GGI SCHOOL: (crash) COURSE ON NUMERICAL RELATIVITY

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Overview on :

1. 3+1 GEOMETRY
2. ARNOWITT-DESER-MISNER-YORK FORMALISM (Darmois, Choquet-Bruhat, ...)
3. EVOLUTION (CAUCHY) PROBLEM IN GR

Notation :

$$c = G = 1$$

signature $- + + +$

$a, b, c, \dots = 0,1,2,3$ abstract indexes

$\alpha, \beta, \gamma, \dots = 0,1,2,3$ indexes (4D)

$i, j, k, \dots = 1,2,3$ spatial indexes (3D)

Ricci scalar, matter tensor

$$\text{EFE}: \quad {}^4G_{ab} := {}^4R_{ab} - \frac{1}{2}\,{}^4R\, g_{ab} = 8\pi T_{ab}$$

Einstein tensor, Ricci tensor, metric (4D)

$\mathcal{L}_v$: Lie derivative along v^a

Sometime use of geometrical notation for tensors, e.g.

$$\gamma \longleftrightarrow \gamma_{ab}$$

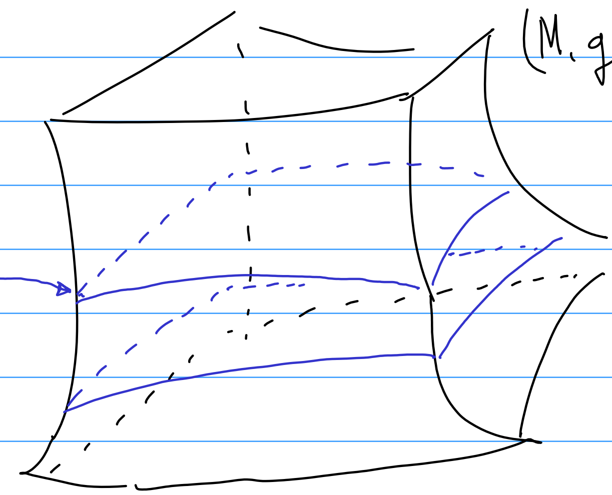
$$n \longleftrightarrow n^a$$

DISCLAIM : Simplified discussion !

① 3+1 GEOMETRY

Spacetime 4D (M, g, ∇)

$^4R_{abcd}$



(Σ_t, γ, D)

R_{ijkh}

Foliation

Scalar field $t = x^0 : M \to \mathbb{R}$: dt^a timelike

↳ spatial hypersurfaces (3D) Σ_t

$$n^a = -\alpha \; dt^a$$ Normal vector $n_a n^a = -1$

w/

$$\alpha = \left(-dt_a dt^a\right)^{-1/2}$$ LAPSE function

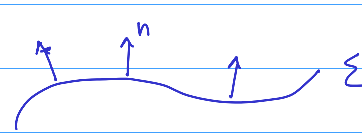
3-metric

$$\gamma = \phi^* g$$

$$\gamma_{ij} = \frac{\partial x^\alpha}{\partial x^i}\frac{\partial x^\beta}{\partial x^j} g_{\alpha\beta} = \dots = g_{ij} + n_i n_j$$

↳ cov. der. D , Riemann (3D)

"another" curvature :



$$K_{ab} := -\gamma^c_a \gamma^d_b \nabla_{(c} n_{d)}$$

Extrinsic curvature

(1) $$\boxed{K_{ab} = -\frac{1}{2\alpha} \mathcal{L}_m \gamma_{ab}}$$ $m^a = \alpha n^a$ $m^2 = -\alpha^2$

(1) "kinematical" equation

"time derivative of γ" = "velocity"

Observations:

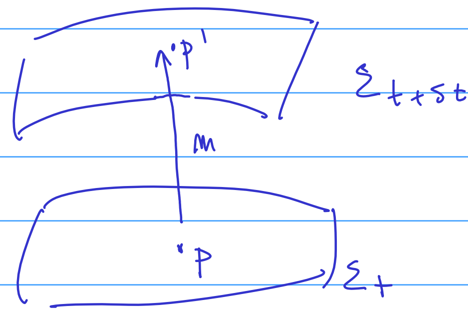
(i) m carries pt from Σ_t to $\Sigma_{t+\delta t}$

(ii) $\mathcal{L}_m$ transports tensors from Σ_t to $\Sigma_{t+\delta t}$

↳ Σ identified by the diffeomorphism generated by m

↳ (M, g) as the "time development" of (Σ, γ)

Consider (i):



$$t(p') = t(p + \delta t \cdot m) = t(p) + \delta t \cdot m^a (dt)_a =$$
$$= t(p) + \delta t$$

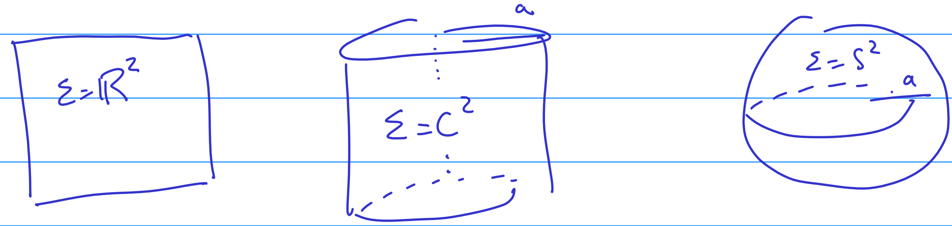
↳ Meaning of lapse:

n (m) : Eulerian obs.

$$\boxed{\delta \tau^2 = -g(\delta t\, m, \delta t\, m) = -m^2 \delta t^2 = \alpha^2 \delta t^2}$$

Example 1: Meaning of K_{ab}

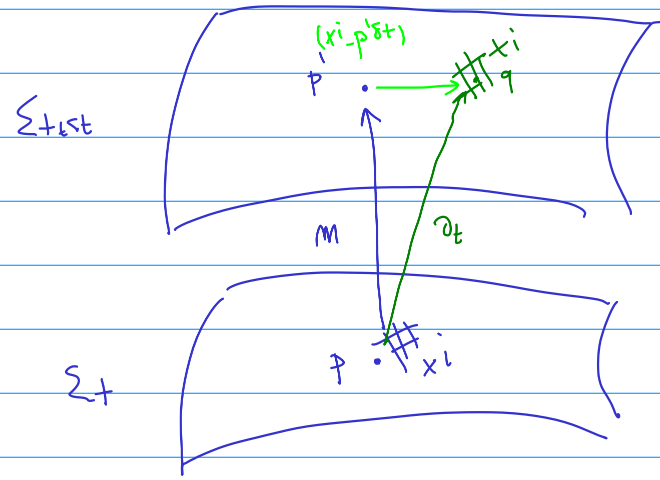
$M = \mathbb{R}^3$



$R_{ijkh} = 0$	$R_{ijkh} = 0$	$R_{ijkh} \neq 0$
$K_{ij} = 0$	$K_{\varphi\varphi} = -a$	$K_{ij} \neq 0$
	$K = -\frac{1}{a}$	$K = -\frac{2}{a}$

Coordinates on M: $x^\mu = (t, x^i) \rightsquigarrow T_p(M)$ basis (∂_t, ∂_i)

└→ Time vector



(i) ∂_t is tangent to $x^i = $ const

(ii) ∂_t drags Σ_t (analogously to m), but is a different one:

$$(\partial_t)^a = m^a + \beta^a$$

w/ β <u>SHIFT VECTOR</u> $(n^a \beta_a = 0)$

In gen. ∂_t <u>not</u> timelike

$$(\partial_t)^2 = -\alpha^2 + \beta^2$$

└→ "controls the coordinates" from Σ_t to $\Sigma_{t+\delta t}$

└→ (α, β^i) gauge freedom.

Metric:

$$\boxed{g = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)} \qquad (g)$$

<u>Example</u>: Weak-field metric

$$g = -(1+2\phi) dt^2 + (1-2\phi) f_{ij} dx^i dx^j$$

Newton potential (ϕ)   └ flat metric (f_{ij})

$\hookrightarrow \alpha = (1+2\phi)^{1/2} \sim 1+\phi$

$\beta^i = 0$

$\gamma_{ij} = (1-2\phi)\,\delta_{ij}$

Example 3: Schw. metric (isotropic coords)

$$g = -\left(1-\frac{M}{2r}\right)^2 \psi^{-2}\, dt^2 + \psi^4 \left(dr^2 + r^2 d\Omega^2\right)$$

w/ $\psi := 1 + \frac{2M}{r}$ (conf. factor)

$\hookrightarrow$ $\alpha = \left(1-\frac{M}{2r}\right)\psi$ $\quad \beta^i = 0$

$\gamma_{ij} = \psi^4 \delta_{ij}$ $\quad K_{ij} = 0$

Note: $\psi = (1-2\phi)^{1/4} \sim 1 - \frac{1}{2}\phi$

Example 4: (g) w/ $\boxed{\alpha \equiv 1 \quad \beta^i \equiv 0}$ GEODESIC GAUGE

$$g = -dt^2 + \gamma_{ij}\, dx^i dx^j$$

$\hookrightarrow$ t proper time of Eulerian obs

BH ev. in geod. gauge $\rightarrow$ "code crash" at $\tau = t = \pi$!

② ADMY

EFE in 3+1 form:

$$g_{ab} \longrightarrow \gamma_{ab},\ \alpha, \beta^a$$
$${}^4\text{Riemann} \longrightarrow \text{Riemann},\ K_{ab},\ D$$

↳ Gauss-Codazzi-Ricci identities (...)

Result (vacuum):

$$(\text{ADMY}) \begin{cases} 0 = R + K^2 - K_{ij}K^{ij} =: C_0 & (H) \\ 0 = D_j K^j{}_i - D_i K =: C_i & (M) \\ \mathcal{L}_m \gamma_{ij} = -2\alpha K_{ij} & (K) \\ \mathcal{L}_m K_{ij} = -D_i D_j \alpha + \alpha (R_{ij} + K K_{ij}) - 2 K_{ik} K^k{}_j & (D) \end{cases}$$

Observations:

(i) ADMY equivalent to EFE

(ii) No eqs for α, β^i → need to prescribe, or define eqs for α, β^i

(iii) Can formulate a Cauchy problem ~

$$\mathcal{L}_m : \text{" time derivative "}$$

Q: What kind of equations?

A: Can think of mixed hyperbolic (evolution) and elliptic (constraints) for:

$$\left(\underline{\underline{\gamma_{ij}, K_{ij}}},\ \alpha, \beta^i \right)$$

More in detail:

(H): $C_0 = 0$ • Scalar op. involving γ, K + spatial derivatives $1^{st}, 2^{nd}$

• Nonlinear eq, elliptic-like

• No time derivatives ($\mathcal{L}_m$)

• Defined on Σ_t

HAMILTONIAN CONSTRAINT

(M) $C_i = 0$ • 1 rank-1 tensor eq (3) involving K_{ij} + spatial 1st derivatives

• Nonlinear eq, elliptic-like

• Defined on Σ_t

MOMENTUM CONSTRAINT

(K) 1 rank-2 symmetric for γ_{ij} involving 1st time derivative ($\mathcal{L}_m$) K_{ij}

KINEMATICAL EQ.

(D) 1 rank-2 symmetric for K_{ij} involving 1st time derivatives + 2nd spatial derivatives of γ

DINAMICAL EQ.

Note: Conformal decomposition of $\gamma = \psi^4 \tilde{\gamma}$

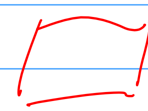
weak field limit of (D) $\Rightarrow$ $\Delta \phi = 4\pi\rho$

w/ $\psi \sim 1 - \frac{1}{2}\phi$

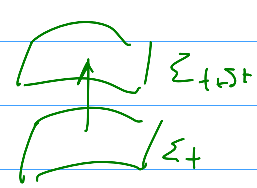
Structure:

$$\left\{\begin{array}{l} \boxed{\begin{aligned} 0 &= C_0 \\ 0 &= C_i \end{aligned}} \\ \boxed{\begin{aligned} \mathcal{L}_m \gamma &= K \\ \mathcal{L}_m K &= \dots \end{aligned}} \end{array}\right.$$

CONSTRAINTS defined on Σ_t



EVOLUTION EQ flow



Example 5: Maxwell eqs for A_α (vector potential)

$$\begin{cases} \boxed{0 = \partial_\alpha E^\alpha = \partial^\alpha(\partial_\alpha A_0 - \partial_t A_\alpha) =: C} & (\partial_t C \equiv 0) \\ \boxed{\Box A_\alpha = 0} & (\text{Lorentz gauge } \partial_\alpha A^\alpha = 0) \end{cases}$$

Note: C is transported along the dynamics

$$C = 0 \quad t = 0 \Rightarrow \quad C = 0 \quad \forall t > 0$$

↳ "free-evolution" schemes (NR).

Q: WHAT HAPPENS IN GR?

A: THE SAME ...

Proof: Consider "extended EFE", Z4 system:

$${}^4G_{ab} + 2\nabla_{(a} Z_{b)} - g_{ab} Z_c Z^c = 8\pi T_{ab}$$

w/ Z^a new vector field.

$Z^a \equiv 0$ (algebraic constraint) $\Rightarrow$ GR

3+1 decomposition (scheme):

$$(Z4) \quad \begin{cases} \mathcal{L}_m \gamma = K \\ \mathcal{L}_m K = [\text{as}(D)] + DZ \\ \mathcal{L}_m (Z_a n^a) = \frac{1}{2} C_0 + \dots \\ \mathcal{L}_m Z_i = C_i + \dots \end{cases}$$

$$\nabla^a \, {}^4G_{ab} = 0 \Rightarrow \boxed{\Box_g Z_a + R_{ab} Z^b = 0}$$

↳ if $t = 0$ $C_0 = C_i = Z^a = \dot{Z}^a = 0$, then $Z4 = GR$ $\forall t > 0$

Summary:

Initial data problem

$$C_\alpha = 0$$

for $\Sigma_{t=0}$

Free evolution

$$\begin{cases} (K) \\ (\gamma) \end{cases}$$

for $\Sigma_{t>0}$

Cauchy IVP

Hamilton formulation of GR

$$S_{GR} = \int d^4x \sqrt{-g}\, {}^4R = \int dt \underbrace{\int d^3x\, \alpha \sqrt{\gamma}\, C_0}_{\text{Lagrangian density, } L} + \text{boundary terms}$$

π^{ij} conj. momenta

$$H = \pi^{ij} \dot{\gamma}_{ij} - L = \int_\Sigma d^3x \sqrt{\gamma} \left(\alpha C_0 + 2\beta^i C_i \right)$$

- $\pi^{ij} \longrightarrow K_{ij}$

  ADM $\qquad\qquad$ York

- α, β^i Lag. multipliers.

③ EVOLUTION PROBLEM

Well-posedness

DEF 1: Well-posed PDE problem iff exists a unique solution that depends continuously on the boundary data

For hyperbolic (wave-like):

DEF 2: A 2^{nd} order PDE system is strongly hyperbolic iff the principal part of the system is in the form:

$$\Box u \simeq 0$$

TH: Strongly hyperbolic $\Rightarrow$ well-posedness of IVP

Q: Is ADMY (K)+(D)+ gauge strongly hyp.?
A: Yes, but:

(i) Prescribe β^i
(ii) α belongs to a certain class (Bona-Masso family) ✓
(iii) $C_i \equiv 0$ is identically satisfied $\forall t$

(i) and (iii) too restrictive for numerical purposes.

↳ Need to find other formulations!

Harmonic gauge (1^{st} proof of well-posedness)

EFE in vacuum:

$$0 = R_{\alpha\beta} = \underbrace{-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} + \nabla_{(\alpha}\Gamma_{\beta)}}_{\text{Highest derivatives; P.P.}} + Q_{\alpha\beta}[g,\partial g]$$

w/ $\Gamma_\alpha := g^{\beta\gamma}\Gamma_{\alpha\beta\gamma} = -\Box x_\alpha$

Choose : $\boxed{\Gamma_\alpha \equiv 0}$ Harmonic gauge (HG)

$$\boxed{0 \doteq \Box_g g_{\alpha\beta}}$$ well-posed !

Observations : (i) Generalized Harmonic gauge $\rightarrow$ BH evolutions !

(ii) Z4 : $Z_\alpha = \Gamma_\alpha$ = GR in HG

Towards BSSNOK/Z4c formulations

Q : In HG gauge is fixed...

Do well-posed formulations with "gauge flexibility" exist ?

A : Yes.

Consider ADMY, combine (K) w/ (D) to obtain a 2^{nd} order in time eq for γ_{ij} :

$$\partial_{tt}\gamma_{ij} \doteq -2\alpha R_{ij} =$$

$$\sim \underbrace{\Delta\gamma_{ij}}_{\text{"good term"}} + \underbrace{\gamma_{ik}\partial_j\partial_h\gamma^{kl} + \gamma_{jk}\partial_i\partial_h\gamma^{kl}}_{\text{"bad term"}} + \dots$$

Introduce a new field : $\boxed{f^k := \partial_h \gamma^{kh}}$

and find an evolution eq. for f^k:

(f) $\quad \partial_t f^k = \partial_t \partial_h \gamma^{kh} = \partial_h \underbrace{\partial_t \gamma^{kh}} \sim \partial_h K^{kh} \sim C^k$

mom. constraint!

↳ this proves a successful strategy ...

<u>Note</u>: (f) $\longleftrightarrow$ $\mathcal{L}_m Z_i = C_i$ in the Z4 system.

BSSNOK | Z4c $\longrightarrow$ obtained with a further <u>conformal decomposition</u> of (γ, K).

<u>SUMMARY</u>

- Split spacetime into space + time
- Formulate GR in 3+1
- Hyperbolicity / well posedness of IVP