Gravitational Radiation, BMS, Soft Theorems, Memory and All That (Part II)

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Gravitational Scattering, Inspiral and Radiation Training Week

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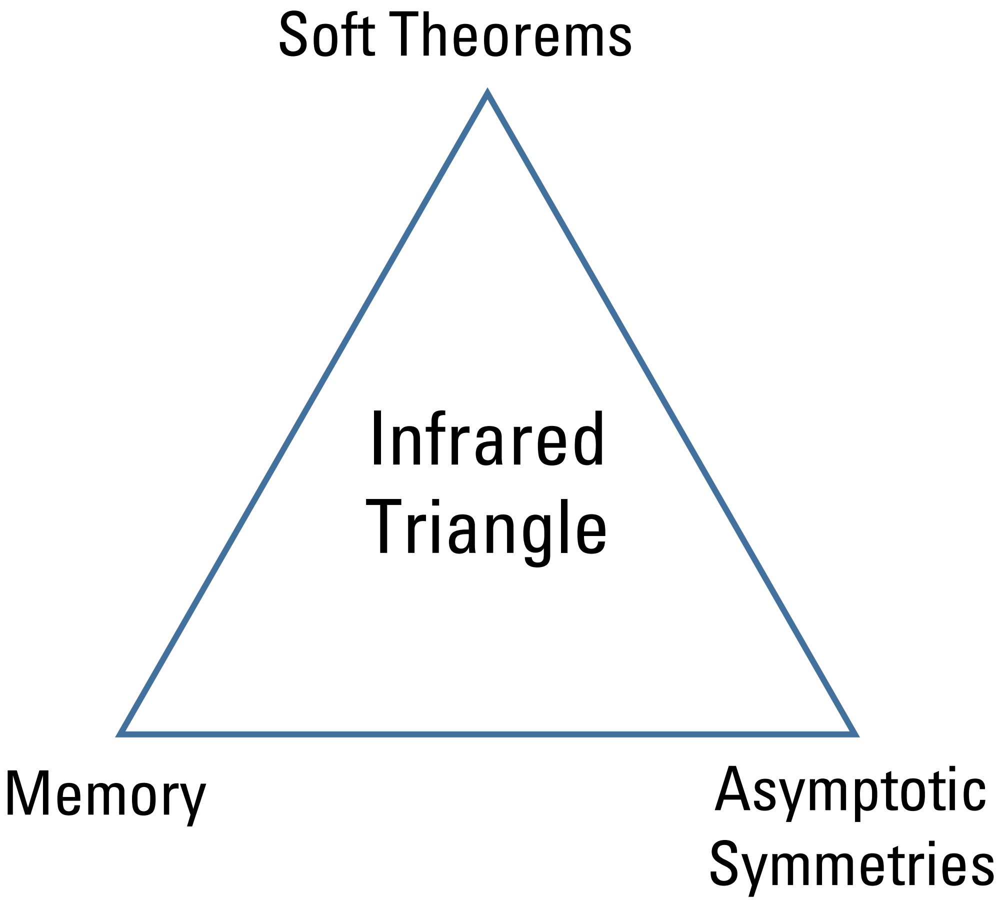
Outline

1. **Soft Theorems (for Scattering Amplitudes)**
   - *Universal behavior of low-frequency radiation*
2. **Ward Identities**
   - *Universal behavior due to symmetry*
3. **Asymptotic Symmetries**
   - *Identification of symmetries implied by soft theorems*
4. **IR Divergences & Faddeev-Kulish States**
   - *Role of vacuum transitions in scattering amplitudes*



QFT approach to Gravitational Radiation

- QFT approach: **Scattering Amplitudes** involving gravitons

$$\mathcal{A}(p_1, p_2, \cdots) \equiv \langle \text{out}|\mathcal{S}|\text{in}\rangle$$

$$\text{initial/final state}: \quad |\text{in/out}\rangle = \underbrace{|p_1, p_2, \cdots\rangle}_{\text{particles}} = a^\dagger(p_1)a^\dagger(p_2)\cdots|0\rangle$$

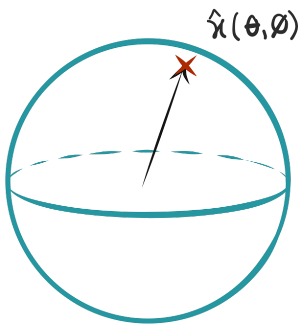
- Behavior of radiation at low-frequency?
  →Amplitudes with emission of **low-energy graviton**
- Universality of infrared behavior = Soft Theorems

$$\mathcal{A}(p_1, \cdots p_n; q) \xrightarrow{\omega \to 0} \left[\frac{1}{\omega} S_{(0)} + \log\omega \; S^{\log}_{(1)} + S_{(1)} + \omega S_{(2)}\right] \mathcal{A}(p_1, \cdots p_n)$$

- **Soft Theorems**: universal relation among scattering amplitudes

Weinberg's Soft Graviton Theorem

➢ *Scattering amplitude for the* emission *of a* soft graviton *has a* pole *in the* energy *of that graviton with a* universal residue:



$$\lim_{\omega \to 0} \omega \langle \text{out} | a_+(\omega \hat{x}) \mathcal{S} | \text{in} \rangle = \sum_{k=1}^{n} S_k(\hat{x}) \langle \text{out} | \mathcal{S} | \text{in} \rangle$$

Graviton Momentum:

$$q^\mu = \omega \hat{q}^\mu = \omega \big(1, \hat{x}(\theta, \phi)\big)$$

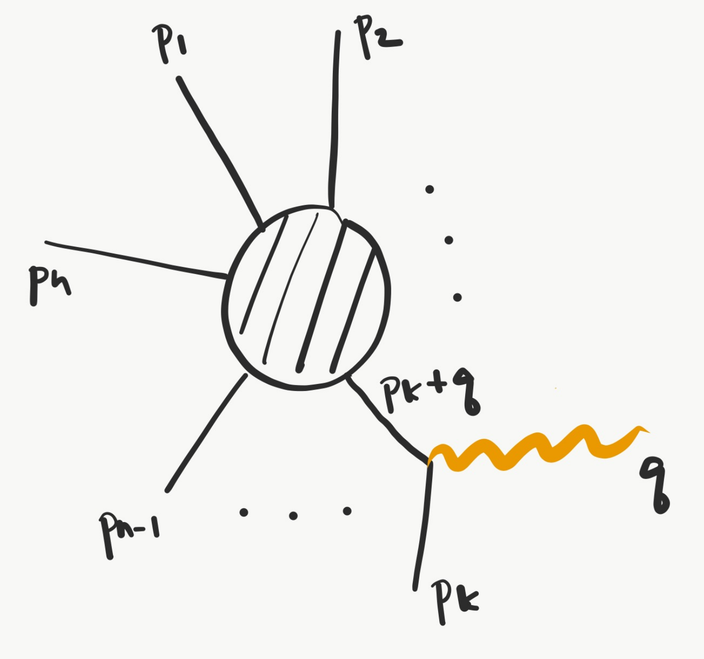
Soft Factor:

$$S_k(\hat{x}) = \frac{\kappa}{2} \frac{\varepsilon^+_{\mu\nu} p_k^\mu p_k^\nu}{\hat{q} \cdot p_k}$$

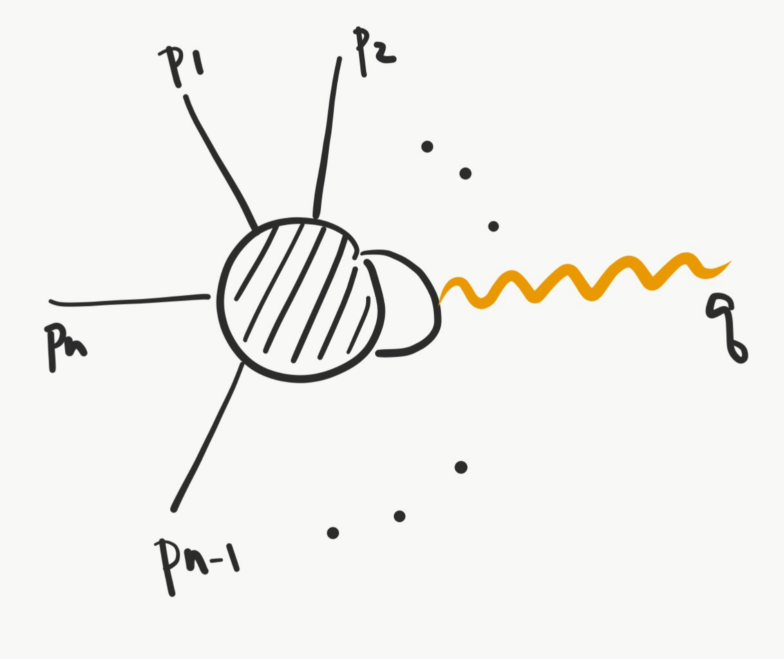
[Weinberg 1965]

Feynman Diagrammatic Derivation

- Consider S-matrix for emission of outgoing graviton of momentum q in the limit $q \to 0$.
- There are two classes of diagrams which contribute:



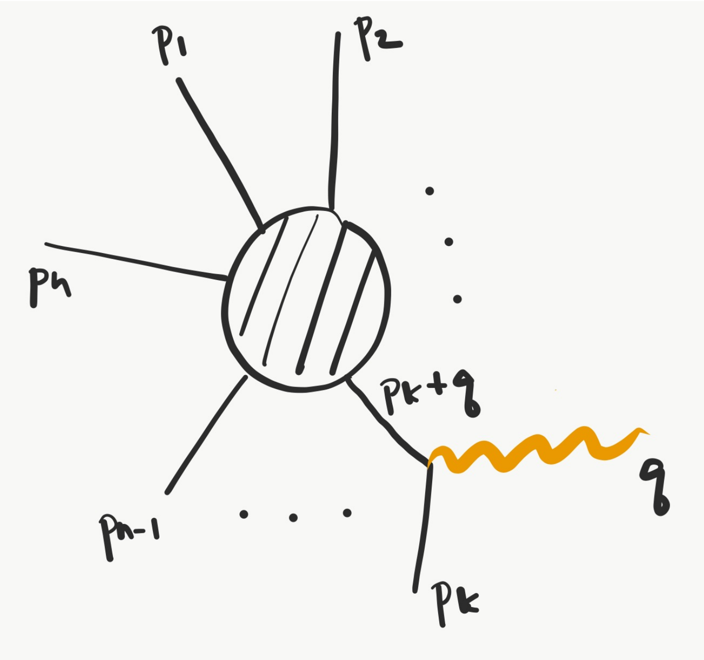
Graviton emitted from external line



Graviton emitted from internal line

Graviton emission from external line

➢ Focus on the first diagram:



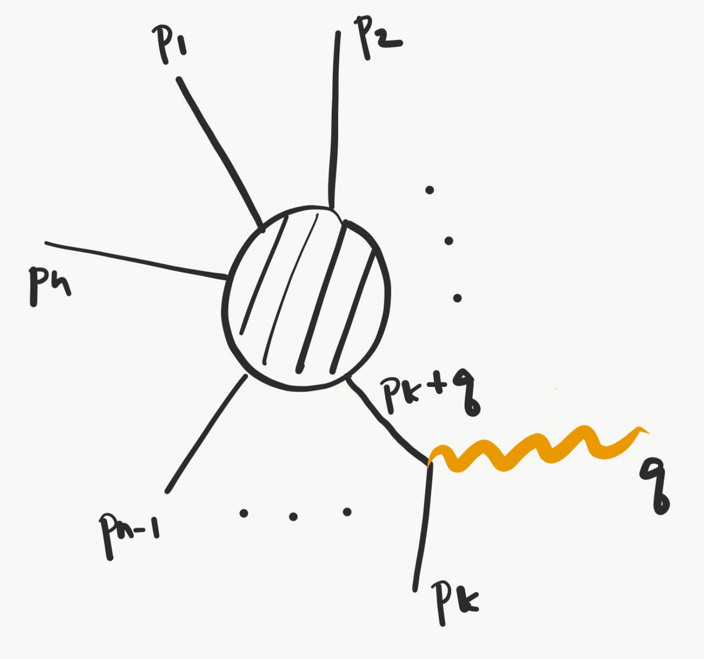
$$= \text{(vertex)}\ \text{(propagator)}\ \mathcal{A}(p_1, \cdots, p_k + q, \cdots, p_n)$$

for particle of type k

matrix element without graviton & with particle k's momentum shifted

Graviton emission from external line

➢ Consider for concreteness, emission from a scalar particle of mass m_k:



$$= \text{(vertex)}\ \text{(propagator)}\ \mathcal{A}(p_1, \cdots, p_k + q, \cdots, p_n)$$

$$= (i\kappa\varepsilon_{\mu\nu}p_k^\mu p_k^\nu)\ \frac{-i}{(p_k+q)^2 + m_k^2 - i\epsilon}$$

$$\times\ \mathcal{A}(p_1, \cdots, p_k + q, \cdots, p_n)$$

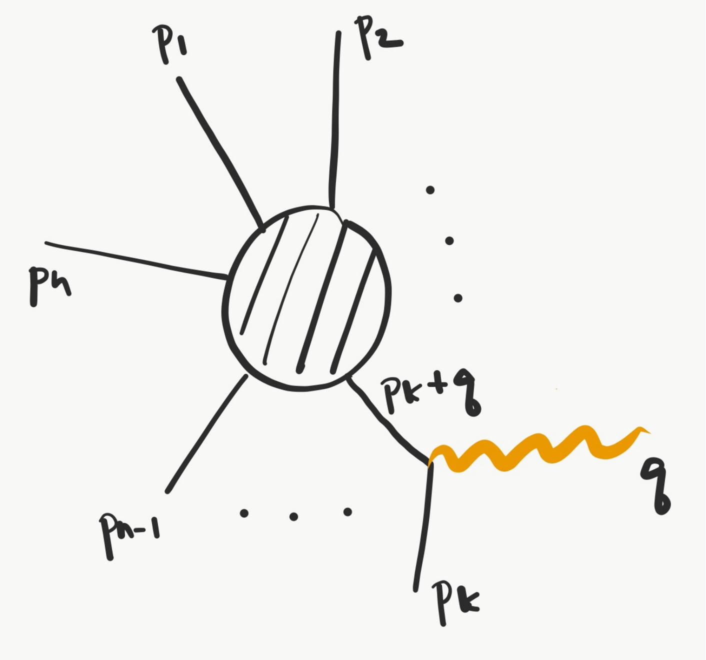
$$\left(\begin{array}{l} \mathcal{L}_{\text{int}} = \frac{\kappa}{2} h^{\mu\nu} T_{\mu\nu} \\ T_{\mu\nu} \sim \partial_\mu \varphi \partial_\nu \varphi \end{array} \right)$$

$$\approx \frac{\kappa}{2} \frac{\varepsilon_{\mu\nu}p_k^\mu p_k^\nu}{p_k \cdot q} \mathcal{A}(p_1, \cdots, p_n)$$

Graviton emission from external line

➢ The net contribution from graviton emission from external line is just given by the sum:

$$\sum_{k=1}^{n}$$

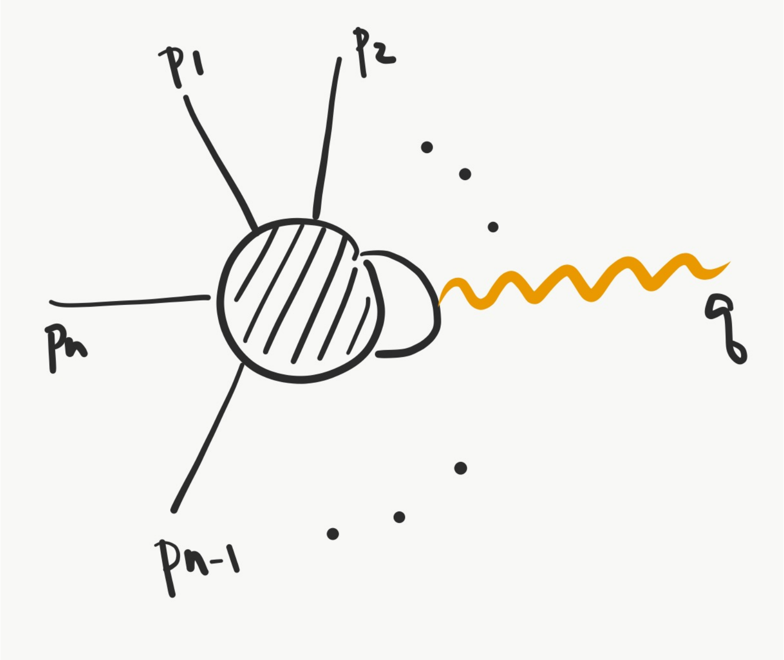


$$\approx \sum_{k=1}^{n} \frac{\kappa}{2} \frac{\varepsilon_{\mu\nu} p_k^\mu p_k^\nu}{p_k \cdot q} \mathcal{A}(p_1, \cdots, p_n)$$

$\Rightarrow$ Leading behavior in $q \to 0$ limit involves a simple pole in q

Graviton emission from internal line

➢ Next, consider second class of diagrams:



Emission from internal line carrying momentum ℓ

$$\sim \frac{-i}{\ell^2 + m^2 - i\epsilon} \, i\kappa\varepsilon_{\mu\nu}\ell^\mu \ell^\nu \underbrace{\frac{-i}{(\ell+q)^2 + m^2 - i\epsilon}}_{\approx \frac{-i}{\ell^2 + m^2 - i\epsilon}}$$

Internal momentum ℓ is off-shell ($\ell^2 \neq -m^2$) so no pole in $q \to 0$ limit.

Weinberg's Soft Graviton Theorem

$$\lim_{\omega\to 0}\omega\langle\text{out}|a_+(\omega\hat{x})\mathcal{S}|\text{in}\rangle = \sum_{k=1}^{n} S_k(\hat{x})\langle\text{out}|\mathcal{S}|\text{in}\rangle$$

$$S_k(\hat{x}) = \frac{\kappa}{2}\frac{\varepsilon^+_{\mu\nu}p^\mu_k p^\nu_k}{\hat{q}\cdot p_k}$$

Soft Factor

- **Infrared:** *Low-energy graviton* probes long-distance properties of scattering process.
- **Universality:** Soft factor is *independent* of precise theory.
  - Only depends on momenta of external particles
  - Same contribution for each external particle
- **Symmetry:** Implies *exact* relationship between scattering amplitudes.

$$\langle\text{out}|\text{U}^{-1}\mathcal{S}\text{U}|\text{in}\rangle = \langle\text{out}|\mathcal{S}|\text{in}\rangle \quad \Longrightarrow \quad \langle\text{out}|[\text{Q},\mathcal{S}]|\text{in}\rangle = 0$$

$$U = e^{iQ}$$

Soft Theorems Imply Symmetries

➢ *Can always interpret* soft theorems *as* statements of invariance *of the $\mathcal{S}$-matrix under an* infinite-dimensional symmetry

Weinberg's Soft Graviton Theorem:

$$\lim_{\omega\to 0}\omega\langle \mathrm{out}|a_+(\omega\hat{x})\mathcal{S}|\mathrm{in}\rangle = \sum_{k=1}^{n} S_k(\hat{x})\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle$$

$$S_k(\hat{x}) = \frac{\kappa}{2}\frac{\varepsilon^+_{\mu\nu}p^\mu_k p^\nu_k}{\hat{q}\cdot p_k}$$

➢ Regard soft factor S_k as eigenvalue of single particle state under operator Q_H

$$S_k(\hat{x})|p_k\rangle = Q_H(\theta,\phi)|p_k\rangle = -i\delta_{(\theta,\phi)}|p_k\rangle$$

➢ RHS gives transformation of single particle states under $\delta_{(\theta,\phi)}$

$$\sum_{k=1}^{n} S_k\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle = \langle \mathrm{out}|[Q_H,\mathcal{S}]|\mathrm{in}\rangle$$

➢ Soft theorem implies that the $\mathcal{S}$-matrix is invariant under the transformation $\delta_{(\theta,\phi)}$ of single particle states provided that *a soft particle is added.*

[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

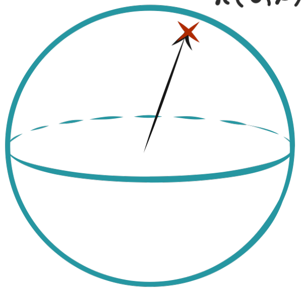
Soft Theorems Imply Symmetries

Weinberg's Soft Graviton Theorem:

$$\lim_{\omega \to 0} \langle \text{out}|\omega a_+(\omega \hat{x})\mathcal{S}|\text{in}\rangle = \sum_{k=1}^{n} S_k \langle \text{out}|\mathcal{S}|\text{in}\rangle = \langle \text{out}|[Q_H, \mathcal{S}]|\text{in}\rangle$$

- Denote operator which **adds soft particles** Q_S

$$Q_S(\theta, \phi) \sim -\lim_{\omega \to 0} \omega \left[a_+(\omega \hat{x}) + a_-^\dagger(\omega \hat{x})\right]$$



- Then, the LHS can be written as

$$\lim_{\omega \to 0} \langle \text{out}|\omega a_+(\omega \hat{x})\mathcal{S}|\text{in}\rangle = -\langle \text{out}|[Q_S, \mathcal{S}]|\text{in}\rangle$$

- Rearranging the soft theorem

$$\langle \text{out}|[Q, \mathcal{S}]|\text{in}\rangle = 0, \qquad Q = Q_H + Q_S$$

$\Rightarrow$ Obtain statement of invariance under symmetry generated by Q

- *Can always interpret* soft theorems *as statements of* invariance *of the S-matrix under an* infinite-dimensional symmetry.

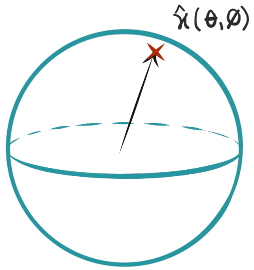
[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

Supertranslations & Weinberg’s Soft Theorem

- ➢ To identify symmetry transformation, parametrize by **functions** $f(\theta, \phi)$ rather than **points** (θ, ϕ).
- Identify soft factor for massless p_k as Green’s function on S^2

$$S_k(\hat{x}) = \frac{\kappa}{2}\frac{\varepsilon^{+}_{\mu\nu}p_k^\mu p_k^\nu}{\hat{q}\cdot p_k}$$

$$\hat{x}(z,\bar{z}) = \left(\frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right)$$

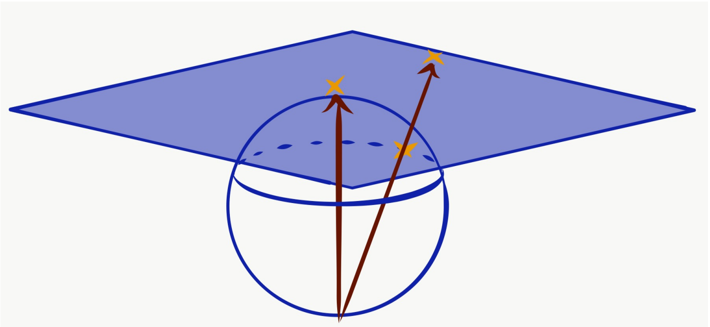


$$\hat{q}^\mu = (1, \hat{x})$$

$$p_k^\mu = \omega_k(1, \hat{x}_k)$$

$$S_k(z,\bar{z}) \sim \frac{\kappa}{2}\omega_k\frac{\bar{z}-\bar{z}_k}{z-z_k}$$

$$z = e^{i\phi}\tan\left(\tfrac{1}{2}\theta\right)$$



$$\mathcal{D}_{(\hat{x})}S_k(\hat{x}) \sim \omega_k\delta^{(2)}(\hat{x},\hat{x}_k)$$

$$\mathcal{D}_{(\hat{x})} \sim \frac{2}{\kappa}\partial_{\bar{z}}^2$$

[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

Supertranslations & Weinberg's Soft Theorem

- Construct charges with ***local*** action (*i.e.* action only depends on f at point (θ_k, ϕ_k))

$$Q[f] = \int d^2\hat{x} \; f(\hat{x}) \mathcal{D}_{(\hat{x})} Q(\hat{x})$$

- Action on massless particles:

$$-i\delta_f |p_k\rangle = \underbrace{Q_H[f]}_{\int d^2\hat{x} \; f(\hat{x}) \mathcal{D}_{(\hat{x})} Q_H(\hat{x})} |p_k\rangle = \omega_k f(\theta_k, \phi_k) |p_k\rangle$$

$$\mathcal{D}_{(\hat{x})} S_k(\hat{x}) \sim \omega_k \delta^{(2)}(\hat{x}, \hat{x}_k)$$

$$Q_H(\hat{x}) |p_k\rangle = S_k(\hat{x}) |p_k\rangle \qquad S_k(\hat{x}) = \frac{\kappa}{2} \frac{\varepsilon^{+}_{\mu\nu} p_k^{\mu} p_k^{\nu}}{\hat{q} \cdot p_k}$$

$\Rightarrow$ Charges act *locally* on massless particles.

[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

Supertranslations & Weinberg's Soft Theorem

- Having identified transformation δ_f as the symmetry implied by the soft theorem, can now determine physical interpretation

$$Q[f] = \int d^2\hat{x}\ f(\hat{x})\mathcal{D}_{(\hat{x})}Q(\hat{x})$$

$$-i\delta_f|p_k\rangle = Q_H[f]|p_k\rangle = \omega_k f(\theta_k, \phi_k)|p_k\rangle$$

- When $f = 1$, find total energy

$$-i\delta_{f=1}|p_k\rangle = \omega_k|p_k\rangle$$

$\Rightarrow Q[f = 1]$ generates *ordinary* time translations

[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

Supertranslations & Weinberg's Soft Theorem

- Generic $f = f(\theta, \phi)$, find energy *weighted* by f

$$-i\delta_f|p_k\rangle = \omega_k f(\hat{x}_k)|p_k\rangle$$

$\Rightarrow Q[f]$ generates translation *weighted* at each angle by f

- f is arbitrary
  - $\Rightarrow$ *Independent* translation symmetry at every angle ("Supertranslations")
  - $\Rightarrow Q_H$ characterizes *local* energy flux at every angle

[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

Locality on the Celestial Sphere

- To understand why transformations that act **independently at every angle** in **momentum space** also act **locally at every angle** in **position space**, consider the **saddle point approximation**:

$$e^{ix\cdot p_k} = e^{-i\omega u - i\omega r(1-\hat{x}\cdot\hat{x}_k)} \overset{r\to\infty}{\sim} \frac{1}{i\omega r} e^{-i\omega u} \gamma^{z\bar{z}} \delta^{(2)}(z - z_k)$$

$\Rightarrow$ Plane waves of massless particles localize to the point on the sphere in the direction of propagation.

$$u = t - r, \qquad \vec{x} = r\hat{x}(z,\bar{z}),$$

$$\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$$

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}x^\mu x^\nu \\ &= -du^2 + 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \end{aligned}$$

[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

Locality on the Celestial Sphere

- Expansion of massless scalar field in plane wave modes

$$\Phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2p^0} \left[e^{ip\cdot x} a(\vec{p}) + e^{-ip\cdot x} a^\dagger(\vec{p}) \right]$$

- Saddle point approximation ⇒ localization in angles at large r

$$\Phi(x) = \frac{1}{r}\phi(u,z,\bar{z}) + \cdots\,, \qquad \phi(u,z,\bar{z}) \sim \int d\omega\ e^{i\omega u} a^\dagger(\omega,z,\bar{z}) + c.c.$$

- Transformation under **angle-dependent time translation**

$$\delta_f \phi \sim f\partial_u \phi \sim \int d\omega\ e^{i\omega u} \underbrace{f\omega a^\dagger(\omega,z,\bar{z})}_{\text{Transformation of momentum state}} + c.c.$$

Transformation of momentum state → $-i\delta_f |p_k\rangle = \omega_k f(\hat{x}_k)|p_k\rangle$

[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

BMS Symmetries

Method: Asymptotic symmetry analysis

$$\begin{array}{c}\text{asymptotic}\\ \text{symmetries}\end{array} = \frac{\text{allowed diffeomorphisms}}{\text{trivial diffeomorphisms}}$$

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &\quad + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + D^zC_{zz}dudz + D^{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z} + \cdots \end{aligned}$$

Minkowski:

$$u = t - r, \qquad \vec{x} = r\hat{x}(z,\bar{z}), \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$$

[Bondi, van der Burg, Metzner (1962); Sachs (1962)]

BMS Symmetries

Result: Supertranslations + Lorentz transformations

$$\xi = f(z, \bar{z})\partial_u + \cdots$$

- Independent translation symmetry at every angle on the celestial sphere
- These are *precisely* the symmetries implied by Weinberg's soft graviton theorem.

[Bondi, van der Burg, Metzner (1962); Sachs (1962)]

Construction of Hard Charge

- With result from saddle point approximation can also give explicit operator expression for hard charge

$$Q_H[f] = \int du \int d^2\hat{x}\; f(\hat{x})\; T_{uu}(u,\hat{x}) \qquad\qquad T_{uu} \sim \partial_u\phi\partial_u\phi$$

- Use canonical commutation relations to verify its action:

$$[\partial_u\phi(u,\hat{x}), \phi(u',\hat{x}')] \sim \delta(u-u')\delta^{(2)}(\hat{x}-\hat{x}')$$

Position space: $$[Q_H[f], \phi(u,\hat{x})] \sim f(\hat{x})\partial_u\phi(u,\hat{x}) \sim \delta_f\phi(u,\hat{x})$$

$$\phi(u,z,\bar{z}) \sim \int d\omega\; e^{i\omega u} a^\dagger(\omega,z,\bar{z}) + c.c.$$

Momentum space: $$Q_H[f]|p_k\rangle = \omega f(\hat{x}_k)|p_k\rangle = -i\delta_f|p_k\rangle$$

$\Rightarrow Q_H$ characterizes *local* energy flux at every angle

[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

Massive Particles

- Previous analysis for construction of charge $Q[f]$ assumed scattering of massless particles.
- Used property of soft factor:

$$\mathcal{D}_{(\hat{x})} S_k(\hat{x}) \sim \omega_k \delta^{(2)}(\hat{x}, \hat{x}_k) \qquad S_k(\hat{x}) = \frac{\kappa}{2} \frac{\varepsilon^{+}_{\mu\nu} p^{\mu}_k p^{\nu}_k}{\hat{q} \cdot p_k} \qquad \begin{aligned} p^{\mu}_k &= \omega_k(1, \hat{x}_k) \\ \hat{q}^{\mu} &= (1, \hat{x}) \end{aligned}$$

- Found $Q_H[f]$ characterized local energy flux through null infinity.

- Massive particles in momentum eigenstates do not reach null infinity
- There is no canonical way to associate massive momenta to points on the celestial sphere

Massive Particles

- On the other hand, argument that soft theorem implies infinite-dimensional symmetries applied *regardless* of whether massive or massless.

Charges parametrized by points (θ, ϕ):

$$S_k(\hat{x})|p_k\rangle = Q_H(\theta,\phi)|p_k\rangle = -i\delta_{(\theta,\phi)}|p_k\rangle$$

$$S_k(\hat{x}) = \frac{\kappa}{2}\frac{\varepsilon^{+}_{\mu\nu}p_k^{\mu}p_k^{\nu}}{\hat{q}\cdot p_k}$$

$$\hat{q}^{\mu} = (1, \hat{x})$$

⇒ This is the transformation of massive particles under the symmetry generated by $Q(\hat{x})$

- What is the transformation of massive particles under the symmetry generated by $Q[f]$?

Massive Particles

- Recall relation between $Q(\hat{x})$ and $Q[f]$:

Localizes soft factor for massless p_k^μ

$$Q[f] = \int d^2\hat{x}\ f(\hat{x})\mathcal{D}_{(\hat{x})}Q(\hat{x})$$

⇒ Implies the following transformation under the symmetry generated by $Q[f]$

$$\begin{aligned}
-i\delta_f|p_k\rangle &\equiv Q_H[f]|p_k\rangle \\
&= \int d^2\hat{x}\ f(\hat{x})\mathcal{D}_{(\hat{x})}Q_H(\hat{x})|p_k\rangle \\
&= \left(\int d^2\hat{x}\ f(\hat{x})\mathcal{D}_{(\hat{x})}\frac{\kappa}{2}\frac{\varepsilon^{+}_{\mu\nu}p_k^\mu p_k^\nu}{p_k\cdot\hat{q}}\right)|p_k\rangle
\end{aligned}$$

[Campiglia & Laddha, hep-th/1509.01406]

Massive Particles

- To further elucidate the action on massive particles, identify momenta with points on 3D hyperboloid

$$p_k^\mu = m_k \hat{p}_k^\mu, \qquad \hat{p}_k^2 = -1, \qquad \hat{p}_k^\mu = \frac{1}{2\rho_k}\big(n^\mu + \rho_k^2 \hat{q}(z_k, \bar{z}_k)\big)$$

$$n^\mu = (1, 0, 0, -1), \qquad \hat{q}^\mu(z, \bar{z}) = (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z})$$

- Similar saddle point analysis shows massive particles in momentum eigenstates localize to this point on unit hyperboloid in *spacetime.*

$$x^\mu = \frac{\tau}{2\rho}\big(n^\mu + \rho^2 \hat{q}^\mu(z, \bar{z})\big), \qquad x^2 = -\tau^2$$

(Covers points inside the lightcone of the origin)

[Campiglia & Laddha, hep-th/1509.01406]

Massive Particles

➢ Then, symmetry transformation takes the form:

$$-i\delta_f|p_k\rangle = \left(\int d^2\hat{x}\ f(\hat{x})\mathcal{D}_{(\hat{x})}\frac{\kappa}{2}\frac{\varepsilon^{+}_{\mu\nu}p^{\mu}_{k}p^{\nu}_{k}}{p_k\cdot\hat{q}}\right)|p_k\rangle$$

Bulk-to-boundary propagator on AdS_3

$$\sim m_k\mathcal{G}_{(3)}(\hat{p}_k;\hat{x}) \sim \frac{m_k}{(\hat{p}_k\cdot\hat{q})^3}$$

$$\Rightarrow \boxed{-i\delta_f|p_k\rangle = m_k\widetilde{f}(\hat{p}_k)|p_k\rangle}$$

$$\widetilde{f}(\hat{p}_k) \equiv \int d^2\hat{x}\ f(\hat{x})\mathcal{G}_{(3)}(\hat{p}_k;\hat{x})$$

Extension of f from null infinity to timelike infinity under de Donder gauge condition

[Campiglia & Laddha, hep-th/1509.01406]

Recap

- Introduced soft theorems as quantum theoretic characterization of the universality of gravitational radiation in the infrared.
- By showing soft theorems are equivalent to Ward identities of infinite-dimensional symmetries, demonstrated that this universal behavior in the infrared was equivalent to universal behavior due to symmetry.
- Studied the action of the hard charge on single particle states to identify the physical interpretation of the symmetry.

- Recall needed to introduce soft charge to interpret the soft theorem as the Ward identity.
  - → What is the role of the soft charge?
  - → What transformation does it implement?
  - → What transforms non-trivially under its action?

Soft Modes & Infinite-Dimensional Symmetries

- Form of hard charge $Q_H[f]$ fixes the form of the soft charge:

$$Q_S[f] \sim \int d^2\hat{x}\ f(\hat{x})\mathcal{D}_{(\hat{x})} \lim_{\omega\to 0} \omega \left[a(\omega\hat{x}) + a^\dagger(\omega\hat{x})\right]$$

- For some choices $\hat{f}$ of f,

$$\mathcal{D}_{(\hat{x})}\hat{f} = 0 \qquad \Rightarrow \qquad Q_S[\hat{f}] = 0$$

- In the case of supertranslations, soft charge vanishes for four ordinary translations.

$$\mathcal{D}_{(\hat{x})} \sim D^z D^z \qquad \Rightarrow \qquad \hat{f} = Y_{\ell m}, \qquad \ell = 0, 1.$$

- *Finitely many* symmetry transformations preserve particle number.
- *Infinitely many* require addition of soft particles.
  - Addition of soft particles was crucial to obtain infinite-dimensional symmetry
  - If restrict to charges with no soft contributions, only find four translational symmetries.

[He, Lysov, Mitra, Strominger, hep-th/1401.7026; Strominger, hep-th/1703.05448]

Construction of Soft Charge

➢ As for hard charge, can identify field operator expression for soft charge

$$Q_S(\theta,\phi) \sim \lim_{\omega\to 0} \omega \left[a(\omega\hat{x}) + a^\dagger(\omega\hat{x})\right] \sim \lim_{\omega\to 0} \int dt\ \varepsilon^{\mu\nu} e^{i\omega t} \partial_t h^{\text{rad}}_{\mu\nu}$$

- The soft charge given is the **zero-frequency component** of the **radiative gravitational field**.

$$Q_S(\theta,\phi) \sim \varepsilon^{\mu\nu} \underbrace{\Delta h_{\mu\nu}}_{h_{\mu\nu}(t_f) - h_{\mu\nu}(t_i)}$$

Follows from

$$\int d\omega\ \frac{e^{i\omega t}}{\omega} \sim \Theta(t)$$

- Equivalently, the soft charge can be written as a **permanent net shift** in the **asymptotic metric**.

Gravitational Memory from Soft Theorems

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Gravitational-wave bursts with memory and experimental prospects

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permanent change in the gravitational-wave field (the burst's memory) δh_{ij}^{TT} is equal to the 'transverse, traceless (TT) part'[36] of the time-independent, Coulomb-type, $1/r$ field of the final system minus that of the initial system. If $\mathbf{P}^A$ is the 4-momentum of mass A of the system and P_i^A is a spatial component of that 4-momentum in the rest frame of the distant observer, and if $\mathbf{k}$ is the past-directed null 4-vector from observer to source, then δh_{ij}^{TT} has the following form:

$$\delta h_{ij}^{TT} = \delta \left(\sum_A \frac{4 P_i^A P_j^A}{\mathbf{k} \cdot \mathbf{P}^A} \right)^{TT}$$

$$\lim_{\omega \to 0} \omega \langle \text{out} | a_+(\omega \hat{x}) \mathcal{S} | \text{in} \rangle = \sum_{k=1}^{n} S_k(\hat{x}) \langle \text{out} | \mathcal{S} | \text{in} \rangle$$

$$\langle \text{out} | \varepsilon_+^{\mu\nu} \Delta h_{\mu\nu} \mathcal{S} | \text{in} \rangle$$

$$S_k(\hat{x}) = \frac{\kappa}{2} \frac{\varepsilon_{\mu\nu}^+ p_k^\mu p_k^\nu}{\hat{q} \cdot p_k}$$

$$\varepsilon_+^{\mu\nu} \langle \Delta h_{\mu\nu} \rangle \equiv \frac{\langle \text{out} | \varepsilon_+^{\mu\nu} \Delta h_{\mu\nu} \mathcal{S} | \text{in} \rangle}{\langle \text{out} | \mathcal{S} | \text{in} \rangle} = \frac{\kappa}{2} \sum_{k=1}^{n} \frac{\varepsilon_{\mu\nu}^+ p_k^\mu p_k^\nu}{\hat{q} \cdot p_k}$$

[Strominger & Zhiboedov, hep-th/1411.5745]

Soft Charges & Degenerate Vacua

In what sense does the soft charge implement an infinitesimal symmetry transformation?

Action on vacuum state:

$$Q_S(\theta,\phi)|\Omega\rangle = |\Omega'\rangle$$

$$Q_S(\theta,\phi) \sim -\lim_{\omega\to 0}\omega\left[a_+(\omega\hat{x}) + a_-^\dagger(\omega\hat{x})\right]$$

- Q_S carries no energy
  - $\Rightarrow Q_S$ maps vacuum states to vacuum states
  - $\Rightarrow Q_S$ implements symmetry transformation *on the vacuum*

[Kapec, Perry, Raclariu, Strominger, hep-th/1703.05448;
Choi, Kol, Akhoury, hep-th/1708.05717;
Choi, Akhoury, hep-th/1712.04551]

Soft Charges & Degenerate Vacua

- Revisit Ward identity:

$$\langle \text{out}|[Q,\mathcal{S}]|\text{in}\rangle = 0, \qquad Q(\hat{x}) = Q_H(\hat{x}) + Q_S(\hat{x})$$

- Label vacuum states by eigenvalue under Q_s

$$Q_S(\hat{x})|\alpha\rangle = \alpha(\hat{x})|\alpha\rangle$$

- Consider scattering states composed of finite-energy particles, built from these vacuum states

$$|\text{out};\alpha\rangle \equiv a^\dagger(p_1)a^\dagger(p_2)\cdots|\alpha\rangle$$

- Finite energy particles do not affect action of the soft charge

$$Q_S(\hat{x}) \sim \lim_{\omega\to 0} \omega\left[a(\omega\hat{x}) + a^\dagger(\omega\hat{x})\right]$$

$$Q_S(\hat{x})|\text{out};\alpha\rangle = \alpha(\hat{x})|\text{out};\alpha\rangle \qquad [a(\vec{p}), a^\dagger(\vec{p}')] \sim p^0\delta^{(3)}(\vec{p}-\vec{p}')$$

[Kapec, Perry, Raclariu, Strominger, hep-th/1703.05448; Choi, Kol, Akhoury, hep-th/1708.05717; Choi, Akhoury, hep-th/1712.04551]

Soft Charges & Degenerate Vacua

➢ Consider the Ward identity between the following scattering states:

$$\underbrace{\langle \text{out}; \alpha^{\text{out}} | [Q_H, \mathcal{S}] | \text{in}; \alpha^{\text{in}} \rangle}_{\substack{Q_H \text{ acts non-trivially on} \\ \text{finite-energy particles}}} = -\underbrace{\langle \text{out}; \alpha^{\text{out}} | [Q_S, \mathcal{S}] | \text{in}; \alpha^{\text{in}} \rangle}_{\substack{Q_S \text{ acts non-trivially on} \\ \text{vacuum states}}}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\left(\sum_{k=1}^{n} S_k(\hat{x}) \right) \langle \text{out}; \alpha^{\text{out}} | \mathcal{S} | \text{in}; \alpha^{\text{in}} \rangle \qquad \left(\alpha^{\text{in}}(\hat{x}) - \alpha^{\text{out}}(\hat{x}) \right) \langle \text{out}; \alpha^{\text{out}} | \mathcal{S} | \text{in}; \alpha^{\text{in}} \rangle$$

$$\Rightarrow \quad \boxed{\left(\sum_{k=1}^{n} S_k(\hat{x}) + \alpha^{\text{out}}(\hat{x}) - \alpha^{\text{in}}(\hat{x}) \right) \langle \text{out}; \alpha^{\text{out}} | \mathcal{S} | \text{in}; \alpha^{\text{in}} \rangle = 0}$$

[Kapec, Perry, Raclariu, Strominger, hep-th/1703.05448; Choi, Kol, Akhoury, hep-th/1708.05717; Choi, Akhoury, hep-th/1712.04551]

Soft Charges & Degenerate Vacua

$$\left(\sum_{k=1}^{n} S_k(\hat{x}) + \alpha^{\text{out}}(\hat{x}) - \alpha^{\text{in}}(\hat{x})\right)\langle \text{out}; \alpha^{\text{out}}|\mathcal{S}|\text{in}; \alpha^{\text{in}}\rangle = 0$$

Solution:

$$\langle \text{out}; \alpha^{\text{out}}|\mathcal{S}|\text{in}; \alpha^{\text{in}}\rangle = 0 \qquad \text{and/or} \qquad \alpha^{\text{in}}(\hat{x}) - \alpha^{\text{out}}(\hat{x}) = \sum_{k=1}^{n} S_k(\hat{x})$$

⇒ A degeneracy of vacuum states is *necessary* for non-trivial scattering!

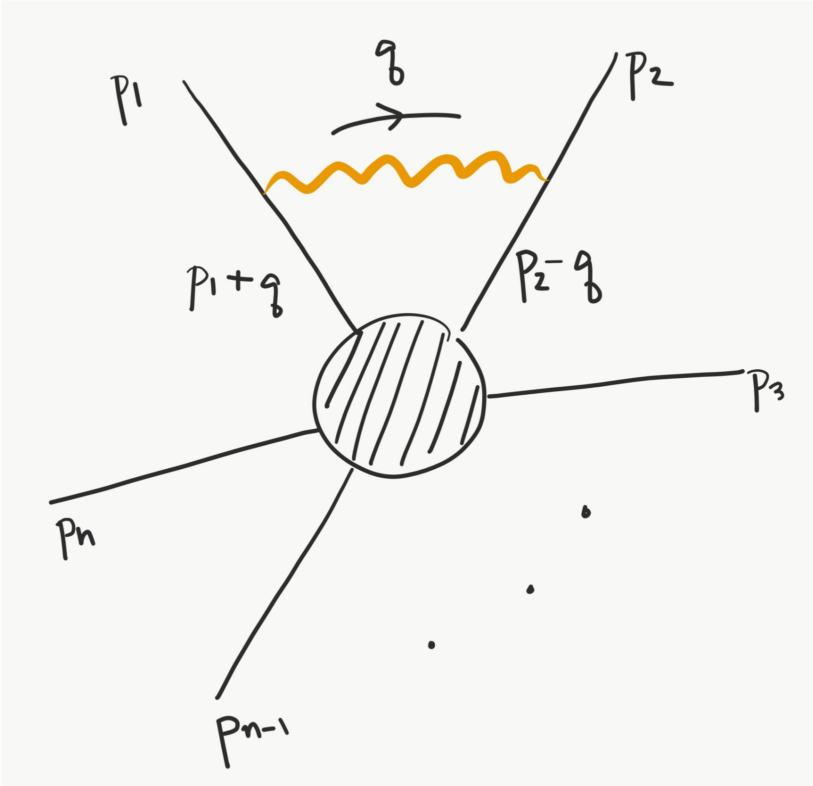
- ➢ Assuming a unique vacuum state, symmetry constraint implies all S-matrix elements vanish.
- ➢ Allowing for vacuum transitions, shift between \`in' and \`out' vacua is determined by the soft factor.

➢ Do standard scattering states account for this degeneracy?

[Kapec, Perry, Raclariu, Strominger, hep-th/1703.05448; Choi, Kol, Akhoury, hep-th/1708.05717; Choi, Akhoury, hep-th/1712.04551]

Infrared Divergences

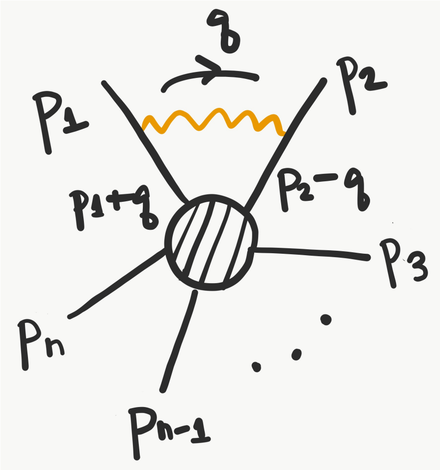
- Recall the story of infrared divergences due to exchange of soft virtual graviton.
- Consider contribution from virtual graviton exchange between external lines.



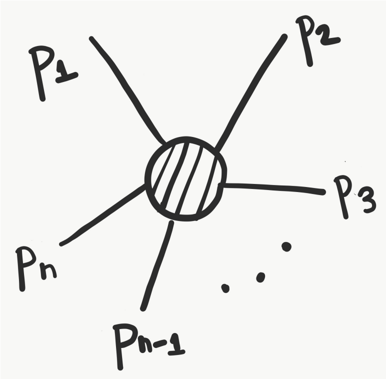
[Weinberg 1965]

Infrared Divergences

- To approximate contribution to loop integral from $q \approx 0$ region, Taylor-expand integrand about $q = 0$:



$$= \int \frac{d^4q}{(2\pi)^4} \left[\frac{\kappa}{2} \frac{p_1^\mu p_1^\nu}{p_1 \cdot q - i\epsilon}\right] \frac{-i\frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)}{q^2 - i\epsilon} \left[-\frac{\kappa}{2} \frac{p_2^\rho p_2^\sigma}{p_2 \cdot q + i\epsilon}\right] \times$$

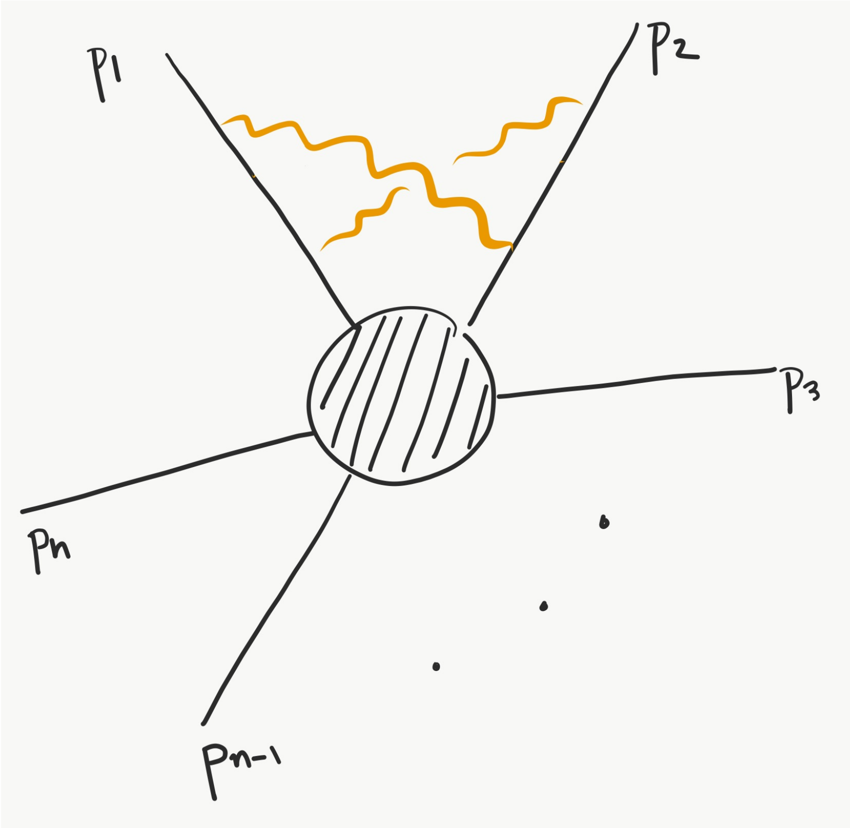


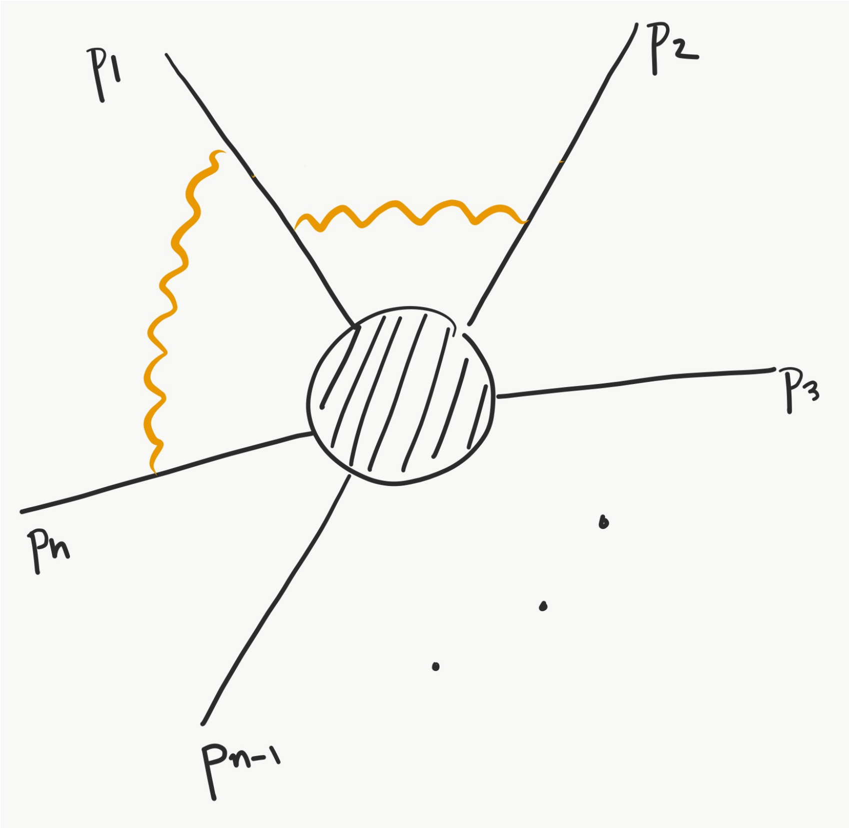
$$+ \ldots$$

- Leading contribution contains logarithmic divergence from $q \approx 0$ region of integration.
- Logarithmic divergence factorizes.
- Vertex rule is soft factor from soft graviton theorem.

[Weinberg 1965]

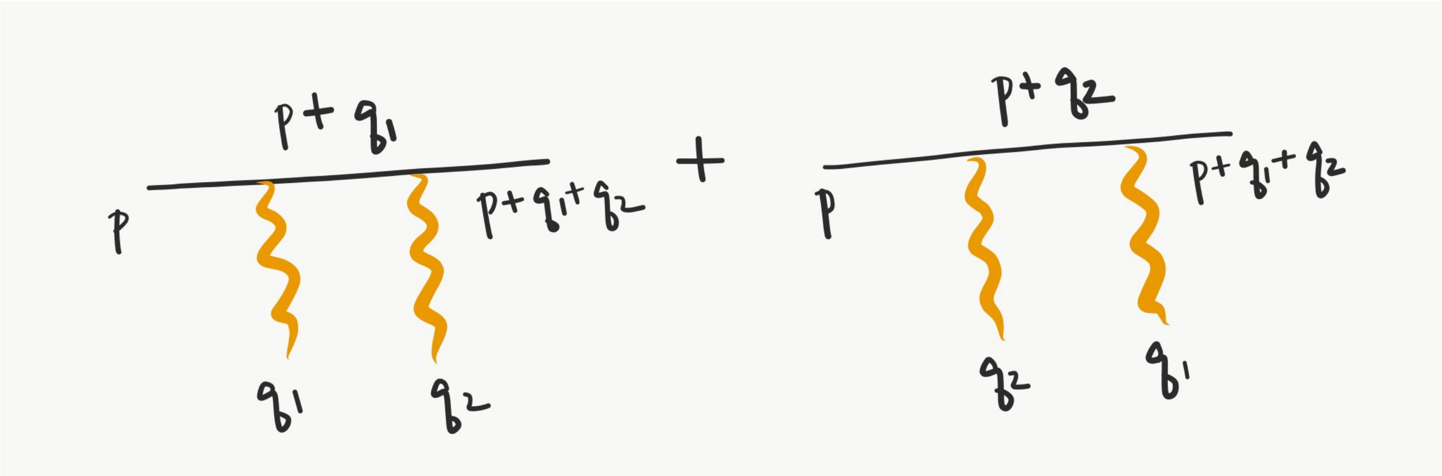
Multiple Soft Exchanges?





[Weinberg 1965]

Multiple Soft Emissions



$$\frac{p^\mu p^\nu}{p\cdot q_1}\frac{p^\rho p^\sigma}{p\cdot(q_1+q_2)}+\frac{p^\mu p^\nu}{p\cdot q_2}\frac{p^\rho p^\sigma}{p\cdot(q_1+q_2)}=\frac{p^\mu p^\nu}{p\cdot q_1}\frac{p^\rho p^\sigma}{p\cdot q_2}$$

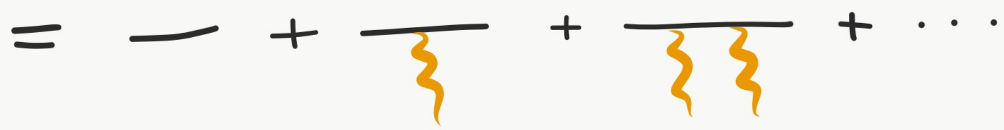
$\Rightarrow$ Contributions from multiple soft exchanges are *multiplicative.*

[Weinberg 1965]

All Soft Exchanges

- An efficient way to sum all soft exchanges is to note that Wilson lines operators reproduce the sum over soft emissions/absorptions.

$$\exp\left[i\int_0^\infty d\tau\ p^\mu p^\nu h_{\mu\nu}(p\tau)\right] = \exp\left[\frac{\kappa}{2}\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2q^0}\frac{p^\mu p^\nu}{p\cdot q}\sum_{\alpha=\pm}\left(\varepsilon^{*\alpha}_{\mu\nu}a_\alpha(\vec{q}) - \varepsilon^{\alpha}_{\mu\nu}a^\dagger_\alpha(\vec{q})\right)\right]$$



$$h_{\mu\nu}(x) = \frac{\kappa}{2}\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2q^0}\sum_{\alpha=\pm}\left(\varepsilon^{*\alpha}_{\mu\nu}a_\alpha(\vec{q})e^{iq\cdot x} - \varepsilon^{\alpha}_{\mu\nu}a^\dagger_\alpha(\vec{q})e^{-iq\cdot x}\right)$$

[Naculich & Schnitzer, hep-th/1101.1524; White, hep-th/1103.2981]

Factorization of Virtual IR Divergences

- The $\mathcal{S}$-matrix factorizes into piece that contains all IR divergences and a piece that is IR-safe (IR-finite).

$$\mathcal{A}(p_1, \cdots, p_n) = \langle W_1 \cdots W_n \rangle_{\lambda_{\mathrm{IR}}, \Lambda} \, \mathcal{A}_\Lambda(p_1, \cdots, p_n)$$

$$W_k = \exp\left[i \int_0^\infty d\tau \; p_k^\mu p_k^\nu h_{\mu\nu}(p_k \tau) \right]$$

$$\langle W_1 \cdots W_n \rangle_{\lambda_{\mathrm{IR}}, \Lambda}$$
$$= \exp\left[\frac{1}{2} \sum_{k,\ell} \int_{\lambda_{\mathrm{IR}}}^{\Lambda} \frac{d^4 q}{(2\pi)^4} \left[\frac{\kappa}{2} \frac{p_k^\mu p_k^\nu}{p_k \cdot q - i\epsilon} \right] \frac{-i\frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)}{q^2 - i\epsilon} \left[-\frac{\kappa}{2} \frac{p_\ell^\rho p_\ell^\sigma}{p_\ell \cdot q + i\epsilon} \right] \right]$$

[Weinberg 1965]

Factorization of Virtual IR Divergences

- Taking the infrared cutoff $\lambda_{\mathrm{IR}} \to 0$,

$$\langle W_1 \cdots W_n \rangle_{\lambda_{\mathrm{IR}},\Lambda}$$
$$= \exp\left[\frac{1}{2}\sum_{k,\ell}\int_{\lambda_{\mathrm{IR}}}^{\Lambda} \frac{d^4q}{(2\pi)^4}\left[\frac{\kappa}{2}\frac{p_k^\mu p_k^\nu}{p_k\cdot q - i\epsilon}\right]\frac{-i\frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma}\right)}{q^2 - i\epsilon}\left[-\frac{\kappa}{2}\frac{p_\ell^\rho p_\ell^\sigma}{p_\ell\cdot q + i\epsilon}\right]\right]$$
$$= \left(\frac{\lambda_{\mathrm{IR}}}{\Lambda}\right)^A$$

$$\boxed{\langle W_1 \cdots W_n \rangle_{\lambda_{\mathrm{IR}},\Lambda} \overset{\lambda_{\mathrm{IR}}\to 0}{\longrightarrow} 0}$$

$\Rightarrow$ Virtual IR divergences cause all exclusive scattering amplitudes vanish.

- Is this same vanishing of scattering amplitudes when the vacuum is taken to be unique?

[Weinberg 1965]

Dressing by Coherent Clouds of Soft Gravitons

- First, consider how to **change** the soft charge carried by the vacuum.
- Can be achieved by finding an operator satisfying:

$$[Q_S(\hat{x}), \mathcal{W}_\alpha] = \alpha(\hat{x})\mathcal{W}_\alpha$$

$$\Rightarrow \qquad Q_S(\hat{x})\mathcal{W}_\alpha|\alpha_0\rangle = (\alpha + \alpha_0)\mathcal{W}_\alpha|\alpha_0\rangle$$

$$\Rightarrow \qquad \mathcal{W}_\alpha|\alpha_0\rangle \propto |\alpha + \alpha_0\rangle$$

[Kapec, Perry, Raclariu, Strominger, hep-th/1703.05448; Choi, Kol, Akhoury, hep-th/1708.05717; Choi, Akhoury, hep-th/1712.04551]

Dressing by Coherent Clouds of Soft Gravitons

- To find $\mathcal{W}_\alpha$, write as:

$$\mathcal{W}_\alpha = \exp\left[i\int d^2\hat{x}\ \alpha(\hat{x})C(\hat{x})\right]$$

$$[Q_S(\hat{x}), \mathcal{W}_\alpha] = \alpha(\hat{x})\mathcal{W}_\alpha \quad \Rightarrow \quad [Q_S(\hat{x}), C(\hat{x}')] = -i\delta^{(2)}(\hat{x}, \hat{x}')$$

Since C is conjugate to Q, it is natural to interpret it as a Goldstone boson.

$$[Q_S[f], C(\hat{x})] \sim -if(\hat{x}) \sim -i\delta_f C(\hat{x})$$

[Kapec, Perry, Raclariu, Strominger, hep-th/1703.05448; Choi, Kol, Akhoury, hep-th/1708.05717; Choi, Akhoury, hep-th/1712.04551]

Dressing by Coherent Clouds of Soft Gravitons

- To find an expression for C, recall the soft charge:

$$Q_S(\hat{x}) \sim \lim_{\omega \to 0} \omega \left[a(\omega \hat{x}) + a^\dagger(\omega \hat{x}) \right]$$

$$\Rightarrow \; C(\hat{x}) \sim i \int_0^\infty d\omega \;\; f(\omega, \hat{x}) \left[a(\omega \hat{x}) - a^\dagger(\omega \hat{x}) \right] \qquad f(\omega = 0, \hat{x}) = 1$$

- Can verify from standard commutation relations:

$$\left[a(\vec{p}), a^\dagger(\vec{p}') \right] = 2p^0 (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \qquad \Leftrightarrow \qquad \left[a(\omega \hat{x}), a^\dagger(\omega' \hat{x}') \right] \sim \frac{\delta(\omega - \omega')}{\omega} \delta^{(2)}(\hat{x}, \hat{x}')$$

$$\begin{aligned} [Q_S(\hat{x}), C(\hat{x}')] &\sim -i \lim_{\omega \to 0} \omega \int_0^\infty d\omega' f(\omega', \hat{x}) \left[a(\omega \hat{x}), a^\dagger(\omega' \hat{x}') \right] \\ &\sim -i \lim_{\omega \to 0} \omega \int_0^\infty d\omega' f(\omega', \hat{x}) \frac{\delta(\omega - \omega')}{\omega} \delta^{(2)}(\hat{x}, \hat{x}') \\ &\sim -i f(0, \hat{x}) \delta^{(2)}(\hat{x}, \hat{x}') \end{aligned}$$

Dressing by Coherent Clouds of Soft Gravitons

- Can change the value of the soft charge by applying

$$\mathcal{W}_\alpha = \exp\left[i \int d^2\hat{x}\ \alpha(\hat{x}) C(\hat{x})\right]$$

$$C(\hat{x}) \sim i \int_0^\infty d\omega\ f(\omega, \hat{x}) \left[a(\omega\hat{x}) - a^\dagger(\omega\hat{x})\right] \qquad f(\omega = 0, \hat{x}) = 1$$

- Recall constraint required:

$$\alpha^{\rm in}(\hat{x}) - \alpha^{\rm out}(\hat{x}) = \sum_{k=1}^{n} S_k(\hat{x})$$

- Satisfy constraint by applying $\mathcal{W}_\alpha$ with $\alpha = -\Sigma\, S_k$ to "out" state.

[Kapec, Perry, Raclariu, Strominger, hep-th/1703.05448; Choi, Kol, Akhoury, hep-th/1708.05717; Choi, Akhoury, hep-th/1712.04551]

Dressing by Coherent Clouds of Soft Gravitons

- Evaluating the dressing:

$$C(\hat{x}) \sim i \int_0^\infty d\omega \; f(\omega, \hat{x}) \left[a(\omega\hat{x}) - a^\dagger(\omega\hat{x})\right]$$

$$f(\omega = 0, \hat{x}) = 1 \qquad S_k(\hat{x}) = \frac{\kappa}{2} \frac{\varepsilon_{\mu\nu} p_k^\mu p_k^\nu}{p_k \cdot q}$$

$$\mathcal{W}_\alpha \Big|_{\alpha = -\sum_k S_k} = \exp\left[i \int d^2\hat{x} \; \alpha(\hat{x}) C(\hat{x})\right]$$

$$= \exp\left[-\frac{\kappa}{2} \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{f(\vec{q})}{2q^0} \frac{p^\mu p^\nu}{p \cdot q} \sum_{\alpha=\pm} \left(\varepsilon^{*\alpha}_{\mu\nu} a_\alpha(\vec{q}) - \varepsilon^\alpha_{\mu\nu} a^\dagger_\alpha(\vec{q})\right)\right]$$

$$\overset{f=1}{=} W_k^\dagger$$

⇒ Contribution from a soft virtual graviton emitted/absorbed from an external line is *precisely cancelled* by emission/absorption from dressing.

⇒ Amplitudes which satisfy the constraint are infrared finite!

[Kapec, Perry, Raclariu, Strominger, hep-th/1703.05448; Choi, Kol, Akhoury, hep-th/1708.05717; Choi, Akhoury, hep-th/1712.04551]

Infrared Divergences & Vacuum Degeneracy

- Infrared divergences which set amplitudes to zero are a consequence of violating the constraint due to symmetry.

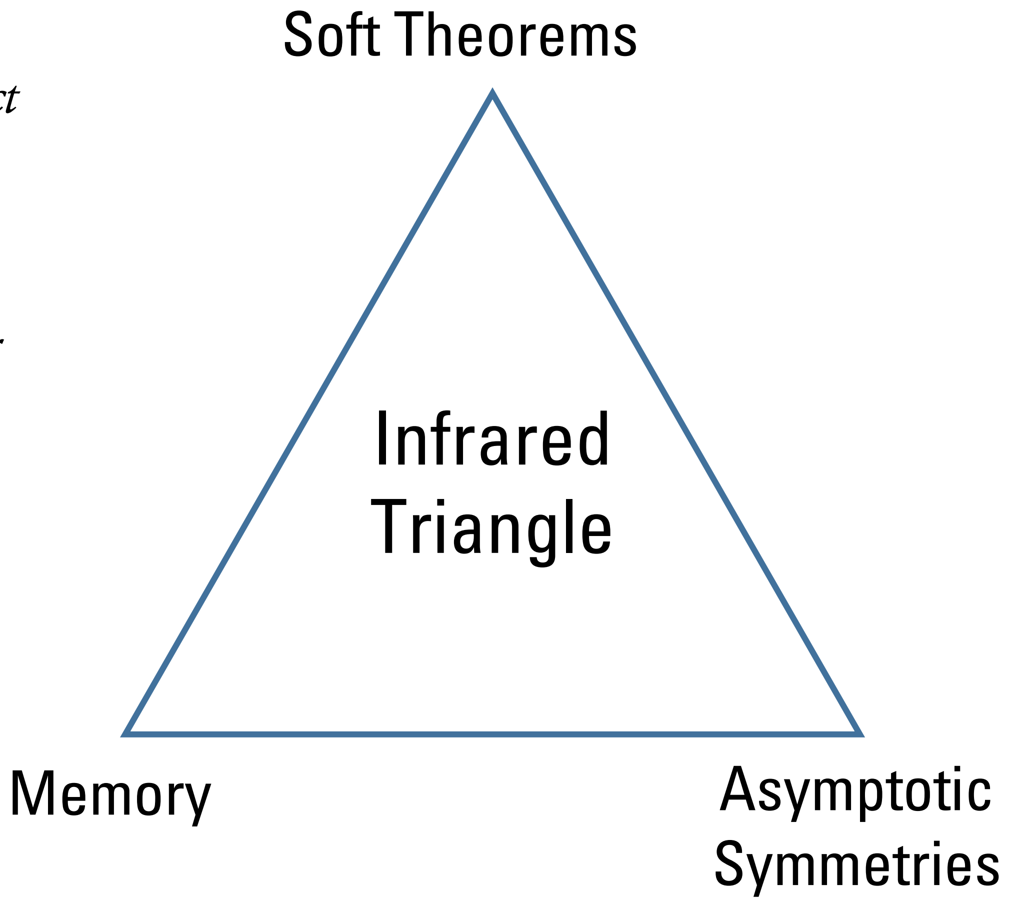
- Physically, the dressings needed to satisfy the constraints represent the gravitational radiation emitted during non-trivial scattering.

- In quantum scattering process, the soft sector is required *by symmetry* to agree with the classical result.

[Kapec, Perry, Raclariu, Strominger, hep-th/1703.05448; Choi, Kol, Akhoury, hep-th/1708.05717; Choi, Akhoury, hep-th/1712.04551]

Summary

1. **Soft Theorems (for Scattering Amplitudes)**
   - *Universal behavior of low-frequency radiation captured by exact relations between scattering amplitudes.*
2. **Soft Theorems = Ward Identities**
   - *Universal behavior in the IR is equivalent to universal behavior due to symmetry.*
3. **Asymptotic Symmetries**
   - *Identification of symmetries implied by soft theorems from the action of the hard charge.*
4. **IR Divergences & Vacuum Degeneracy**
   - *Soft charge implements symmetry transformation on vacuum.*
   - *IR divergences are the consequence of scattering states that violate the symmetry constraint.*



Thank You!