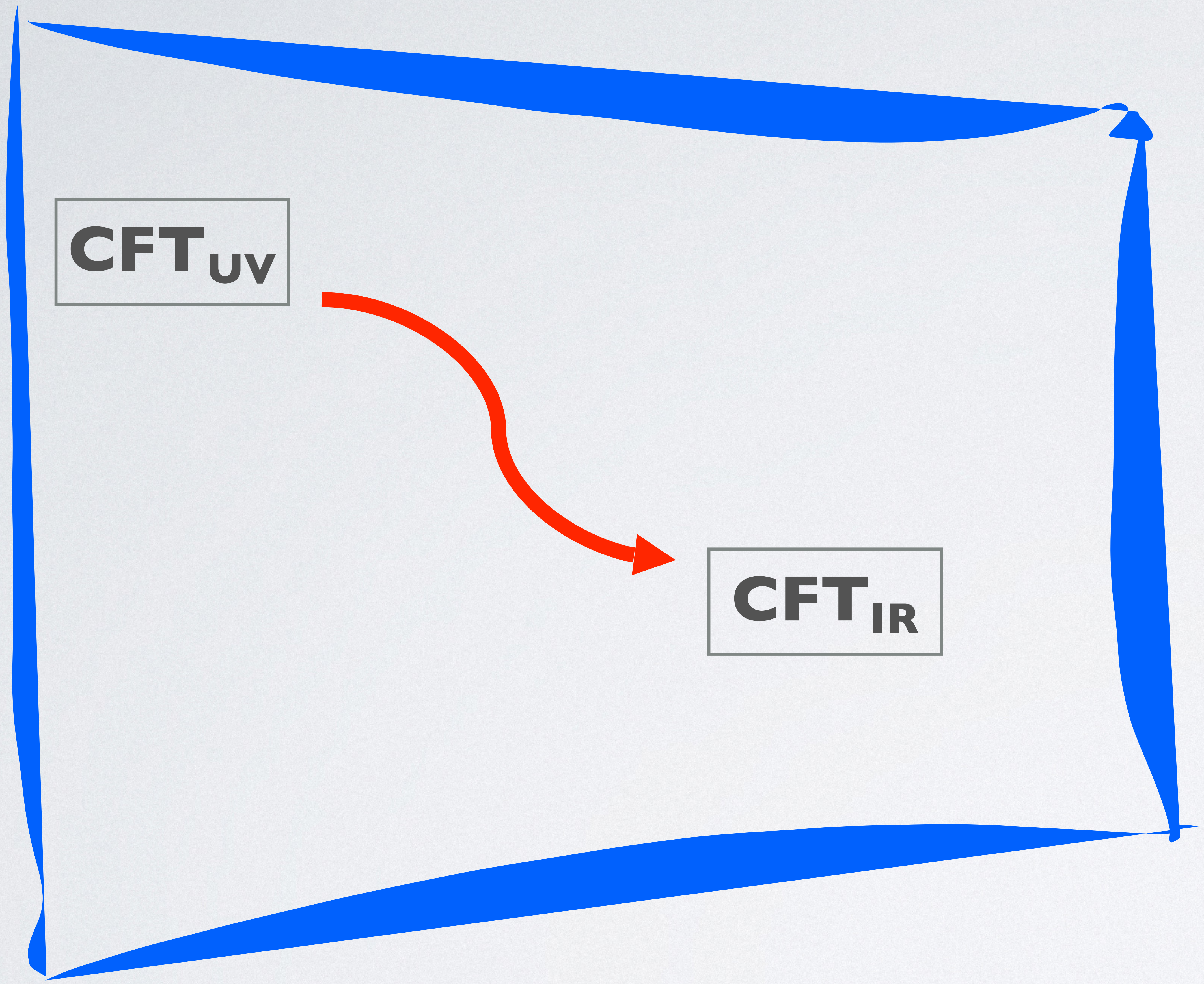
$$T_{yx} = T_{xy} = \frac{i}{2\pi}(\bar{T} - T)$$

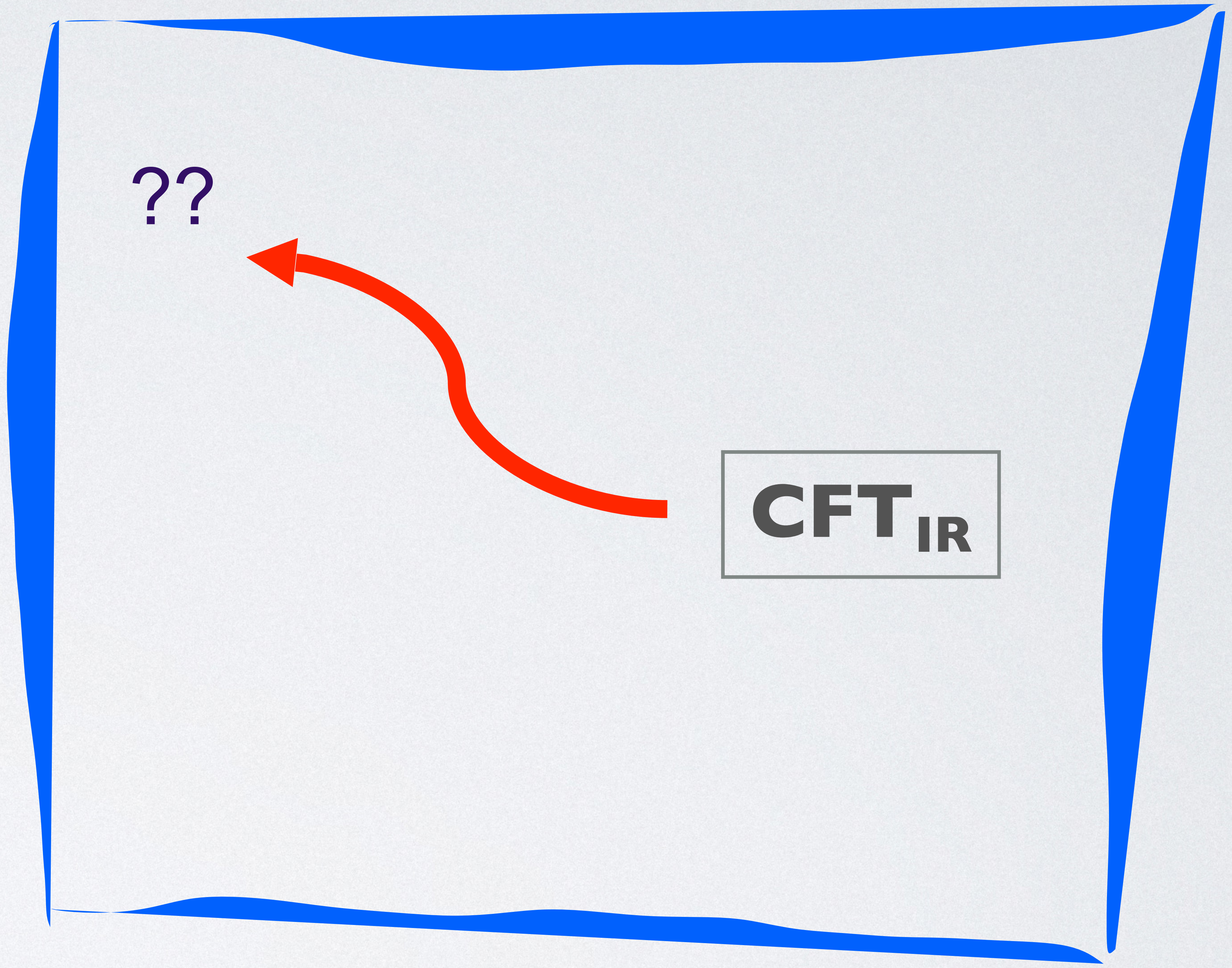
and

$$T\bar{T}(z, \bar{z}) = T(z)\bar{T}(\bar{z})$$

$$(z = x + iy,\ \bar{z} = z - iy)$$

Can we reverse the renormalisation group trajectory?





Let us try with the $T\bar{T}$ perturbation …

We need the correct definition of $T\bar{T}$ outside a CFT fixed point:

$$T_{xx} = -\frac{1}{2\pi}(\bar{T} + T - 2\Theta)\,, \quad T_{yy} = \frac{1}{2\pi}(\bar{T} + T + 2\Theta)\,, \quad T_{xy} = \frac{i}{2\pi}(\bar{T} - T)\,.$$

Sasha Zamolodchikov (2004):

$$\mathrm{T\bar{T}}(z,\bar{z}) := \lim_{(z,\bar{z})\to(z',\bar{z}')} T(z,\bar{z})\bar{T}(z',\bar{z}') - \Theta(z,\bar{z})\Theta(z',\bar{z}') + \text{total derivatives}$$

Therefore, up to total derivatives

$$T\bar{T}(z,\bar{z}) := -\,\pi^2 \det(T_{\mu\nu}(z,\bar{z}))$$

At first order

$$\mathscr{L}(x,y;\tau) = \mathscr{L}(x,y;0) + \tau \det(T_{\mu\nu}(x,y;0))$$

$T\bar{T}$ **Lagrangian and Hamiltonian flow equations**:

$$\partial_\tau \mathscr{L}(\tau) = \det(T_{\mu\nu}(\tau))\,, \qquad T_{\mu\nu}(\tau) = \frac{-2}{\sqrt{g}}\frac{\partial \mathscr{L}(\tau)}{\partial g^{\mu\nu}}\,, \qquad \partial_\tau \mathscr{H}(\tau) = \det(T^{\mu\nu}(\tau))$$

(Euclidean space-time)

Massless boson field theories

$$\mathscr{L}(0) = \partial\overrightarrow{\phi} \cdot \bar{\partial}\overrightarrow{\phi}$$

(In complex coordinates: $\partial = \partial_z \, , \bar{\partial} = \partial_{\bar{z}}$)

$(z = x^1 + ix^2, \bar{z} = x^1 - ix^2)$

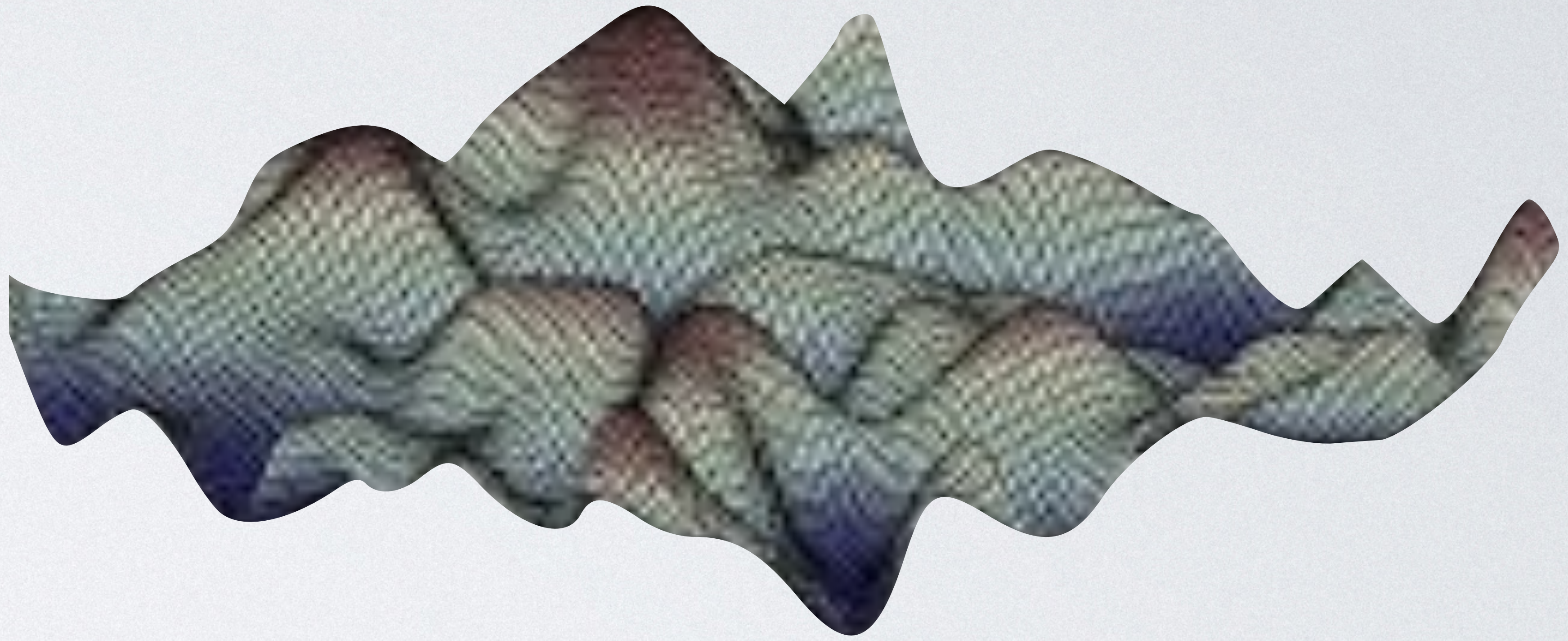
$$\Downarrow$$

$$\mathscr{L}(\tau) = \frac{1}{2\tau}\left(-1 + \sqrt{1 + 4\tau\partial\overrightarrow{\phi} \cdot \bar{\partial}\overrightarrow{\phi} - 4\tau^2 \,\mathscr{B}}\right) = -\frac{1}{2\tau} + \mathscr{L}_{NG}^{static}(\tau)$$

$$\mathscr{B} = \sum_{i=1}^{N} \left(\partial\phi_i\right)^2 \sum_{j=1}^{N} \left(\bar{\partial}\phi_j\right)^2 - \left(\sum_{i=1}^{N} \partial\phi_i\bar{\partial}\phi_i\right)^2$$

The Nambu-Goto model

$$\mathscr{A} = \int dA = \int d^2x\, \mathscr{L}_{NG} \;= \frac{1}{2\tau}\int \sqrt{\det\left(\sum_{\mu=1}^{D} \partial_\alpha X^\mu(x^1, x^2)\, \partial_\beta X^\mu(x^1, x^2)\right)}\; d^2x$$



in the static gauge

$$X^1 \to x^1\,, \quad X^2 \to x^2\,, \quad X^i \to \tau^{\frac{1}{2}}\,\phi^{i-2}\,, \quad (i = 3,\ldots D)$$

we have

$$\mathscr{L}_{NG} \to \mathscr{L}_{NG}^{static}$$

Boson field theories with generic potential

$$\mathscr{L}^V(0) = \mathscr{L}(0) - V$$

$$\mathscr{L}(0) = \partial\overrightarrow{\phi} \cdot \bar{\partial}\overrightarrow{\phi}\,,\; V = V(\overrightarrow{\phi})$$

$$\Downarrow$$

$$\mathscr{L}^V(\tau) = \frac{-V}{1+\tau V} + \frac{1}{2\bar{\tau}}\left(-1 + \sqrt{1 + 4\bar{\tau}\mathscr{L}(0) - 4\bar{\tau}^2\,\mathscr{B}}\right)$$

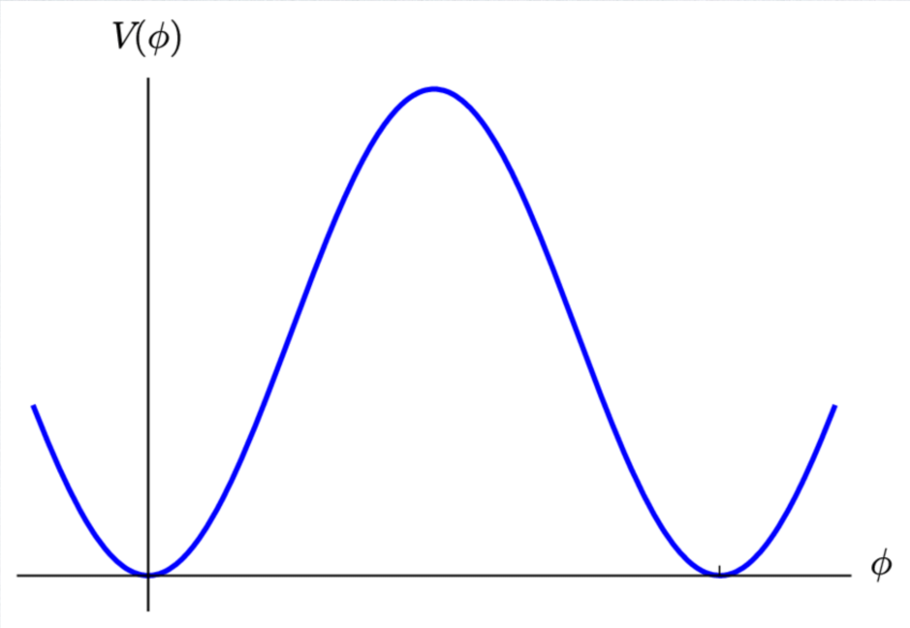
with

$$\bar{\tau} = \tau\,(1 + \tau V)$$

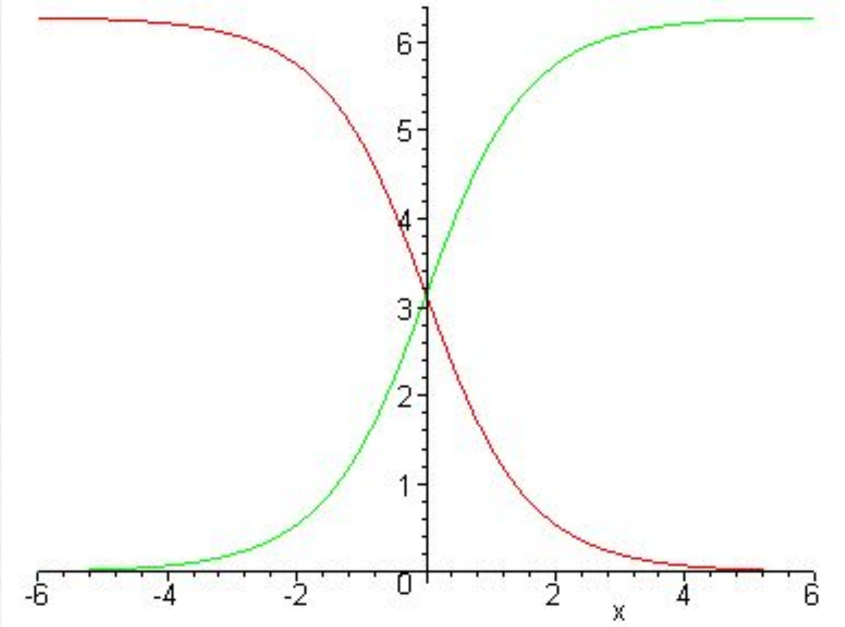
The sine-Gordon model

$$\mathcal{L}_{sG}(\phi,\tau) = \frac{-V}{1+\tau V} + \frac{-1+S(\phi)}{2\tau(1+\tau V)} \qquad S(\phi) = \sqrt{1+4\tau\,(1+\tau V)\,\partial\phi\bar{\partial}\phi}$$

with



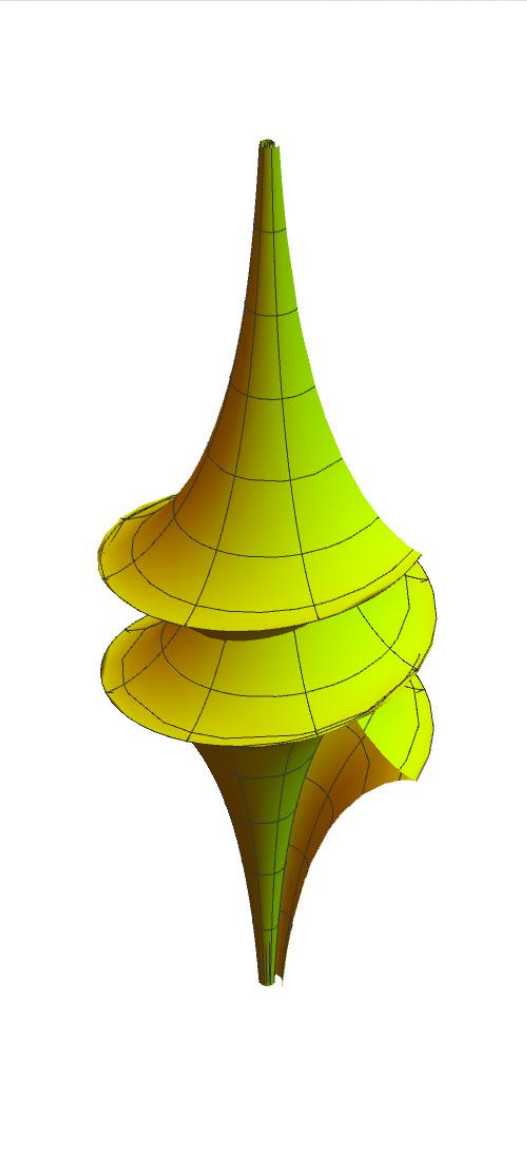
$$V = 2\frac{m^2}{\beta^2}\left(1-\cos(\beta\phi)\right)$$



and EoM

$$\partial\left(\frac{\bar{\partial}\phi}{S}\right) + \bar{\partial}\left(\frac{\partial\phi}{S}\right) = -\frac{V'}{4S}\left(\frac{S+1}{1+\tau V}\right)^2 \qquad V' = 2\frac{m^2}{\beta}\sin(\beta\phi)$$

A local change of coordinates

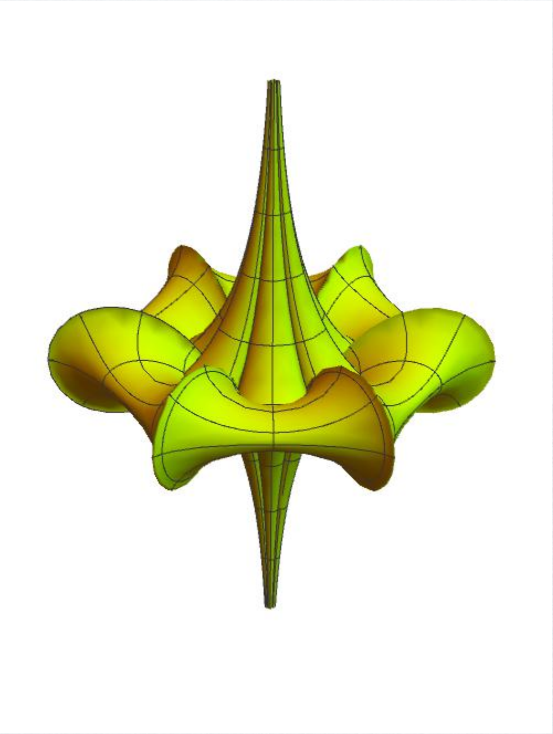


$$\mathcal{J}^{-1} = \begin{pmatrix} \partial_w z & \partial_w \bar{z} \\ \partial_{\bar{w}} z & \partial_{\bar{w}} \bar{z} \end{pmatrix} = \begin{pmatrix} 1+\tau V & -\tau \left(\frac{\partial\phi}{\partial w}\right)^2 \\ -\tau \left(\frac{\partial\phi}{\partial \bar{w}}\right)^2 & 1+\tau V \end{pmatrix}$$

$(z = x^1 + ix^2, \bar{z} = x^1 - ix^2)$

$(w = y^1 + iy^2, \bar{w} = y^1 - iy^2)$

$$\phi^{(\tau)}(\mathbf{z}) = \phi^{(0)}(\mathbf{w}(\mathbf{z})) \ , \quad \mathbf{z} = (z, \bar{z}) \, , \quad \mathbf{w} = (w, \bar{w})$$

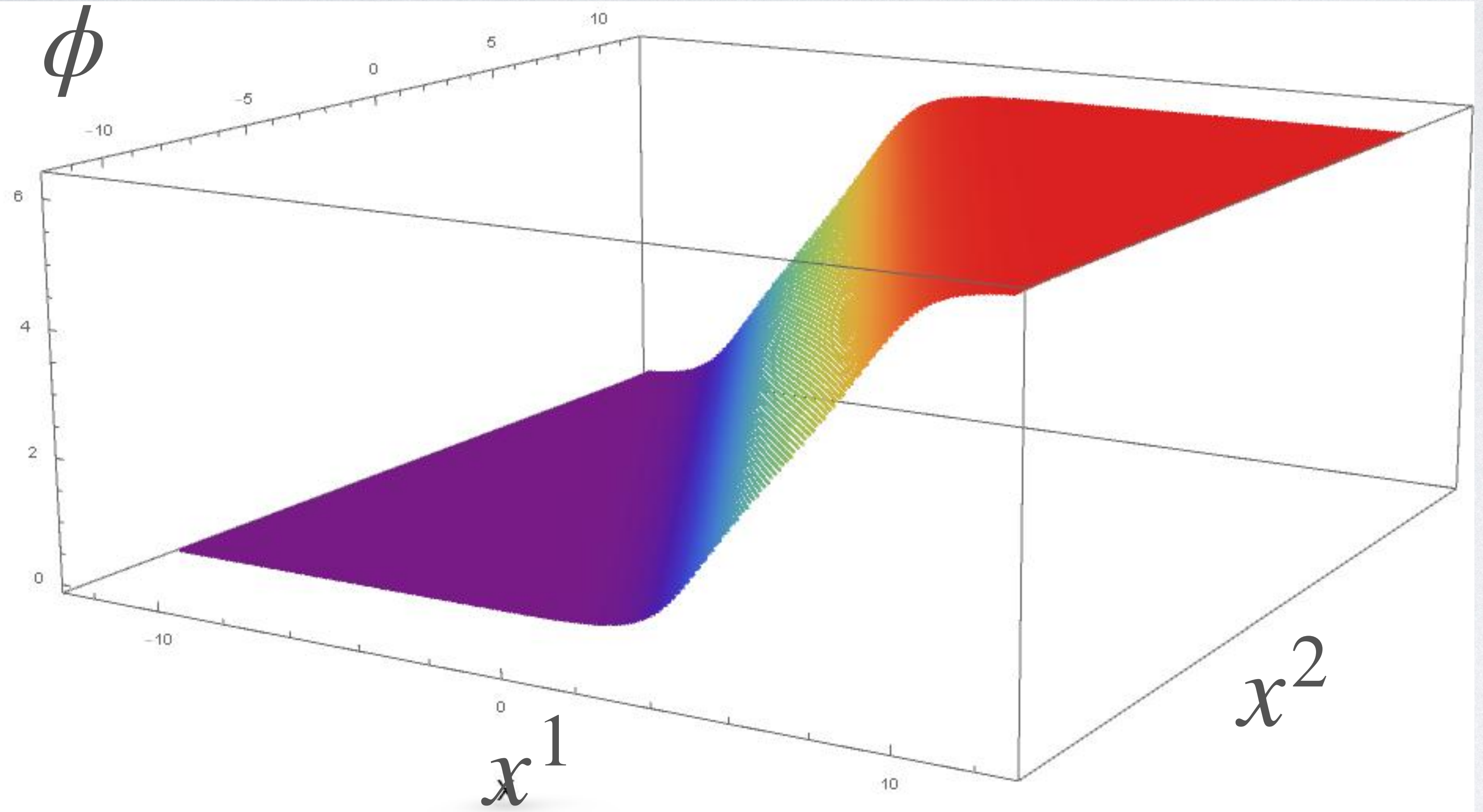


$$\partial\left(\frac{\bar{\partial}\phi}{S}\right) + \bar{\partial}\left(\frac{\partial\phi}{S}\right) = -\frac{V'}{4S}\left(\frac{S+1}{1+\tau V}\right)^2 \longrightarrow 2\partial_w\partial_{\bar{w}}\phi = -V'$$

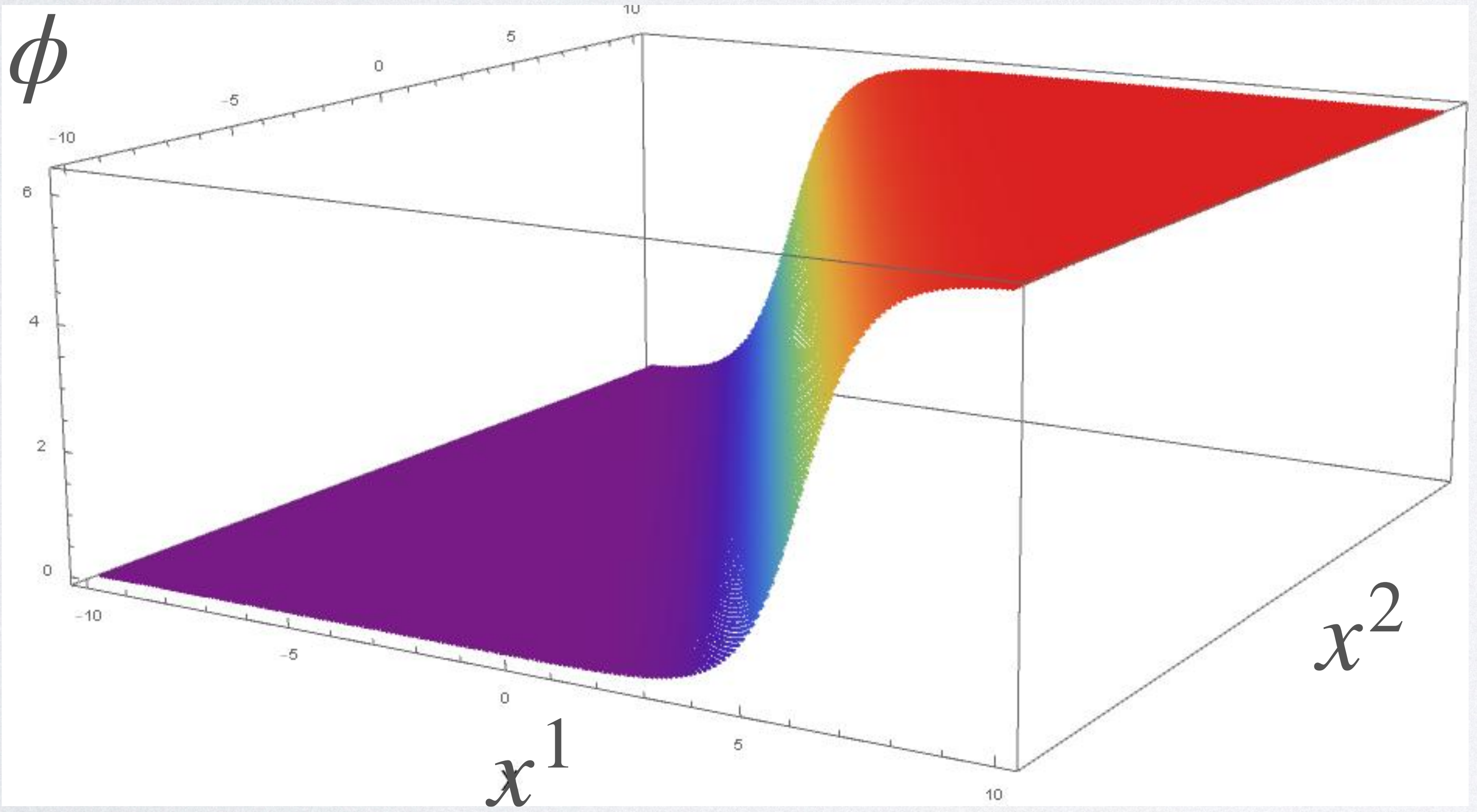
(Pseudo-spherical solitonic surface: the kink and the breather)

The deformed kink solution

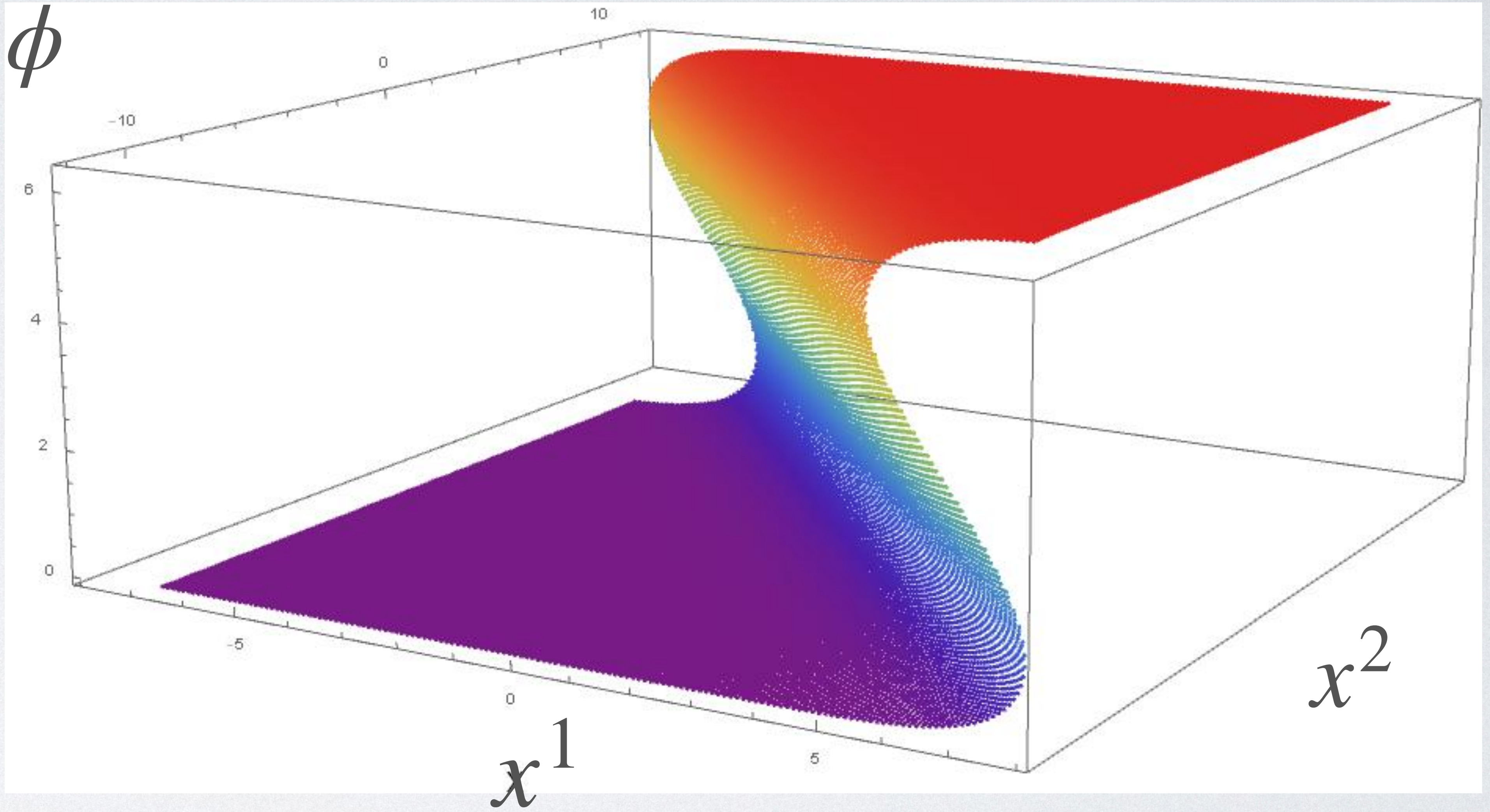
$$\phi^{(0)}_{\text{1-kink}}(\mathbf{w}) = 4\arctan\left(e^{\frac{m}{\beta}\left(aw+\frac{1}{a}\bar{w}\right)}\right)\ ,\ a = \sqrt{\frac{1-v}{1+v}}$$



$\tau < 0$

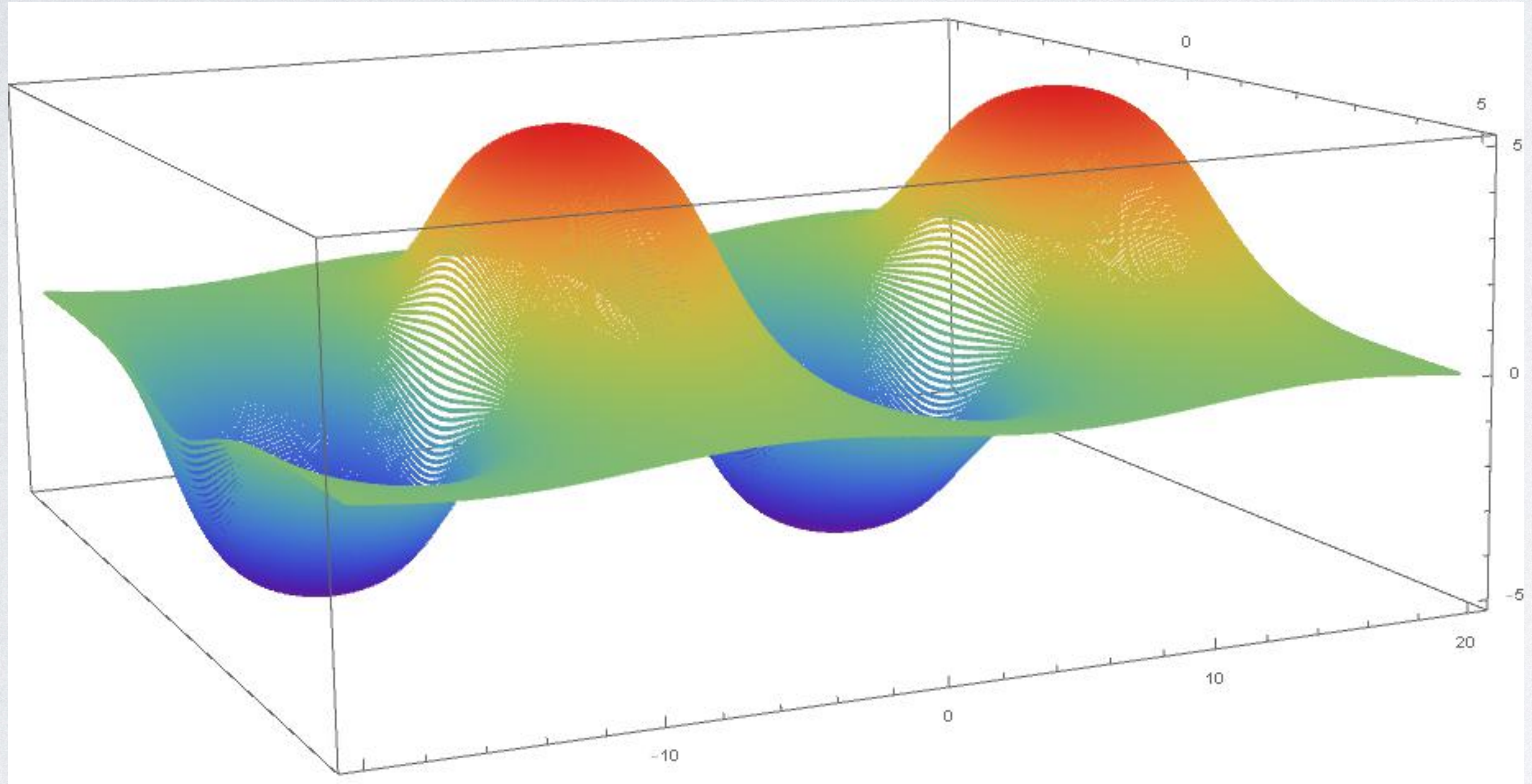


$\tau = 0$

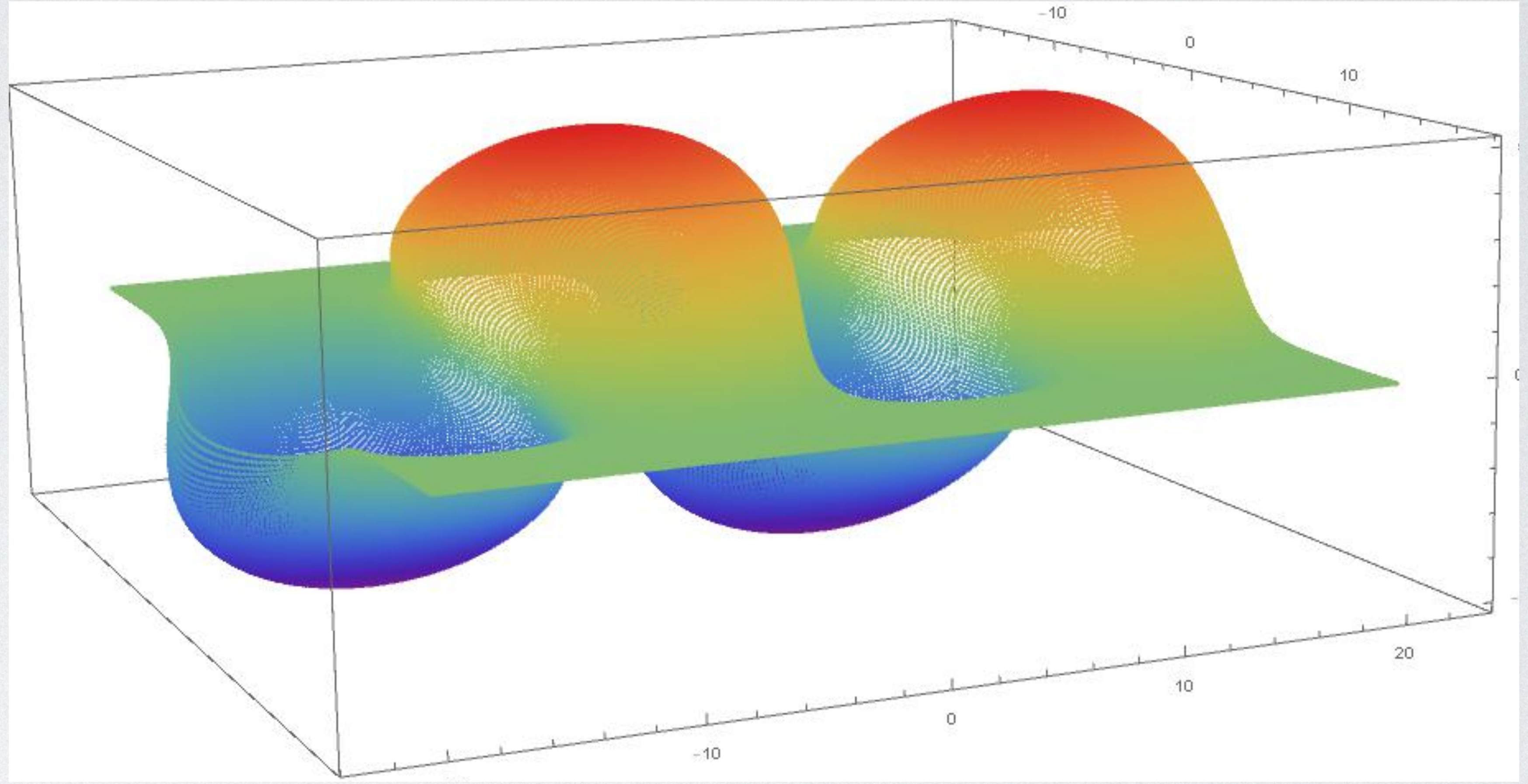


$\tau > 0$

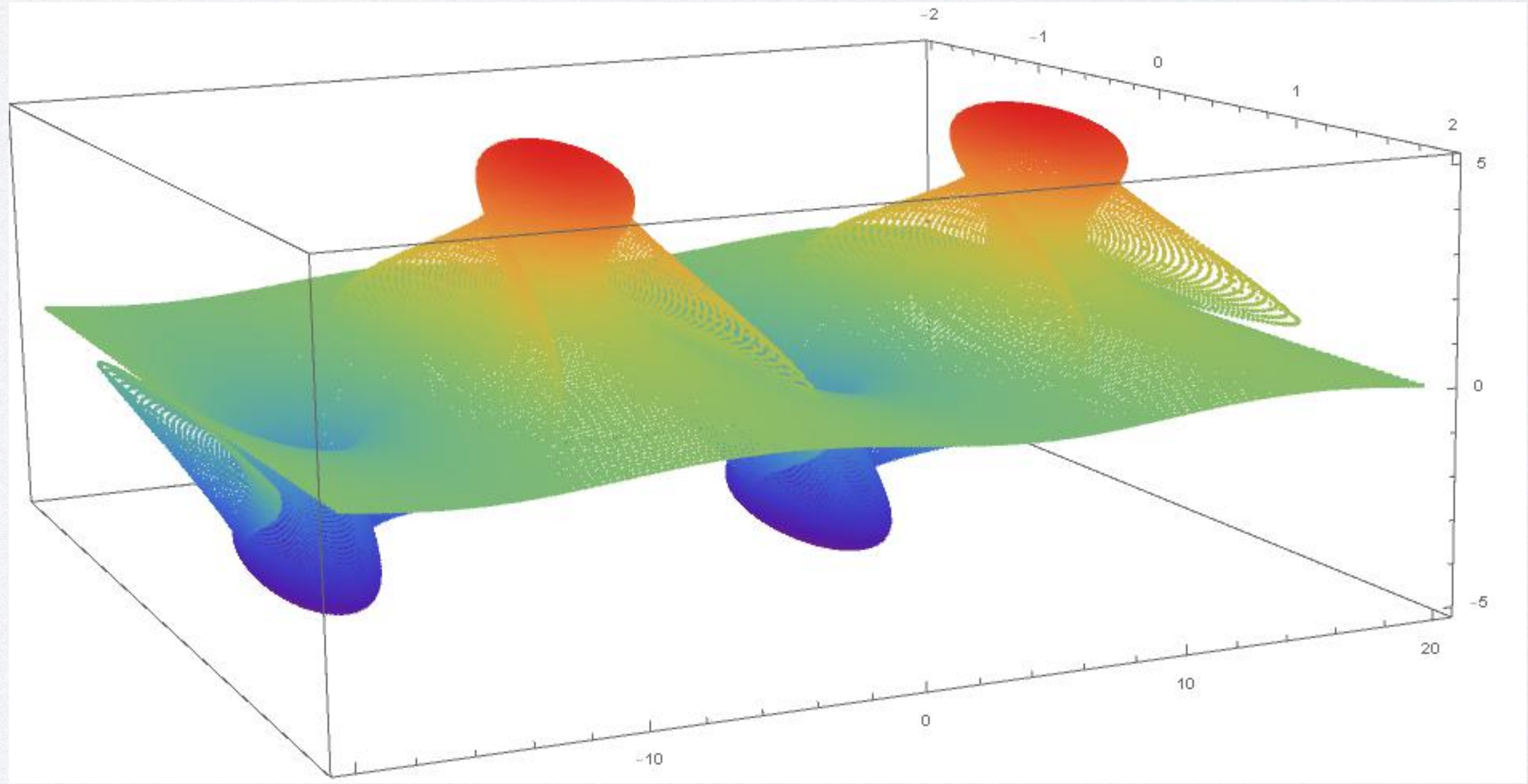
The deformed sine-Gordon breather



$\tau = 0$



$\tau < 0$



$\tau > 0$

Generic $T\bar{T}$-deformed models

$$\mathcal{J}^{-1} = \begin{pmatrix} \partial_w z & \partial_w \bar{z} \\ \partial_{\bar{w}} z & \partial_{\bar{w}} \bar{z} \end{pmatrix} = \begin{pmatrix} 1+\tau V & -\tau\left(\frac{\partial\phi}{\partial w}\right)^2 \\ -\tau\left(\frac{\partial\phi}{\partial \bar{w}}\right)^2 & 1+\tau V \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 1-\tau\Theta(\mathbf{w}) & -\tau\bar{T}(\mathbf{w}) \\ -\tau T(\mathbf{w}) & 1-\tau\Theta(\mathbf{w}) \end{pmatrix}$$

Notice that

$$\boxed{\frac{\partial^2 x^\mu}{\partial y^\rho \partial y^\sigma} = \frac{\partial^2 x^\mu}{\partial y^\sigma \partial y^\rho} \quad \Longleftrightarrow \quad \partial_\mu \mathbf{T}^\mu{}_\nu = 0}$$

$$\mathcal{A}[\phi] = \int dz\, d\bar{z}\, \mathcal{L}^{(\tau)}(\mathbf{z}) = \int dw\, d\bar{w}\, \left|\det\left(\mathcal{J}^{-1}\right)\right| \mathcal{L}^{(\tau)}(\mathbf{z}(\mathbf{w}))$$
$$= \int dw\, d\bar{w}\, \left(\mathcal{L}^{(0)}(\mathbf{w}) + \tau\, \mathrm{T\bar{T}}^{(0)}(\mathbf{w})\right)$$

Quantum $T\bar{T}$-deformations on infinite cylinder of circumference R

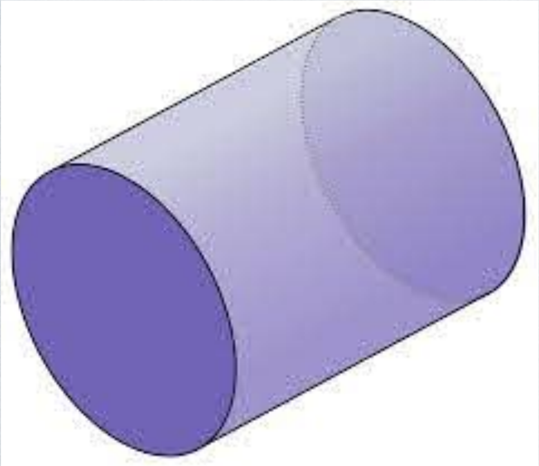
$$\partial_\tau \mathscr{H}(\tau) = \det(T_{\mu\nu}(\tau)) \to \partial_\tau \langle n \,|\, \mathscr{H}(\tau) \,|\, n \rangle = \langle n \,|\, \det(T_{\mu\nu}(\tau)) \,|\, n \rangle$$

$$\langle n \,|\, \det(T_{\mu\nu}(\tau)) \,|\, n \rangle = \langle n \,|\, T_{11} \,|\, n \rangle \langle n \,|\, T_{22} \,|\, n \rangle - \langle n \,|\, T_{12} \,|\, n \rangle \langle n \,|\, T_{21} \,|\, n \rangle$$

[Zamolodchikov 2004]

$$E_n(R,\tau) = -R \langle n | T_{22} | n \rangle \ , \ \partial_R E_n(R,\tau) = -\langle n | T_{11} | n \rangle \ , \ P_n(R) = -\mathbb{i} R \langle n | T_{12} | n \rangle$$

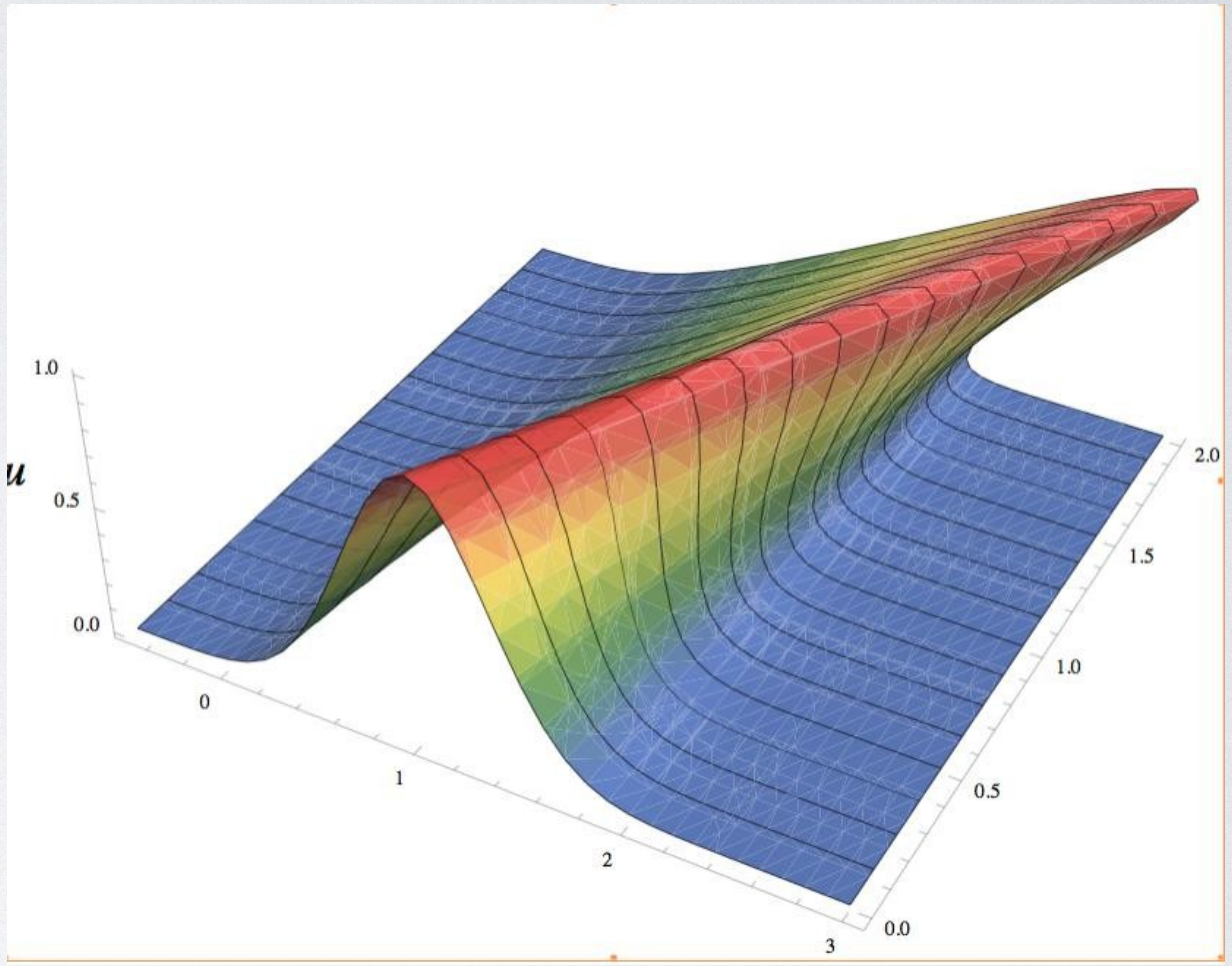
$$P(R,\tau) = P(R) = \frac{2\pi k}{R}, \quad k \in \mathbb{Z}$$



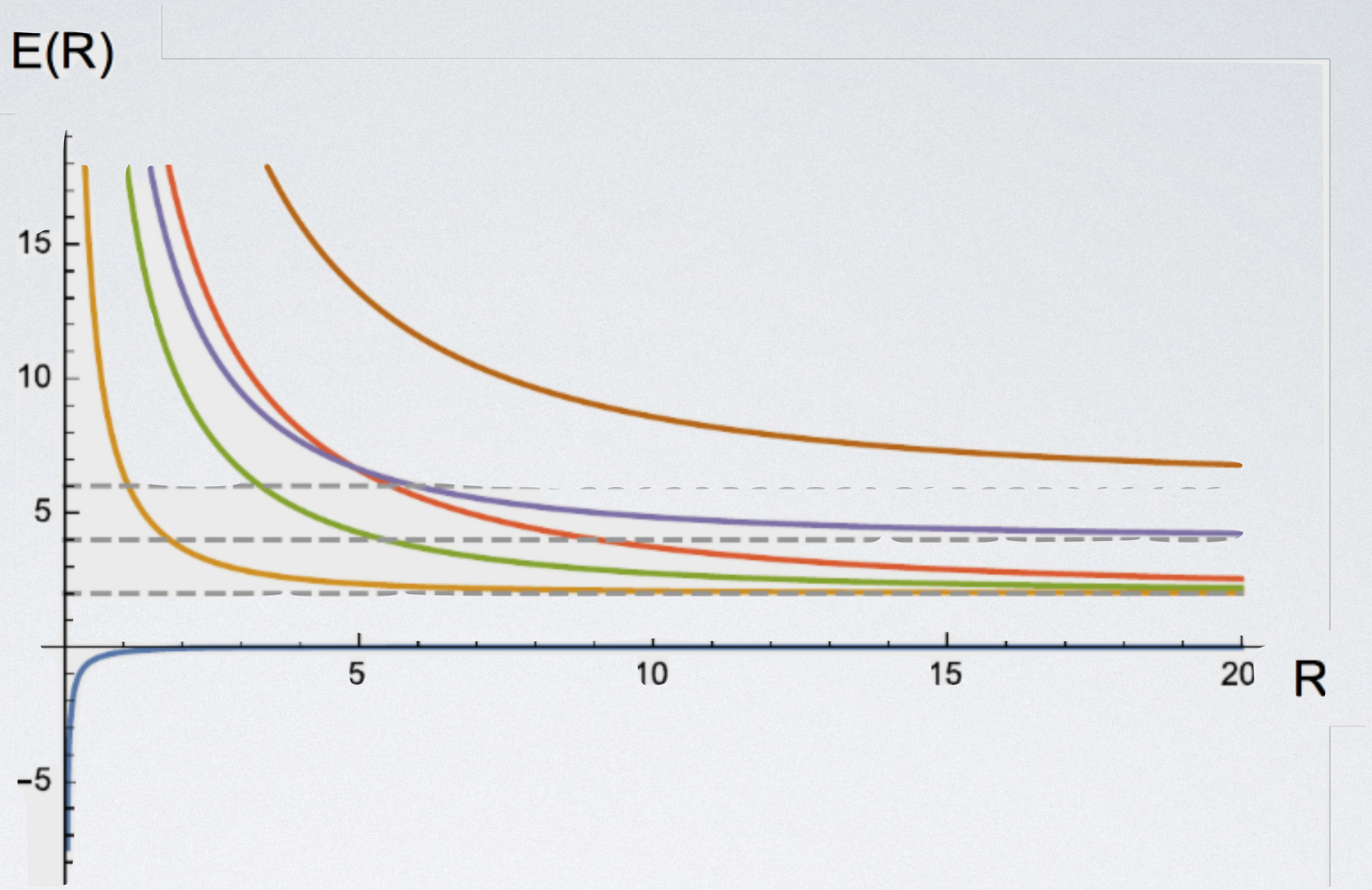
The inviscid Burgers equation for the quantum spectrum

$$\partial_\tau E_n(R,\tau) = E_n(R,\tau)\partial_R E_n(R,\tau) + \frac{P_n^2(R)}{R}$$

source term



$$P_n = 0 \rightarrow E_n(R,\tau) = E_n(R + \tau E_n(R,\tau),0)$$

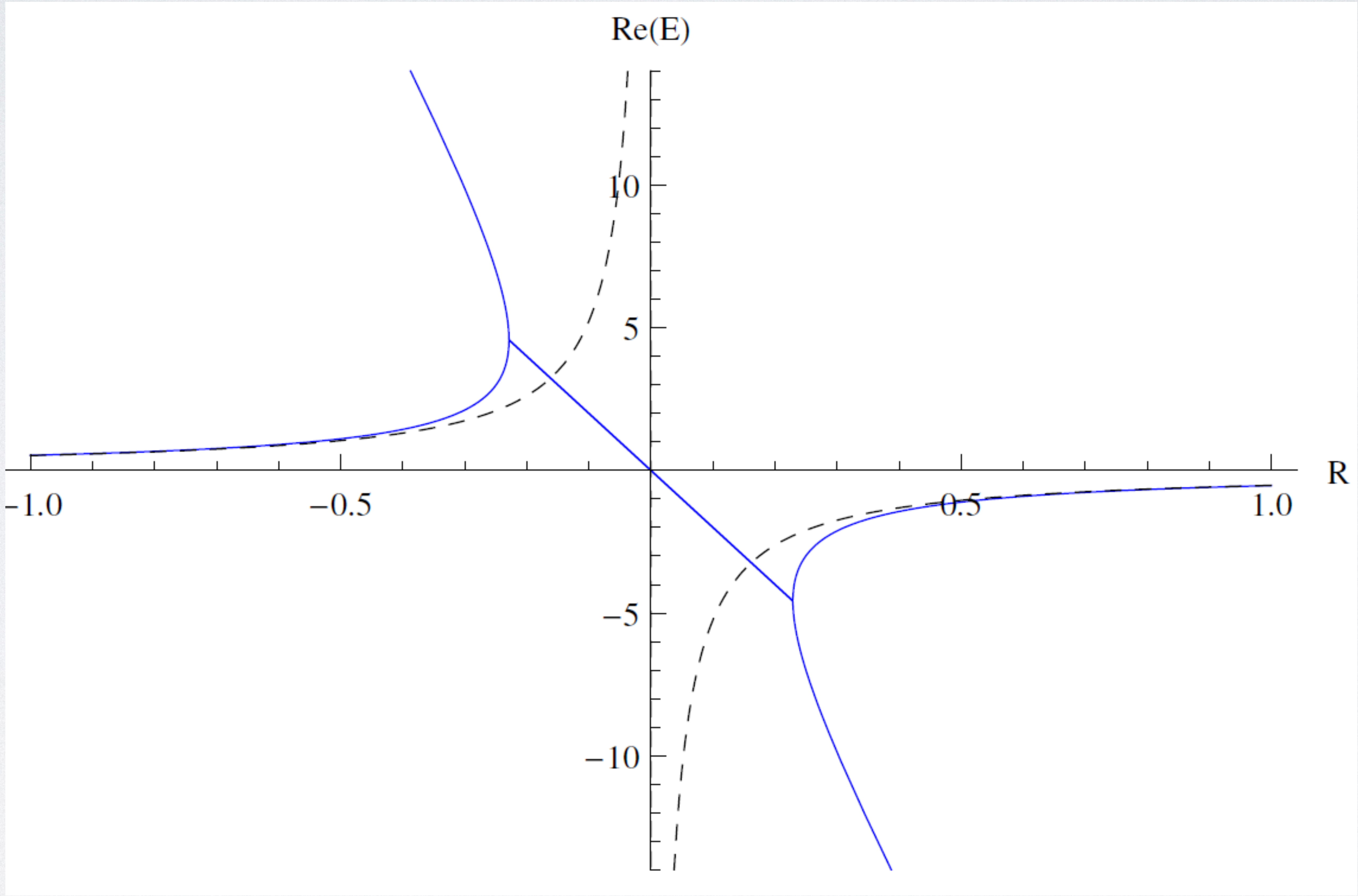


(Typical $\tau = 0$ finite-volume spectrum)

$$E(R,0) \sim -\pi \frac{c_{\rm eff}}{6\,R}, \quad R \sim 0,$$

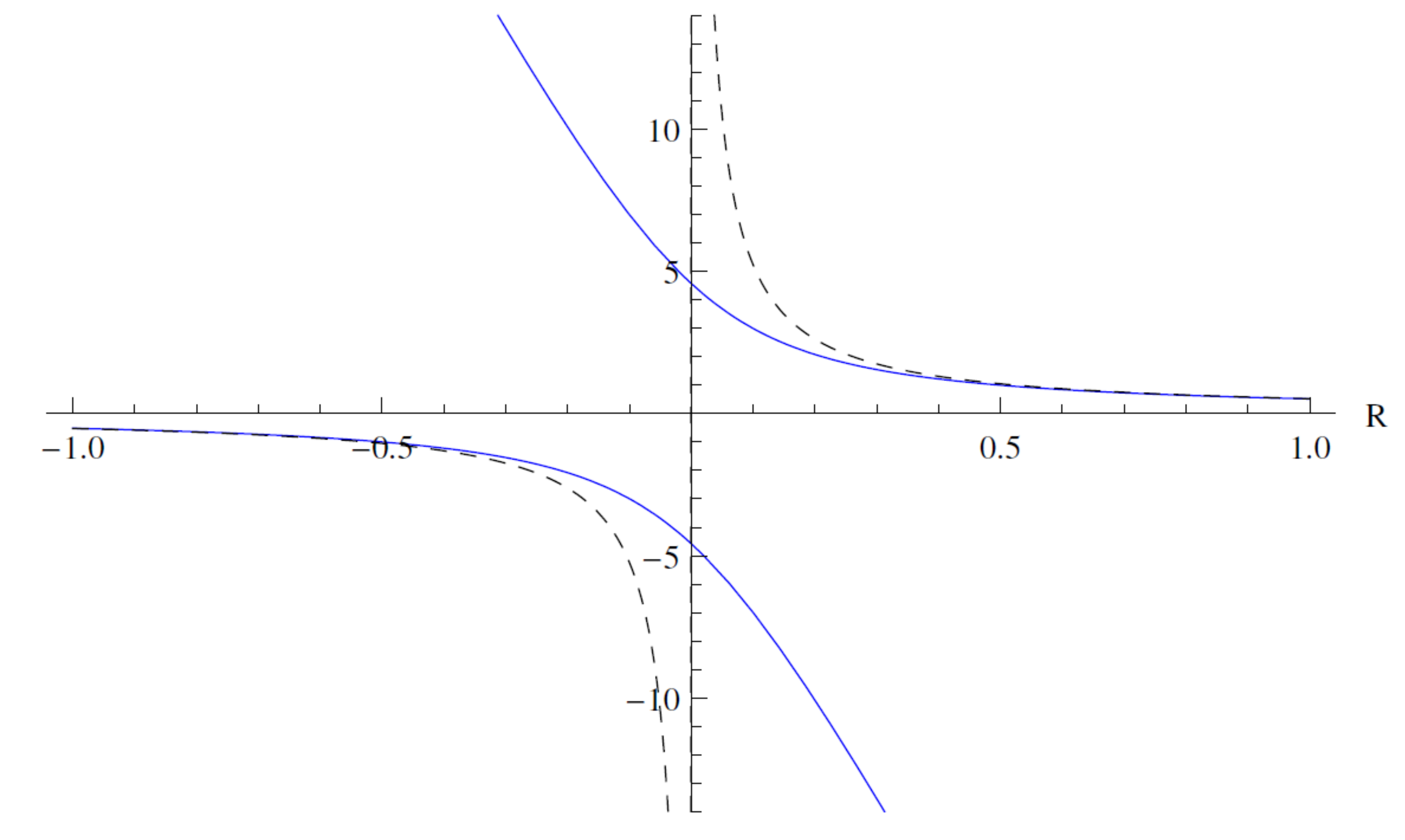
where, $c_{eff} = c - 24\Delta$ is the "effective central charge" of the UV CFT state.

For $c_{eff} > 0$ (i.e. the ground-state energy) we have a "wave-breaking" phenomena

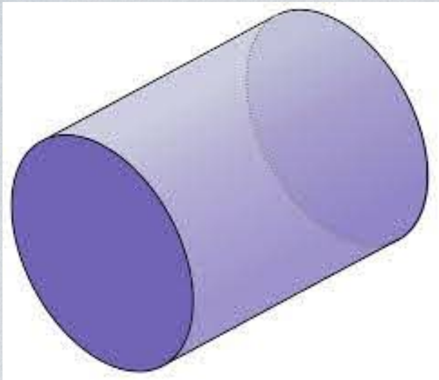


$(\tau > 0)$

For $c_{eff} < 0$ (i.e. generic excited state) the branch points move off, along the imaginary axis



$(\tau < 0)$



The CFT case

The total energy:

$$
\begin{aligned}
E(R,\tau) &= E^{(+)}(R,\tau) + E^{(-)}(R,\tau) \\
&= -\frac{R}{2\tau} + \sqrt{\frac{R^2}{4\tau^2} + \frac{2\pi}{\tau}\left(n_0 + \bar{n}_0 - \frac{c_{\rm eff}}{12}\right) + \left(\frac{2\pi(n_0 - \bar{n}_0)}{R}\right)^2}
\end{aligned}
$$

Dubovsky-Flauger-Gorbenko 2012

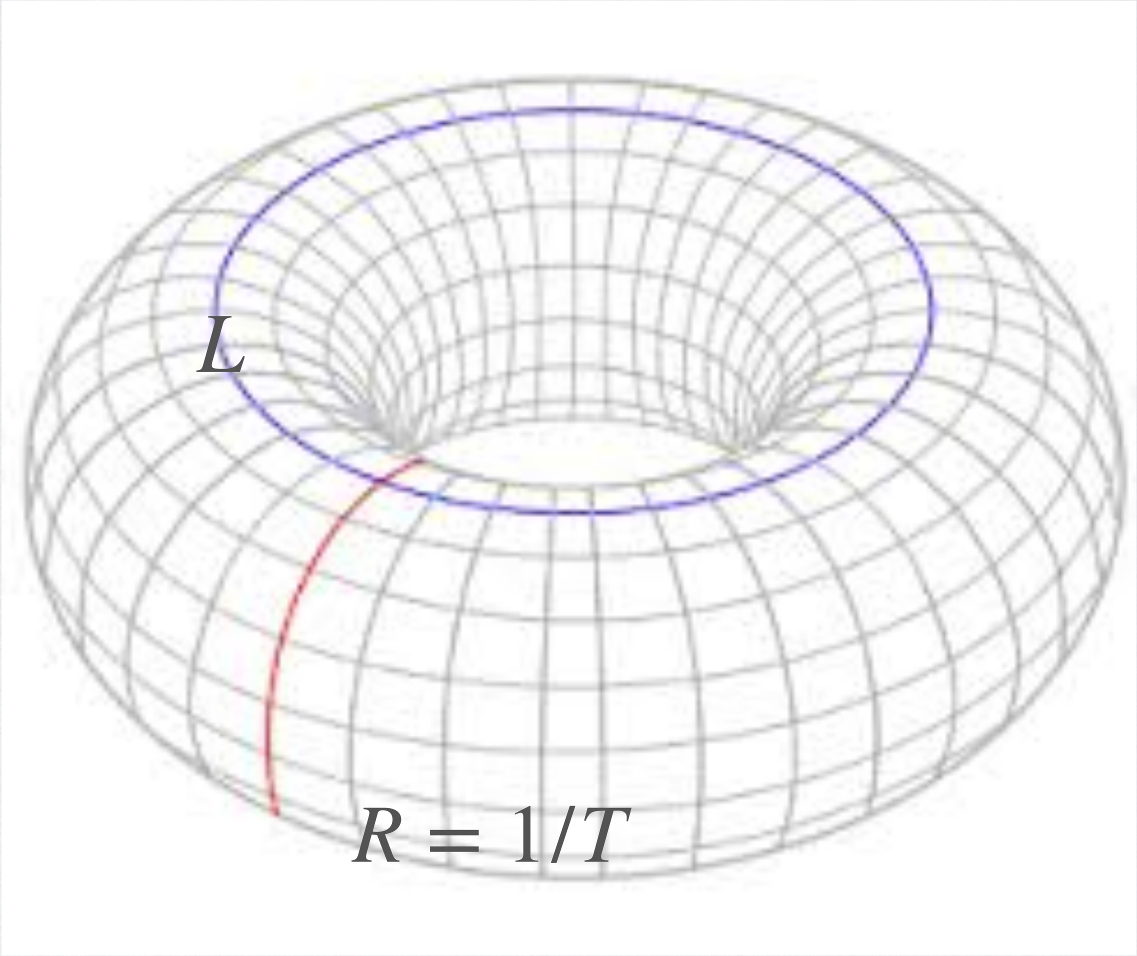
Caselle-Gliozzi-Fioravanti-Tateo 2013

$c_{eff} = c - 25\Delta$ **(primary)**

which matches the form of the (D=26, $c_{eff} = 24$) Nambu Goto spectrum, for a generic CFT.

Notice that there are spectral singularities connecting the two branches.
The most evident being the tachyonic critical point at

$$R_{cr} = \sqrt{\frac{2\pi c\tau}{3}}$$



From the point of view of a QFT at finite temperature $T = 1/R$, this critical point is consequence of an exponential growing of the degeneracy of the energy levels at large energy E

Consider the degeneracy of a free (massless) fermionic system on a circle, with $c = 1/2$ and circumference $L \to \infty$

The asymptotic behaviour of the level degeneracy for large $n_0 = \bar{n}_0 = n$ is

$$\boxed{\rho(n) = 3\left(\frac{\pi T_H}{3E}\right)^3 e^{E/T_H} = \rho(E)\frac{dE}{dn},} \qquad T_H = \sqrt{\frac{3}{\pi\tau}}, \qquad E(n) \simeq \sqrt{4\pi n/\tau}$$

$$\delta S(E) = \frac{\delta E}{T} \to T(E) = \frac{1}{\partial_E S(E)}, \quad S = \log \rho(E)$$

Indeed, T_H coincides with the upper limit temperature of the system:

$$T_H = \sup(T(E))$$

Comparing this result with the tachyonic singularity at R_{cr} we obtain:

$$R_{cr} = 1/T_H.$$

Thank you for your attention!